REPRESENTATIONS AND LEARNING OF FRACTIONS

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Abstract: In this paper the ability of grade 5 pupils to use external systems of representation regarding the concepts of equivalence and addition of fractions is examined. The pupils' ability to move from one system of representation to another is also investigated. The underlying assumption of this study is that pupils who develop deep understandings of the concepts of equivalent fractions and the addition of fractions are also able to identify and represent the concepts in different representations and are flexible in moving from one representation to another. The results of this study reveal that pupils have not acquired sufficient abilities for transformation from one representation system to another.

Introduction *The present study investigates the ability of primary school pupils to use external representations and move from one system of representation to another in the context of fractions. The understanding of fractions does not appear to be easy, given the diversity of representations associated with this concept. Streefland (1991, p.6) suggests that fractions "are without doubt the most problematic area in mathematics education". By a representation, we mean a mental structure consisting of the tools used for representing mathematical ideas such as tables, graphs, and equations (Confrey & Smith 1990).*

In mathematics teaching and problem solving, five types of external systems of representations are used: Texts, concrete representations/models, icons or diagrams, languages and written symbols. These external representations are associated with internal representations (Lesh et al., 1987; Duval, 1987; Kaput, 1987 Janvier, 1987; Even, 1998; Hitt, 1998; Gagatsis, Demetriou, Áfantiti, Michaelidou, Panaoura, Shiakalli & Christoforides, 1999; Gagatsis et al., 2000).

By a *translation process,* we mean the psychological process involving the movement from one representation to another (Janvier, 1987). In the last two decades, several researchers have addressed the critical problem of translation between and within representations, and emphasized the importance of moving among multiple representations and connecting them (Goldin, 1998). Researchers have also found that the translations among representations are important for students' learning (Lesh, Post & Behr, 1987), since each representation yields its own insights into mathematical concepts (Confrey & Smith, 1991). Yerushalmy (1997) showed that most students do not take into consideration the movement from one type of representation to another and thus are unable to generalize the concept. In some cases, students identify a mathematical concept with its representations but do not seem to abstract the concept from them (Vinner, 1992).

Aims of the study

We aimed to investigate the ability of grade 5 pupils to handle three forms of external systems of representation, that is symbolic, diagrammatic and verbal, regarding the concepts of equivalence and addition of fractions. We also intended to investigate their abilities to translate the concepts of equivalence and addition of fractions from one system of representation to the other. Two research questions were formulated:

1. Is there a form of representation regarding the concepts of equivalence and addition of fractions that pupils tend to handle more effectively?
2. Is there a form of translation among representations regarding the concepts of equivalence and addition of fractions that pupils tend to handle more effectively?

Hypotheses *of the study*

1. The ability to identify and represent the same concept- equivalence and addition of fractions- in different representations, and the flexibility in moving from one representation to another, allow pupils to see rich relationships and develop deeper understandings.
2. The external representational systems (symbolic, diagrammatic, verbal) regarding the concepts of equivalence and addition of fractions are connected to each other.

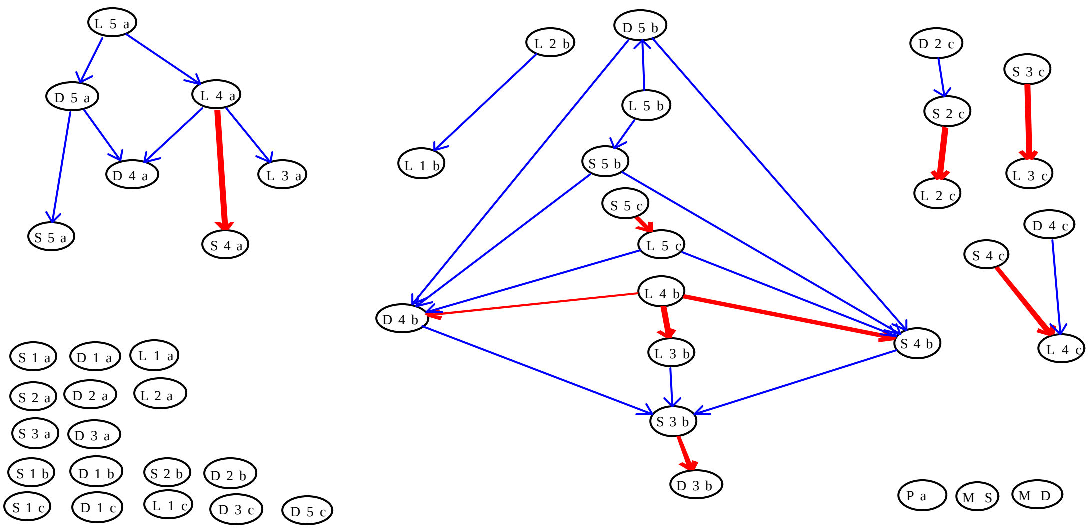
Method *Participants* A written test was administered to 104 pupils of grade 5 (approximate age 10), drawn from two primary schools in Cyprus, during their regular class work. A time limit of 80 minutes was imposed. At the time the written test was administered pupils had been taught the equivalence of fractions but not the addition of unlike fractions.

Ôhe written test The test consisted of four parts. At the beginning of the test, it was stated that information in mathematics could be given in three representations: symbols (symbolic representation), diagrams (diagrammatic representation), and written words (verbal representation). Two examples of equivalence and addition of fractions in symbolic, diagrammatic and verbal form, were presented, in order to help pupils understand what was expected from them.

The first part of the written test consisted of 5 tasks in which pupils were given the symbolic representation of equivalence and addition of fractions and were asked to solve the symbolic tasks and translate them into their diagrammatic and verbal form. The first and second tasks concerned the concept of equivalence of fractions (1/4 = /8, 2/3= /9), whereas tasks 3, 4 and 5 concerned the concept of addition of fractions. Tasks 3 and 4 contained multiplicative denominators (3/4 + 1/8 = , 1/3 + 3/9 =) in contrast with task 5 which contained non multiplicative denominators (2/3 + 3/4 =).
The second part of the written test also consisted of 5 tasks, which were isomorphic to the tasks of the first part. Specifically, pupils were given the diagrammatic representation of equivalence and addition of fractions and were asked to solve the diagrammatic tasks and translate them into their symbolic and verbal form.
The third part of the written test also consisted of 5 tasks, isomorphic to the tasks of the first two parts, in which pupils were given the verbal representation of equivalence and addition of fractions and were asked to solve the verbal tasks and translate them into their symbolic and diagrammatic form.
The fourth part consisted of a problem given in its verbal form. The problem concerned the addition of three unlike fractions with multiplicative denominators (1/4 + 3/8 + 1/16 =). Pupils were asked to solve it with the mode of representation they preferred. The fourth part was added in order to investigate whether there was a form of representation which student prefer and whether there was any connection between problem solving and the translations of the tasks used in the first three parts of the written test.
Analysis Procedure The R. Gras Implicative Statistical Analysis, which is based on a statistical interpretation of the concept of implication, was applied to the data. According to the Gras representation, A→B denotes that "when a student performs successfully in task A, then it is 99% (bold consequences) probable that he/she will also perform successfully in another task B".
With the use of the software "CHIC", which stands for Cohesive Hierarchical Implication Classification, the consequence statistical analysis yielded two diagrams: The implication diagram, where the implications between the tasks are shown, and The similarity tree where the variances are connected depending on the similarity presented by these variances.

Results

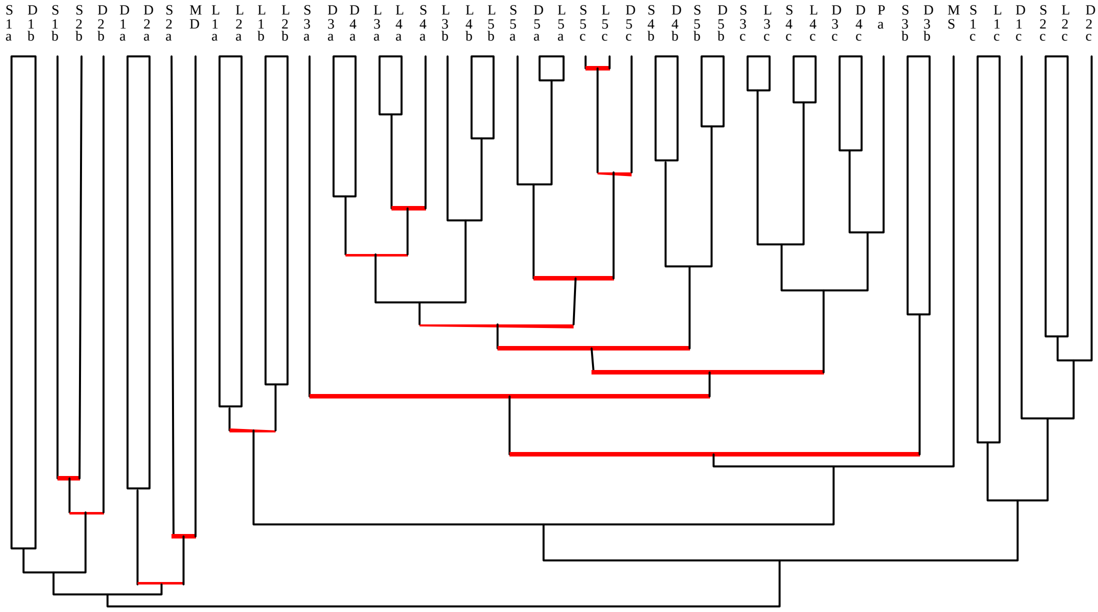
According to the implication diagram, the observed implications concern tasks that belong only to the first part of the written test (e.g. L5a→L4a→L3a) or only to the second part (e.g. L4b→L3b→S3b→D3b) or only to the third part (e.g.D2c→S2c→L2c). In other words, the tasks are divided in three separate groups: Task L5c seems to be an exception, as it appears between the implications of tasks that belong to the second part of the written test.
Additionally, according to the implication diagram, the observed implications mostly concern the same concept: equivalence of fractions of a particular task / addition of fractions of a
particular task (e.g. task 5a:L5a→D5a→S5a or task 2c: D2c→S2c→L2c) or the same representation (e.g. verbal: L2c→L1b). In other words, all the correlations that appear between variables are intra-conceptual (concern the same concept) or intra-representational (concern the same representation). Finally, two remarkable observations are that no implications appeared between the concepts of equivalence and addition of fractions and that the problem of the fourth part of the written test was not correlated with other tasks.



Implication diagram

S, D, L: Form of representation (Symbolic, Diagrammatic, Verbal, respectively) á, b, c: Part of the written test (first, second, third, respectively) 1,2,3,4,5: Number of task Pa: Problem answer (Part 4) MS: Effort in symbolic method (Part 4) MD: Effort in diagrammatic method (Part 4)
Four similarity groups appear in the similarity tree:

1. Tasks of the first and second part of the test that concern the concept of equivalence of fractions and examine handling of symbolic and diagrammatic representation and translations from symbolic to diagrammatic and from diagrammatic to symbolic representation (tasks S1a, D1b, S1b, S2b, D2b, D1a, D2a, S2a).
2. Tasks of the first and second part of the test that concern the concept of equivalence of fractions and examine translations to verbal form (tasks L1a, L2a, L1b, L2b)
3. Tasks of the first, second and third part of the test that concern the concept of addition of fractions (tasks S3a, D3a, D4a, L3a, L4a, S4a, L3b, L4b, L5b, S5a, D5a, L5a, S5c, L5c, D5c, S4b, D4b, S5b, D5b, S3c, L3c, S4c, L4c, D3c, D4c, Pa, S3b, D3b, Ms).
4. Tasks of the third part of the test that concern the concept of equivalence of fractions and examine the handling of verbal representations and translations from verbal form (tasks S1c, L1c, D1c, S2c, L2c, D2c).



Groups A and B are connected with group D. The connection is considered natural since the three groups concern conceptually relative tasks (equation of fractions), although it appears to be in a lower level of importance compared to the connections between the tasks of group C. This means that pupils behave more similarly towards the tasks that concern addition of fractions and less similarly to the tasks that concern equivalence of fractions.

The variable Pa, that stands for the answer to the problem of the fourth part of the test, is connected with tasks of the third part of the test. This result is reasonable since both the task of the fourth part and the tasks of the third part concern the same translation process: from verbal representation to symbolic or diagrammatic representation.

The most important finding associated with the similarity tree is that tasks that concern the concept of equivalence of fraction are, in no case, connected with tasks that concern the concept of addition of fractions. This indicates that pupils do not connect the concept of equivalence with the concept of addition of fractions.

The hierarchical tree diagram indicates that the consequences of priority between the tasks of the first three parts of the test appear in different sequence. For example, the three representational modes appear in a sequence D→S→L for the first part of the text, whereas L→S→D appears for the second part of the test. In the third part of the test, the sequence of priority reappears as D→S→L. The different sequences of priority reveal that the difficulty relating to the tasks of each part of the test is experienced differently by pupils. In other words, pupils cannot realise that the three forms of representation, i.e., symbolic, diagrammatic and verbal, constitute different ways of expressing the same concept.

CONCLUSIONS In summarising the results of the present investigation, the most important finding is that the pupils´ flexibility in moving from one representation to another is poor and thus they are unable to see rich relationships and develop deeper understandings regarding the concepts of equivalence and addition of fractions. The results that emerge from the Gras analysis provide good support for the above conclusion: The implication diagram indicated that there is no connection between the three parts of the test, which shows that there is no connection between the symbolic, diagrammatic and verbal representations. In other words, pupils cannot realise that the three forms of representation, symbolic, diagrammatic and verbal, constitute different ways of expressing the same concept and, in contrast, they think that every representation concerns a different concept. According to the first hypothesis, pupils develop deeper understandings of the concepts of equivalence and addition of fractions if they are able to handle successfully the concepts in at least two different fields of representations. The possession of this ability is an important indication of the understanding of the concepts of equivalence and additions of fractions. The participants of the present study do not possess this ability, so the concepts of equivalence and addition of fractions are not completely constructed.

The second important finding of the present study, which relates to the second hypothesis, is that the external representational systems (symbolic, diagrammatic, verbal) regarding the concepts of equivalence and addition of fractions are connected to each other as long as they concern the same task-concept. For example, the diagrammatic representation is connected with the verbal in the case of task 4 in the third part of the test (e.g. D4c→L4c). The only case that conceptually different tasks are connected to each other is when they appear to be in the same form of representation (e.g. verbal: L2c→L1b). In other words, all the implications - except the ones that concern task 5c (L5c→D4b, L5c→S4b) – that appear between variables are intra-conceptual (concern the same concept) or intra-representational (concern the same representation). This result indicates that pupils have not acquired sufficient abilities for transformation from one representation system to the other.

Finally, SPSS analysis gave answers to the defined research questions: Pupils handle the symbolic representation more effectively as long as it concerns the concept of equation of fractions. In contrast, in the case of the addition of fractions, they handle the diagrammatic representation more effectively. In both cases, pupils handle the verbal representation less effectively (at this point we must remind the reader that pupils had been taught the equivalence of fraction but not the addition of fractions). Moreover, pupils handle more effectively the translation from diagrammatic to symbolic representation and less effectively the translation from verbal to diagrammatic representation.

The above findings have some practical bearing on instruction: First, teachers should start teaching equivalence and addition of fractions from the diagrammatic representation, then proceed to the symbolic representation and conclude with the verbal representation which seems to be the most difficult for the pupils. Second, and this may be more important, teachers, in the process of learning and understanding, should use as many representational systems as possible since the ability to identify and represent the same concept in different representations and the flexibility in moving from one representation to another is intertwined with the deeper understanding of the concept.

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