Students' Constructivist Paradigm in a Spatial Problem-Solving Inquiry-Based Mathematics Classroom

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Abstract: In this report, psychological and sociocultural components of the constructivist theory have been employed to present an analysis of how students' cognitive constructions have been created, modified, and altered as they proceed in an inquiry–based spatial problem –solving mathematics classroom. The classroom work of pre-service high school mathematics future teachers has been carefully described on an instructional activity involving spatial structuring. In addition, I have attempted to clarify how the mental processes of spatial structuring and coordination in identifying a shape or object, identifying its spatial components, combining components into wholes, and establishing interrelationships between and among components and wholes, brought about a rich learning environment.

PROBLEM-SOLVING IN A CONSTRUCTIVIST PARADIGM

Psychological Mechanisms and Reasoning Based on the constructivist theory (von Glasersfeld, 1995), when individuals deal with the physical world, their minds construct, through certain mental mechanisms, collections of cognitive structures that enable them to conceptualize, reasoning, and coordinate their engagements. Abstraction is critical among these mental mechanisms (Battista, 1999). *Abstraction* is the process through which the mind selects, coordinates, unifies, and registers in memory a collection of mental acts. Abstraction has several levels. At its *perceptual level* (basic), abstraction isolates an item in the stream of an experience and seizes it as a unit (von Glasersfeld, 1995). Material or entity is said to have reached *internalized level* whenever it has been sufficiently abstracted so that it can be re-presented (re-created) in the absence of perceptual input. Material or entity is said to have reached *interiorized level* whenever it has been disembedded from its original perceptual context and it can be freely operated on in imagination, including being "projected" into other perceptual material and utilized in novel situations (Battista, 1999, p. 418). Interiorization is "the most general form of abstraction; it leads to the isolation of structure (form), pattern (coordination), and operations (actions) from experiential things and activities" (Seffe & Cobb, 1988, p. 337).

Understanding requires more than abstraction. It requires reflection. *Reflection* is the conscious process of re-presenting experiences, actions, or mental processes and considering their results or how they are composed. *Reflective abstraction* takes mental operations performed on previously abstracted items as elements and coordinates them into new forms or structures that, in turn, can become the content -what is acted upon- in future acts of abstraction (von Glasersfeld, 1995). Battista (1999), in reporting his 3D cube arrays' study, suggested that besides von Glasersfeld's (1995) list of mental mechanisms that includes abstraction and reflection mechanisms there are three additional mechanisms that are fundamental to understanding students' reasoning. They are spatial structuring, mental models, and schemes (Battista,1999; Battista & Clements 1996). *Spatial structuring* is the mental act of constructing an organization or form for an object or set of objects. It determines an object's nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. *Mental models* are nonverbal recall-of-experience-like mental versions of situations; they have structures isomorphic to the perceived structures of the situations they represent (Battista, 1994). Mental models consist of integrated sets of abstractions that are activated to interpret and reason about situations that one is dealing with in action or thought. A *scheme* is an organized sequence of actions or operations that has been abstracted from experience and can be applied in response to similar circumstances. It consists of a mechanism for recognizing a situation, a mental model that is activated to interpret actions within the situation, and a set of expectations (usually embedded in the behavior of the model) about the possible results of those actions (Battista, 1999). Meaningful learning occurs as students make adoptions to their current cognitive structures as a result of their reflection on an experience (Steffe, 1988; Battista, 1999). An accommodation is triggered by a *perturbation* which is described as a disturbance in mental equilibrium caused by an unexpected result or a realization that something is missed or does not work (von Glasersfeld, 1995). Perturbation arises when students interact with other individuals or with the physical world (Battista, 1999).

Sociocultural Components In accordance with the sociocultural components of a constructivist paradigm, students or individuals construct mathematical meaning as they participate in a variety of communities within which particular mathematical practices, reasoning, conceptions, beliefs, and interaction patterns are shared (Cobb & Yackel, 1995). Throughout the classroom sessions in this study, one-on-one interactions between the instructor and the students, and dialogues between a student and his/her neighbors were encouraged. These aspects of the classroom environment are essential in the present activity in this study. They were affective components for learning (NCTM, 2000, pp. 345-46). Throughout these interactions there was continuous refining and change of the students' points of view and hence changing the tendencies of their analyses for sense making.

THE CLASSROOM INSTRUCTION

There were three instructional classroom sessions devoted for this investigation with duration of 80 minutes for each session running one session per week. The students involved in the investigation were pre service high school mathematics future teachers. ***First Session***: Aiming to create a classroom culture of problem solving, investigation, and sense making, students were taught, through the first 80 minutes' session, a spatial unit consisting of a comprehensive revision of the topic of isometries: translation, rotation, reflection and their possible combinations. As well, the van Hiele's levels (0, 1, 2, 3, & 4) of thought development in geometry were discussed. The notions of *simple polygon* and *nonsimple polygon* were also illustrated to the class in the sense that a polygon is called simple provided that its sides (segments) intersect only at their endpoints (that is, only two sides generate each vertex); no two sides with a common point are collinear. Otherwise, the polygon is nonsimple - there is at least one vertex with more than two segments meet at it (O'Daffer, P. & Clemens, S., 1992). Dissection Theory (Eves, 1972) was also highlighted ***Second Session***: In the second session, part one of the activity was introduced (see Figure 1). The instructor explained to the class their task for this part of the activity. The students were instructed that their task was initially to find a way to decompose a square region modeled by a plain sheet of paper into two or more spatial components and then to try to combine these components, without overlapping, into a spatial composite that would resemble a right triangular region. During this session, the instructor's focus was to maintain a climate of discussing, questioning, and listening that would lead to establishing interrelationships between and among components and composites. Each student had his or her own activity sheet for recording answers, a square region of plain paper, a scissors and a ruler. Extra copies of the square region were available for the students to pick, as they needed. The instructor was moving about the classroom, encouraging students to communicate both with him on one-on-one basis and with their neighbors. Students were busy attempting their own dissections, investigating the resulting components, and moving and putting these components together into a composite to establish what would look like a right angle triangular region.

Shape-To-Shape: Part One Square to Right Triangle (Figure 1 below: Student's activity sheet-part one.)

1. Use the modeled plain paper square region provided; try to cut it into two or more pieces such that a right triangular region can be formed while preserving the area of the original square region.Describe in detail all your attempts & steps, highlighting your reasons.

2. Keep an anecdotal description/record of your thoughts and steps.

Throughout the instructional period, the instructor sat with each student who initiated a query and offered feedback, observing, or asking clarifying questions. At the end of the session, the students were asked to hand in their activity sheets to the instructor for the purpose of reviewing and offering interaction comments. The activity sheets were returned at the beginning of the third session.

Third Session: In the third session, firstly the students' activity sheets, part one, were returned. The instructor briefly stated that there were three different dissections appearing in the returned activity sheets and asked one of the students to help present them on the blackboard (see Figure 2a). The students were first asked to review their returned sheet activity in the light of the blackboard's displayed information and the instructor's feedback before revising and resubmitting their work. The instructor requested that the resubmission should describe: (1) the initial steps for their dissection and motion part and the original formation of their composites; (2) any possible modification made in the light of the dialogues and suggestions made in the classroom. The number of responses on each of the three dissection were: seven students used vertex to midpoint dissection (case 1); eight students used diagonal or vertex to vertex dissection (case 2); and one students used midpoint to midpoint of two adjacent sides dissection (case 3).

Secondly, the students were provided with the activity sheet - part two (see Figure 3) and instructed to initially implement it in the class. The instructor made it clear as to what the students were supposed to do. In particular, the students were instructed that in dealing with polygons they were to consider only simple polygons. Also, in this part, the students were encouraged to work collaboratively in pairs if they wished.

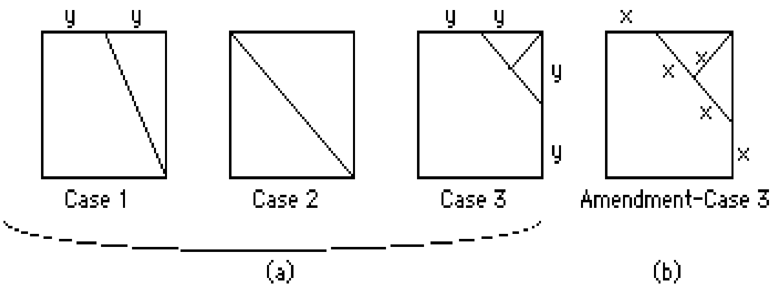

Figure 2 : Students' Dissection Strategies

They were asked to find all possible composites' formation using the pieces they had in part one of the activity (there were copies of the modeled square region available). The instructor circulated around the classroom, interacting with the student pairs or individuals, encouraging collaboration and communication, and promoting sense making. At the end of the session the instructor asked the students to take their work home and complete a revised report on their thoughts, attempts, steps, and findings.

Shape-To-Shape: Part Two Square to other possible Polygon (Figure 3 below: Student's activity sheet-part two.)

1. Use the pieces you obtained in Part One and try to form a different shape while preserving the area of the original square region. Investigate all shape formations possible like a parallelogram, a trapezoid, a quadrilateral without parallel sides, a pentagon, a hexagon, a heptagon, and an octagon (if possible).Explain how far you may be able to go on in terms of the number of sides of the resulting shape; point out if there is any rule here that you may be able to discover.

2. *Keep a detailed anecdotal description of your attempts, ideas, and steps.*

The following section presents excerpts from a few case-study students' work on the activity. The students' work was considered as they evolved, from the initial responses to the final refined form based on student-instructor and student-student classroom interactions. Students were selected for case studies if they did not use a trial and error strategy on all shape formation problems. Because a research assistant and the instructor made numerous observations of the students' responses, it can be said with some confidence that the strategies, thinking, and reasoning of the case-study students were typical of students in the class as a whole (Best & Kahn, 2003, pp. 249-251).

THE EVOLUTION OF THE STUDENTS' THINKING

Case Study 1: Vertex to Vertex Dissection Episode 1, Shape-To-Shape: Part One - Student W

Student W was a Math major/Physics minor. W started his anecdotal notes based on his work in session 2.

W: As I stared at the square, my thoughts were drawn to the hypotenuse of the future triangle. In order to form a triangle,the diagonal of the square would have to be made longer. W provided a figure (see Figure 4a) illustrating his thought.

W: I played with a few ideas and quickly realized, mostly in an intuitive way, that the only reasonable cut was corner to corner. I wanted desperately to conserve symmetry. I had not yet foreseen that the hypotenuse would be created with the two sides next to the right angles, yet I still had a sense that the triangle would work. So, without a real plan, I cut the square corner to corner. I played around for about ten seconds, and realized that the hypotenuse was formed by the *sides* [W typed it in italic] of the square, not the diagonal, and that the right angle came by adding 45's. So, the triangle is: [W provided another figure (see Figure 4b).

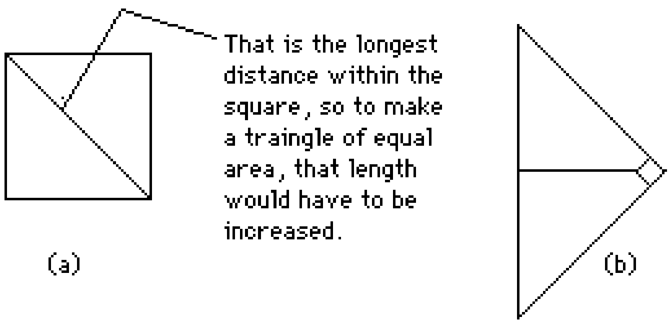

Figure 4 : W's Argumentative Illustration

On the three possible dissections displayed on the blackboard at the third session, W had this to say "I neglected to see the other case, because I was stuck on keeping symmetry". As W applied his strategy he was focussing on a certain attribute of the "future triangle"- its hypotenuse and the given square region – its diagonal. Using spatial structuring mechanism, W was able to conclude that the diagonal of the square would have to be made longer as he indicated in Figure 4b. In other words, he was thinking to establish part-to-part relationships in the process of producing whole-to-whole relationships. W's reflection throughout the problem-solving activity seemed more focused on preserving symmetry and determined by his current mental model of the final product. W's scheme focused on structuring the right angle triangle so that he could correctly dissect the square piece of paper; he seemed intent on determining where to start his dissection and how. While W was engaged in implementing his scheme, the concept of symmetry was high in his priorities; he was so keen to preserve it. Of course, no one knows for sure what brought about W's insight and persistence to preserve symmetry to the point that he entirely overlooked any other alternative dissection of the modeled square. However, while W adopted the mental mechanism of spatial constructing the "future triangle" by determining the nature of the given shape, the square, identifying its components, combining its components into a composite, and establishing interrelationships between and among components and composites, a mental model of symmetry abstraction was present. This combination of a mechanism and a mental model together with his expectation to form the "future triangle" would seem to constitute his present scheme for interpreting and reasoning about the situation he was dealing with.

Episode 2, Shape-To-Shape: Part Two- Student W Student W submitted his final anecdotal notes based on his in-seat work in session 3, dialogue with neighbors/instructor, and the instructor's briefing made in the class at the beginning of session 3 of the three different dissections appeared in the returned activity sheets, part one (see Figure 2a). W displayed the three possible dissections shown in Figure 2a as case 1, case 2, and case 3 respectively. He reported each shape formation in each of the three cases simultaneously. W made the following labels throughout his following notes; he labeled the required shapes by the alphabet a, b, c, etc. and the three dissection cases by the numerals 1, 2, & 3. Considering Figure 2a, W replaced case 3 with the amended version (see Figure 2b) to show his indication that the midpoint dissection would not form a right angle triangle.

W: a) Parallelogram: (1) By trial and error, and without too much thinking, the parallelogram took form. Essentially, I just "saw" it: [W provided a figure] (see Figure 5a). (2) Similarly, the parallelogram for case 2 formed by trial and error: [W

provided a figure] (see Figure 5b). (3) I realized after a few attempts that incorporating the irregular sides of this case into the simple shape of a parallelogram was impossible.
b) Trapezoid: (1) Again by trial and error, the trapezoid was a simple shape to form. There was no thinking or planning, it just happened: [W provided a figure] (see Figure 5c. (2) While doing the trapezoid, I realized that there were only three possible distinct shapes to form with the two triangles. A square, a right triangle, and a parallelogram. No solution for the trapezoid. (3) Similar to the parallelogram, there are few simple shapes that can be made with the three shapes [W referring to the three pieces in case 3], due to the irregular corners and added number of sides to deal with.
c) Quadrilateral (with no parallel sides): (1) I had to start thinking to form this quadrilateral. Two lines in the larger shape are parallel [W referring to the trapezoid piece in case 1], so to form a shape without parallel sides, the triangular shape would have to connect to one of them. It fit only on the "bottom" to form: [W provided a figure] (see Figure 5d). (2) No solution– it is not possible to make a shape without parallel sides. (3) No solution – large number of sides to deal with prevent making low order shapes.
d) Pentagon: (1) In order to form the pentagon, I first had to decide if the classical shape one assumes for a pentagon was necessary, or if any five sided shape would do. I quickly realized that there was no way to make the classical pentagon shape, so the assignment would end without making irregular shapes. The pentagon became [W provided a figure] (see Figure 5e). (2) Building on this concept of irregular shapes, I played with my triangles again, and discovered the pentagon: [W provided a figure] (see Figure 5f). (3) Finally some shapes for case 3 are possible. Placing the triangles against the large shape, and counting sides till they added to five: [W provided a shape (see Figure 5g).
e) Hexagon: (1) Once again, placing the shapes in different forms until it added up to six: [W provided a figure] (see Figure 5h). (2) The trend continued for the rest of the assignment… placing the shapes in different positions and counting sides. The hexagon for case 2: [W provided a figure] (see Figure 5i). (3) Since there are so many sides to work with in case 3 (11 sides), the way I approached these shapes (6 sides and up) was to start with the 5 sided large piece and decided how best to add the correct number of sides to achieve the result. For a hexagon, 1 side needs to be added: [W provided a figure] (see Figure 5j).
f) Heptagon: (1) After playing around a bit: [W provided a figure] (see Figure 5k). (2) There is no shapes possible beyond the six sided polygon, since there are only six sides to work with. (3) For a heptagon, two sides need to be added to the large 5 sided central shape. [W provided a shape] (see Figure 5l).
g) Octagon: (1) There are no solutions, since there are only seven sides to work with. (2) Again, no solutions – only six sides to work with. (3) Starting with the five sided shape, three sides need to be added: [W provided a figure] (see Figure 5m).

W did not provide the rest of the possible figures for case 3.

W's description of his strategies on activity sheet-part two revealed that he, at first, was not showing that much insight into the nature of the pieces. Only at the third shape and beyond, W started giving more attention to the nature of the pieces. In forming the required quadrilateral (no parallel sides) W adopted a comprised of congruence abstractions of the sides of the pieces involved. As W progressed in his strategy of forming composites, he adopted an enumeration scheme and a spatial structuring mechanism. W's reflection on his spatial structuring mechanism seemed to help him to recognize that he had to pick up the largest piece in size and number of sides to be his initial piece for putting together the required composites.

The evolution of W's thinking, therefore, initially started with merely putting the pieces, as intact entities, together in a trial and error strategy overlooking their specific characteristics and their interrelationships. W then realized that there was more meeting the eye, more than patching the pieces together. He started showing more attention to the part-to-part interrelationships among the parts of pieces, and whole-to-whole interrelationships among the component pieces, and the composites; W had started re-presenting his mental processes and considering their results. In effect, W seemed to adopt reflective abstraction that takes mental operations performed on previously abstracted items as elements and coordinates them into new structures or forms (von Glasersfeld, 1995). W did come to know that the maximum number of sides possible for a composite (as a simple polygon) in each of the three cases was the sum of the number of sides of the corresponding pieces.

EPILOGUE: INSTRUCTIONAL IMPLICATIONS

Using the constructivist theory of abstraction, the observations made in this investigation suggest that designing materials for such mathematics instruction would have to include into consideration the students' need for repeated and varied opportunities to reflect and share their ideas to properly construct none trivial concepts. Further, this research suggests that the level of success of this instructional culture depend on the repeated opportunities for students to construct and refine their construct in the process of establishing the required concept. Also, this study suggests that it is equally imperative to note that the level of success of the instructional culture depend on the ability of the students to examine, individually, the feasibility of their spatial developing mental models.

This study explains how meaningful learning can take place in spatial problem-solving inquiry-based instruction. In a suitable classroom environment of inquiry students construct, refine, and share ideas in the process of solving a set of problems. Battista (1999) described students in such culture of inquiry "like scientists, construct, revise, and refine theories to solve and make sense of perceived problems." The distinction here, however, is that students' knowledge building is guided by teachers and instructional materials. Through this culture of investigation that had been created in the classroom, students were asked to construct a variety of "non-classical", none familiar shapes—shapes that are not commonly used in the curriculum like most of shapes shown in Figure 5.

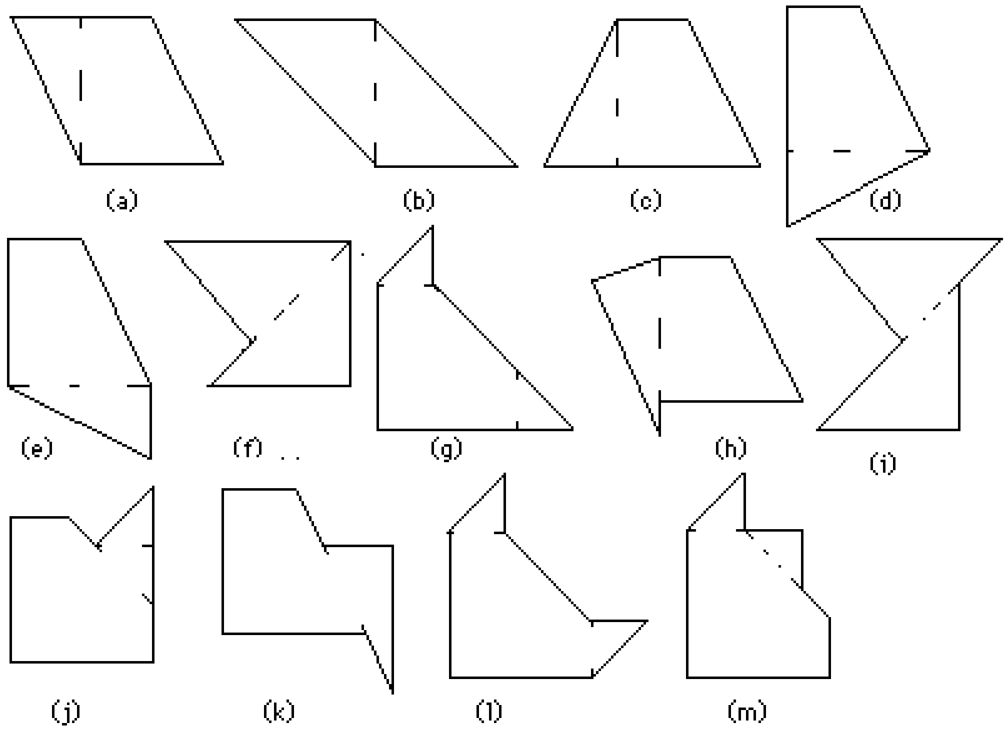

Figure 5: W's Coposites for the Spatial Activity - Part Two

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