

MATHEMATICS TOWARDS PERCEPTION OF THE WORLD

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Abstract: The paper concerns mathematical tools to describe effectively real life situations. Problems of performing relevant and precise information by reasoning process are discussed and existing mathematical tools are mentioned. Consequent mathematics teaching aspects are pointed out.

Introduction

The human existence is connected with a permanent perceiving of the world through the mind. All information acquired by our senses is performed by "individual" reasoning process, which is formally an algorithm yielding certain *outputs* as "results". Outputs possess various forms, e.g. a sentence (or a declaration) in natural language, a number stored in a mind, the way of execution etc. The reasoning process has many faces and dimensions - it may be natural or naive, sophisticated, goal-seeking, etc. In any case, it is strongly determined by an individual's education and experience. To impart knowledge and information, we use our everyday natural language.

When employing natural language to express ideas or formulate judgements, there is a great deal of imprecision and vagueness. Working with some data, we automatically take them as "precious". It tackles the essence of "preciousness" itself. In principle, an "absolute" preciousness need not be attainable. For instance, there can not exist two materials with exactly the same amount of their components. There is another aspect we encounter when describing the reality. It concerns the proportion between the relevancy and the preciousness of information. Increasing the level of preciousness we reach the point when relevancy and preciousness become mutually excluding characteristics. For instance, the principles of functioning of a school may be described by means of several sentences. Trying to get more precise information, we may ask for the names of scholars, subjects, timetable, even the names of parents of scholars, etc. Hence, we must work with more and more precise data and employ more and more sophisticated methods. Obviously, collecting so many facts the information becomes not relevant to meet that was required and we must politely return to concentrated verbal form using natural language. L.A.Zadeh (1973a) formulated it as "*The principle of incompatibility*". In practical situations, we must decide between the relevancy of information, that is less precise, and its preciousness, that is less relevant. For instance, if we want to describe the functioning of a car for the purpose of safety driving, the relevant information should contain "minimal" most important facts, not necessarily all construction details that are more precise. From this viewpoint, a natural language plays a crucial role. Natural language is regarded as the best tool to express effectively relevant information (e.g. Zadeh (1973b)). Natural language serves not only for communication between people. It provides surprisingly effective possibility to carry very important and fast information (although vague) to control human activities. For instance, the order "slow the speed" is vague, but apparently relevant when instructing driving.

In similar considerations, also words "uncertainty", "insecurity" occur in the literature (for the details e.g. Novak(2000), Nguen and Walker(2000)). The need to employ such "mysterious" words is the complexity of real situations around us. In the sequel, we will deal with some views on tools and the capability of mathematics to cope with the mentioned problems.

Mathematical tools to perceive the world

The complexity of real events and consequent impossibility to include all details and existing factors led to the use of procedure of simplification in order to be able to investigate real-life situations. The procedure is currently known as *modelling*. The results of modelling are *models*. In case that modelling is based on mathematical tools, we speak about *mathematical modelling* or *mathematical models*, respectively (in the sequel we omit "mathematical", because other types will not be considered). Theory of modelling is extensively developed discipline with widespread applications. Particularly, in modern trends in mathematics teaching, the phenomenon of modelling plays an important role. From the viewpoint of practical didactic a certain classification of modelling types were suggested (e.g. Meznik(1999)), namely quantitative, qualitative, visual, analytic and abstract. All the mentioned types of modelling and corresponding models were considered under assumption that the investigated collection of objects forms a set. Taking into account contemporary potential of mathematical disciplines, strong and large tools are at the disposal.

In the following we will keep the most natural scheme of perceiving a real situation - starting from some verbal statement (declaration) and ending with a model (mostly a visual model). Also, the choice of examples will be directed at instructing of mathematics.

Example 1.: *The observation claims :* (In Figures 1-4 visual models of the statement are depicted.)

- (a) "Demand is falling sharply" (Fig. 1)
- (b) "Price is increasing slowly" (Fig.2)
- (c) "Profit is static" (Fig.3)
- (d) "Sales rose in 1989, then reached a peak in 1990, and fell back in 1991" (Fig.4)

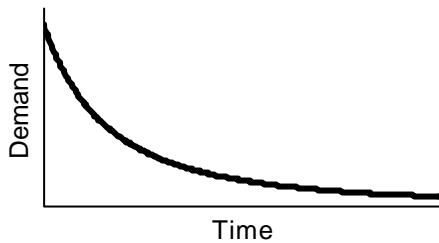


Figure 1.

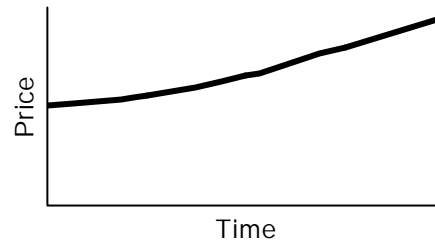


Figure 2.

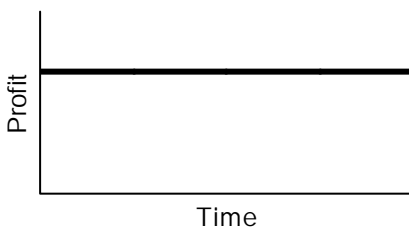


Figure 3.

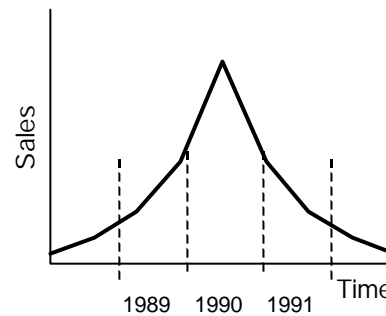


Figure 4.

Example 2.: A child is eating chocolate bonbons and feels a satisfaction. Its satisfaction is measured in numbers as the utility U . So, the utility U is a function of Q (the number of bonbons), $U = U(Q)$. Obviously, when $Q=0$, then $U=0$. Further, if the number of bonbons increases, then utility increases, but only to some maximum level U_{max} , then it decreases (potentially reaching the aversion to chocolate). Still more- because of the maximum point, the increase in utility due to a one unit increase in bonbons must eventually decline (this property is known as the law of diminishing returns). In other words, once the number of bonbons reaches a certain threshold level U_0 , the rate of change in utility will go down (up to U_0 the rate of change in utility may increase). Visual model of this verbal statement is depicted in Figure 5.

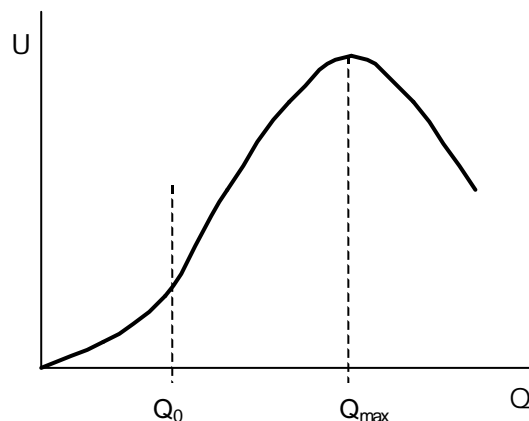


Figure 5.

As mentioned above, the collections of objects undergoing modelling procedure formed sets. In fact, in the real physical world, we frequently encounter collections of objects of non-set nature. To

specify a collection of object as a set we need a unique criterion (say membership criterion) deciding whether an arbitrary object belongs to the collection or not. For instance the collection of people getting income higher than 500USD per month is a set, but the collection of people getting very low income can not be specified as a set. Apparently, a unique generally accepted membership criterion in the latter case does not exist. In such collections an object need not necessarily either belong to or not belong to a collection; there may be "intermediate" degrees of membership. It concerns particularly objects of vague or imprecise or not sharply defined type specified using natural language as mentioned in Introduction. For this purpose the concept of a fuzzy set was introduced by L.A. Zadeh in 1965. A *fuzzy set* (analogous to Zadeh(1965)) is a couple (U,A) , where U is a set (called also a *universum*) and A is a function of U into the interval $< 0,1 >$. A is called a *membership function* (also a *characteristic function*) and the value $A(x)$ is called the *degree of membership* of x in the fuzzy set. For the simplicity, we identify the fuzzy set with its membership function, the universum U is usually clear from the context. So, the fuzzy set A has the membership function A . Obviously, an ordinary set is a fuzzy set with membership function whose images of elements form the two element set $\{0,1\}$. From the definition of a fuzzy set we observe that the nearer the value $A(x)$ to unity, the higher the degree of membership of x in A . The choice of the function A is subjective and context dependent and can be a delicate one. On the other hand, the flexibility in the choice of A is useful in reasoning.

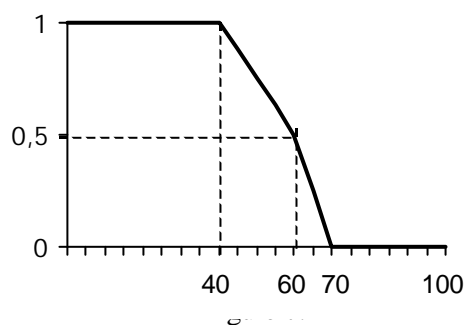
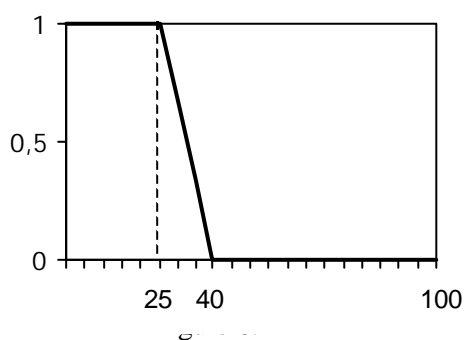
Example 3 : Consider the fuzzy set "young man". Different people may suggest different models of membership function . One such model, decided upon by a young man might be

$$A(x) \begin{cases} 1 & \text{if } x < 25 \\ (40 - x)/15 & \text{if } 25 \leq x \leq 40 \\ 0 & \text{if } x > 40 \end{cases}$$

An older person might model it with the membership function

$$B(x) \begin{cases} 1 & \text{if } x < 40 \\ (80 - x)/40 & \text{if } 40 \leq x \leq 60 \\ (70 - x)/20 & \text{if } 60 < x \leq 70 \\ 0 & \text{if } x > 70 \end{cases}$$

The corresponding fuzzy sets A and B are depicted in a standard way in Figures 6 and 7.



Fuzzy set theory has a very wide scope of applicability in solving various kinds of real physical world problems, particularly in the fields of pattern recognition, information processing, control, system identification, artificial intelligence, decision processes involving incomplete or uncertain data etc. A consumer may find the word "fuzzy" in manuals of current products (washing machines, cameras, cleaners, cookers), where fuzzy set theory is applied in the construction of controlling mechanisms.

When fuzzy set theory is used to set up models of real phenomena, we speak about *fuzzy modelling* and *fuzzy models* .

One of the most valuable contributions of fuzzy modelling to perceive the world consists in effective use for the interpretation of semantics of natural languages. Fuzzy sets proved to be suitable mathematical tool to apprehend the vagueness of natural language. The concept of a *linguistic variable* was introduced (Zadeh(1973b)), whose values may be words or some expressions of a natural language. In this manner we can model the semantics of certain types of words, called *linguistic operators*. Linguistic operators are interpreted (and defined) as operations on the membership function. The most illustrative example is the use of adverbs (so called intensifying adverbs) as linguistic operators. Such typical adverbs are "very", "more or less". Their influence on the membership function of the fuzzy set "small" is shown in Figure 8 (visual fuzzy model).

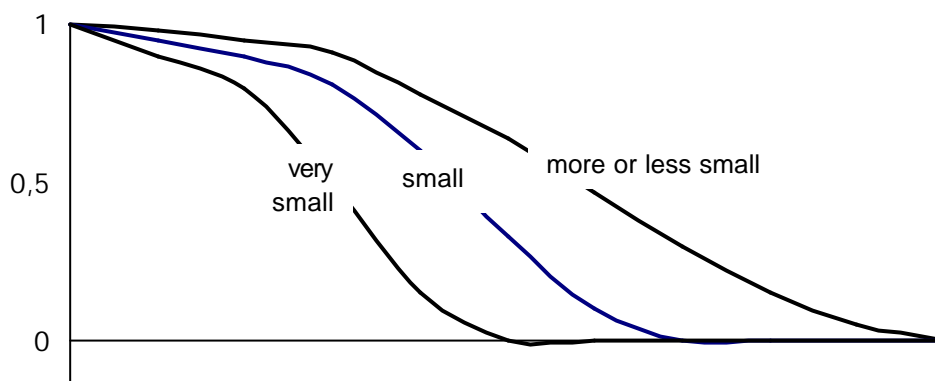


Figure 8.

There is an inconvenience when creating constructions on natural languages. Each natural language possesses specific non-transferable building elements and also natural languages differ in the potentiality and possibility to express slight modifications of the meaning of words. So, the attempts to express mathematically in a standard way the influence of linguistic operators in terms of operations on membership function were not much successful. So, such constructions should be closely related to a specific natural language. With a view to the fact that the mathematics is not so sophisticated to perform associated operations, the use of fuzzy modelling in natural language constructions may significantly contribute to develop the ability to work with quantities of vague or uncertain feature. To my opinion simple and intuitively motivated tasks of the mentioned type using linguistic tools may positively enrich the spectrum of mathematically modelled real-life problems.

References

- Dilwyn, E., Hamson, M.(1989): *Guide to Mathematical Modelling*. Macmillan Press Ltd., London
- Krantz, G.S.(1999): *How to Teach Mathematics*. American Mathematical Society, Providence, Rhode Island
- Mezник, I.(1999): *Modelling as a Support in Teaching of Mathematics*. In: Proceedings of the International Conference on Mathematics Education into 21th Century: Societal Challenges, Issues and Approaches(Ed. A.Rogerson), Volume II, Third World Forum Project Egypt 2000, Cairo 1999, 95-100
- Nguyen, H.T., Walker, E.A.(2000): *A First Course in Fuzzy Logic*. Chapman and Hall/CRC, New York
- Novak, V.(2000): *Foundations of Fuzzy Modelling*. BEN Publ.Comp., Praha (in Czech)
- Zadeh, L.A.(1965): *Fuzzy sets*. Information and Control 8, 338-353
- Zadeh, L.A.(1973a): *Outline of a New Approach to the Analysis of Complex Systems and Decision Processes*. IEEE Trans.Syst.Man.and Cybern. 1, 28-44
- Zadeh, L.A.(1973b): *The Concept of a Linguistic Variable and its Applications to Approximate Reasoning*. Elsevier Publ.Comp., New York