Authentic Tasks and Mathematical Problem Solving

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Approaching mathematics teaching from a humanistic approach involves dynamic interactions supporting meaningful and relevant mathematics learning. A primary goal of such learning is to prepare students to solve everyday problems. Authentic tasks are a critical tool in developing the level of mathematical understanding and conceptualizing indicative of mathematical literacy. Authentic tasks can be characterized by in-depth analysis along four dimensions: thinking and reasoning, discourse, mathematical tools, and attitudes and dispositions. Each of these dimensions, collectively and singularly, must be an explicit focus of the mathematics that engages the students. This paper will present a model demonstrating how these dimensions contribute to a balanced instructional and curricular focus that is at the heart of humanistic mathematics education.

In order to support students' development of mathematical literacy, a standards-based mathematics curriculum should provide for experiences that differ from those indicative of the traditional, typical classroom. Such a curriculum provides opportunities for the students to explore important mathematical topics in the form of situational problems that actively engage students. These lessons, grounded in real-life problems, include four dimensions:

- Thinking and reasoning – engaging in such activities as gathering data, exploring, investigating, interpreting, reasoning, modeling designing, analyzing, formulating hypotheses, using trial and error, generalizing, and verifying outcomes
- Discourse – engaging in individual, small group, and whole class interactions; role of language and interactions in the construction of mathematical meaning
- Mathematical tools – using symbol systems such as tables, graphs, and drawings and technological tools such as calculators, computers, and manipulatives
- Attitudes and dispositions – developing self-regulation, persistence, reflection, and enthusiasm (Pandey, 1990).

Mathematics that is embedded in authentic tasks frequently mix hard and soft data, provide students opportunities to approach the problem with multiple approaches and reach different conclusions grounded in sound mathematical reasoning are the types of tasks that are typical in life and work. While such tasks frequently do not appear to be 'school' mathematics problems, they serve as an effective means of developing important mathematical understanding while also supporting the development of basic skills.

Thinking and Reasoning

Students' experiences with mathematics should support the development of their ability to reason and use logic progressing on a continuum from concrete to formal to abstract. Tasks should build inductive reasoning skills through making conjectures based on generalizations from patterns in observations. In addition, students should employ deductive reasoning in testing those conjectures. In all cases, students' thinking and reasoning is enhanced when they are members of classroom communities that require them to explore, make conjectures, test their ideas, defend the validity of the outcomes, and convince others of their approach. Authentic tasks provide the necessary framework for integrating both mathematics content and process objectives into meaningful learning situations. Consider the thinking and reasoning possible in the following example:

This lesson emerged when the school had a tornado drill and students discovered that the hallway was too small to fit them all. Students investigate the concept of area by figuring out how many people will fit in areas in the school building. Specifically, students investigate how many of them will fit in the following areas: (1) their classroom, (2) a hallway, (3) a lobby, and (4) the cafeteria tables. Students are divided into groups of four and assigned to one of the three areas. The groups choose the materials they want to use and are accompanied to their designated areas by classroom volunteers. The groups estimate the number of students who will fit in an area and then develop a strategy for determining the exact number. During their investigations, students organize and record their solutions and use measuring, counting, and addition to find the total number of people. At the end of the lesson, the class reconvenes and students share their strategies and results. (Annenberg/CPB, 2002).

Thinking and reasoning should be an explicit element in teaching mathematics. The United Kingdom's Cognitive Acceleration through Mathematics Education project is an example of a successful program with theoretical bases in social psychology and developmental psychology. The lesson provides appropriate mediation using cognitive conflict to engage students in higher levels of mathematical thinking (Mok & Johnson, 2001). On-going research shows that CAME supports metacognitive development through raising one's awareness of how they learn and solve problems. Explicit focus on reasoning and thinking supports the intent of the NCTM standards (NCTM, 2000) that advocate students should recognize reasoning and proof as fundamental to mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments, and select and use various types of reasoning. Experience with mathematical modes of though builds cognitive capacity to read critically, identify fallacies, detect bias, assess risk, and suggest alternatives (National Research Council, 1989). Authentic tasks must include substantive emphasis on developing such habits of thought.

Discourse In mathematical communities "we continually and actively build and rebuild our worlds not just through language, but language used in tandem with actions, interactions, non-linguistic symbol systems, objects, tools, technologies, and distinctive ways of thinking, valuing, feeling, and believing." (Gee, 2000, p. 11) Well-balanced mathematics instruction allows students to explore the big ideas of mathematics in a real-world context. For example, students in one middle school classroom used a model of similar triangles to find the estimated distance of a new road connecting two-cities. The task required students to employ a mathematical model but also realize that in this real-world context meant that their responses were 'best' estimates of the actual distance. More importantly, the classroom experiences reinforced mathematical reasoning and communication. Students were asked to write about their solution processes. Peer evaluations of the responses supplied feedback on the mathematical 'soundness' of the approaches and the extent to which good mathematical communication had been modeled. The importance of discourse is evident as students in this classroom engaged in tasks embedded in experiences requiring them to justify their mathematical reasoning and communicate such thinking to their peers and teacher (Pugalee, 2001; Lajoie, 1999).

One means of promoting discourse communities involves a three-phase approach applicable to group discussions. In the first phase, the intent is to get as many student ideas out in the open as possible. What do you think? Why? are good prompts to get the discussion 'geared-up'. In the second phase, the class moves to comparing and evaluating ideas. Open discussion is encouraged, but the teacher focuses on the content of the discussions so that the discussion can be facilitated to promote mathematical understanding. Is your idea the same as …? What do you think about…? are ways to structure this level of the discussion. The third or focusing phase of the discussion focuses the range of ideas. Teachers filter and direct ideas to help the class progress on the issue (Sherin, 2000). Teachers of mathematics should develop tasks that support the use of a variety of forms of discourse, including oral, written, pictorial, symbolic, and graphic. Teachers should take an active role in advancing and organizing discourse so student learning is maximized, including encouraging students to make conjectures so their ideas can be assessed and others can consider the reasonableness of the ideas. Student communication should include multiple modalities including written and oral work. Various structures provide collaborative and cooperative experiences where student listen to and react to others, clarifying their own thinking while asking questions that develop standards for mathematical reasoning and communication (Pacific Regional Education Laboratory, 1996).

Mathematical Tools Authentic tasks allow students to explore multiple modes of representation in experientially real contexts using a variety of technological tools. Abrams (2001) argues that school mathematics has ignored the development of mathematical models in applied settings resulting in a lack of skills such as abilities to analyze units, make choices among multiple representations, and recognize common structures. He advocates moving away from traditional school mathematics as self-contained to curricula that emphasizes mathematics as a tool for solving important problems from other disciplines. NCTM (1991) advocates the use of various tools to aid students in obtaining mathematical power: computers, calculators, and other technology; concrete materials used as models; pictures, diagrams, tables, and graphs; invented and conventional terms and symbols; metaphors, analogies, and stories; written hypotheses, explanations, and arguments; oral presentations and dramatizations.

Gravemeijer, Cobb, Bowers, and Whitenack (2000) describe one approach (formulated by Kaput, 1994) that provides a bridge from formal mathematical symbolizations through the use of computer-based representation systems. Instead of using ready-made graphs of time and distance, students would use their own authentic experiences to understand symbolizing. MathCars provides a computer-simulated driving experience that creates distance-time graphs, velocity-time graphs, and data tables of a simulated trip. Various mathematical symbolizations are linked to dashboard displays. Such investigations allow everyday experiences of motion to be linked to formal graphical representations.

The Middle-school Mathematics through Applications Program (MMAP, see http://mmap.wested.org/.) is a good example of how tools and authentic problem solving provide central curricular tenets. The project-based materials use various software tools and engage students with real-world problems requiring them to apply the mathematics they have learned. The units emphasize proportional reasoning, algebraic expressions, and functions as well as statistics, probability, measurement, and geometry. One of the goals of the program is to increase students' conceptual understanding of mathematics, to improve their ability to use standard symbolic notation, and to improve their mathematical communication skills. In one unit on Antarctica, students play the roles of architects designing living space for themselves or a client or the design of a research station (depending on grade level). Projects such as these emphasize the role of mathematical tools in developing students' capacity to explore, investigate, and formulate conclusions through explorations in problem situations grounded in real world situations.

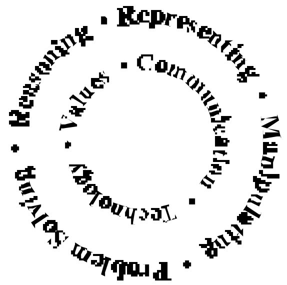
Attitudes and Dispositions

Attitudes and dispositions have historically not been considered in problem solving research. Many students experience "'math phobia', a disconnect between the mathematics and its applications, and mathematics getting in the way of learning (inward view of math, mathematics for its own sake, too abstract, fuzziness, etc.)" (Henderson et. al., 2001). (Lajoie (1999) who worked with the K-12 Authentic Statistics Project, designed to prepare students for decision making in the real work, found that "when students work on tasks they find engaging and are expected not merely to give the 'right' answers but to understand the statistical relationships they use and manipulate, they own that knowledge, stay interested in the mathematics, and do not fear working on problems in new contexts." (p. 131). In one problem at the middle school level, students calculated their pulse rate, used computer software to analyze and chart class data, and engaged in discussions about differences in the data representations. Likewise, an elementary project reported that students engaged in an authentic task involving water quality monitoring felt that they had done important work and were proud of their accomplishments. Scientists who reviewed the culminating presentations related that they were impressed by the quality of the students' thinking as well as their presentation skills (Vye et. al., 1998).

DeBellis and Goldin (1997) posit that beliefs, attitudes, emotions, and values interplay with cognition and as such can either facilitate or hinder monitoring during problem solving. While emotions are more temporary, the others have more stability serving as self-regulating structures. Affective pathways are either positive or negative and impact problem-solving behavior. If a positive pathway is experienced initially during problem solving experiences, curiosity may serve as a motivating factor leading the individual to a deeper understanding of the problem and the enactment of exploratory heuristics. Frustration may lead to an impasse resulting in an ineffective revision of strategies. Experiencing a negative pathway may lead to bafflement and the individual resolving to use 'safe' procedures rather than exploration. Should these efforts fail, the initial frustration may result in anxiety and reliance on others, including teachers and peers, as authorities as well as to avoidance behaviors. Expert problem solvers handle affective issues better. Whereas some students feel they should follow established procedures during problem solving, others employ originality and self-assertiveness. While 'good" problem solvers enact responses that are productive when faced with insufficient understanding, other students resort to guessing or using likely, and often inappropriate, procedures perhaps because they feel that their personal mathematical knowledge is deficient.

NCTM's position statement, "Guiding Students' Attitudes and Decisions Regarding their Mathematics Education" (see www.nctm.org), elaborates on what school programs should do to foster positive attitudes and dispositions. The elements are consistent with those explicated throughout this paper regarding the dimensions of effective authentic learning tasks. Teachers should build on the curiosity and eagerness to learn expressed by elementary students by using real-life experiences emphasizing the value and usefulness of mathematics. Middle school students need encouragement that fosters confidence in their ability to make sense of mathematics. Real-life examples help this age group make connections between mathematics and their future educational and vocational options. Secondary students should recognize the relationships between academic and career choices. Students' beliefs in the relevance and value of mathematics should be reinforced, as should their views about their ability to "do" mathematics, and their need to be life long learners. All students should be provided with current information on the relationship of what they are learning and future options, the increasing number of career opportunities available through mathematical study, and how careers in other field depend on mathematical knowledge.

Conclusions Authentic tasks are important in the development of students' mathematical literacy and their ability to engage in mathematics applied in diverse fields as adult members of society. In considering the nature of such important tasks, four dimensions were described: thinking and reasoning, discourse, mathematical tools, and attitudes and dispositions. Another way to view the importance of these elements is through a model of mathematical literacy (Pugalee, 1999). Problem solving, representing, manipulating, and reasoning are represented in the outer circle of the model. These are the processes of being mathematically literate and are essential components to address in developing and/or selecting authentic mathematical tasks. The enablers of communication, technology, and values play a crucial and central role as depicted in the model. These enablers facilitate the "doing of mathematics". Each of the circles is interdependent underscoring the complex and intricate nature of mathematics. Authentic tasks should be a substantive part of mathematics problem solving; however, such tasks must provide extensive connections to mathematics content and processes that develop mathematical understanding and power. The characteristics of such problems as elucidated in this paper provide a framework for guiding the design and implementation of such tasks into mathematics classrooms.



References

Abrams, J. P. (2001). Teaching mathematical modeling and the skills of representation. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 269-282). Reston, VA: National Council of Teachers of Mathematics.

Annenberg/CPB. (2002). How many people will fit? *Teaching math: A video library K-4*. Cambridge, MA: Author.

DeBellis, V. A. & Goldin, G. A. (1997). The affective domain in mathematical problem solving. In *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 209-216), Lahti, Finland.

Kaput, J. J. (1994). The representational roles of technology in connecting mathematics with authentic experience. In R. Biehler, R. W. Scholz, R. Straber, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 379-397). Dordrecht, The Netherlands: Kluwer.

Gee, J. P. (2000). *An introduction to discourse analysis: Theory and method.* London: Routledge.

Gravemeijer, K., Cobb, P., Bowers, J., Whitenack, J. (2000). Symbolizing, modeling, and instructional design. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 225-274). Mahwah, NJ: Lawrence Erlbaum Associates.

Henderson, P. B. (Chair), Working Group, Striving for Mathematical Thinking. (2001). 6th Annual Conf. on Innovation and Techn. in Com. Sci. Edu., Canterbury, UK. (http://www.cs.geneseo.edu/~baldwin/math-thinking/).

Lajoie, S.P. (1999). Understanding of statistics. In E. Fennema & T.A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 109-132). Mahwah, NJ: Lawrence Erlbaum Associates.

Mok, I., & Johnson, D. (2000). Reasoning algebraically with IT: A cognitive perspective. *Mathematics education research journal, 12*(3).

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: author

National Research Council, Mathematical Sciences Education Board. (1989). *Everybody counts: A report to the nation on the future of mathematics education.* Washington, DC: National Academy Press, 1989.

Pacific Regional Education Laboratory. (1996). *Pacific standards for excellence*. Honolulu: Author. (http://prel.hawaii.edu.)

Pandey, Tej. (1990). *Authentic mathematics assessment*. Washington, DC: ERIC Clearinghouse on Tests, Measurement, and Evaluation, American Institutes for Research (ED354245).

Pugalee, D. K. (2001). Using communication to develop students' mathematical literacy. *Mathematics teaching in the middle school, 6*(5), 297-299.

Pugalee, D. K. (1999). Constructing a model of mathematical literacy. *Clearing house 73*(1), 19-22.

Sherin, M. (2000). Facilitating discussion. *Mathematics teaching in the middle school (6)*2, 122-125.

Vey, J. N., Schwartz, D.L., Bransford, J.D., Zech, L. (1998). SMART environments that support monitoring, reflection, and revision. In D. J. Hacker, J. Dunlosky, & A. C. Graesser (Eds.), *Metacognition in educational theory and practice* (pp. 305-346). Mahwah, NJ: Lawrence Erlbaum Associates.