Arithmetic, algebra and technology: a study on beginner pupils

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Abstract The role of mediation of the Computer Algebra Systems (C.A.S.) in the learning of mathematics has recently been the subject of a wide debate, overall with respect to its potentiality of use and to the influence of its use on algebraic competencies. In fact, the use of C.A.S. software on the one hand allows the use of algebra as an instrument of codification and solution of problems also in complex situations, on the other hand requires managing different modalities of representation (writing on line, the usual algebraic writing, the writing of transformed expressions, possible graphic representations, according to the complexity of the situation) and could set the beginners in front of a transformation that they aren't always able to control. In this framework we will present a study carried out on 14 year old pupils, not expert in the algebraic transformation, to which we have suggested confronting, with the support of a C.A.S., some questions of divisibility and other problems whose generalization requires algebraic competencies.

Introduction

The establishing of Computer Algebra Systems and their diffusion both in utilisable form, also by non-expert users, and in graphic-symbolic calculators has pointed out the problem of their didactic use and has provoked a wide debate. A great amount of research has been concerned with their potentialities which could make meaningful problems, from the point of view of their contents, accessible to pupils, whose solution would require advanced mathematical skills. Much other research has been done on the consequences of the diffusion of such systems on the skills of the pupils in algebra and calculus. (Herget et al. 2000, Heugl 1999). Of particular interest are the studies which concentrate their attention on the linguistic, symbolic and graphic aspects which characterise the didactic use of the C.A.S. (Winsløw 2000, Hershkowitz & Kieran 2001). Within the framework of these problems, the study presented here forms part of some research about the influence of the use of symbolic manipulators in the learning of algebra and proposes confronting the theme relatively to the phase of approach to algebra presenting a study carried out with 14 year old pupils who were inexpert both in algebraic transformations and in the use of symbolic manipulators. In particular, we will examine the behaviour of pupils in the study of some arithmetic problems, a fundamental moment for a conscious approach to algebra, which they are asked to solve having at their disposition software for symbolic elaboration; thus being able to concentrate their attention on the procedures and the interpretation of the results rather than on the calculation. A specific objective of the didactic unit is that of strengthening the reaching of several algebraic goals, in particular the abilities to:

- recognise the necessity of the conventions of writing in arithmetic and algebra
- translate normal algebraic language into another symbolic language (such as the necessary way of writing for entering expressions in the software) and vice versa
- use letters in a conscious way in meaningful contexts
- manage, with confidence, the first elements of literal calculation without losing sight of their meanings
- generalise using algebraic formalism

The goal of the research is that of examining if the presence of C.A.S. encourages these objectives and of studying the kind of use that the pupils make of the software. In this paper, I will examine the didactic unit and comment qualitatively on the reactions of the pupils and the role of the system.

Methodology

For the performance of this didactic unit, amongst the numerous C.A.S. that can be utilised on a personal computer, we chose *Derive*, in a version (4.09 or 5) for Windows in that it is particularly simple to use and, moreover, it is available in Italian.

The use of the software is limited to a few commands, and exclusively to the "algebra window". In particular, the software is used for allowing the pupils to verify the properties under examination both using possibly very large numbers and making the most of the potentialities for factoring or for trying to generalise by means of "putting into a formula". In essence the software in the work unit is used as a support for elaboration and examination of conjectures. The simplicity of use of the system makes it sure that the necessary skills can be acquired by the pupils without difficulty.

The work was carried out, as mentioned, with 14 year old pupils who had one computer at their disposition for every two pupils. For each activity, a work card was proposed to the children, on which was written the problem situation, at times accompanied by instructions. Each one of the pupils had his own card and was able to decide to compile it alone or in agreement with his workmate at the computer. The children were, however, invited to collaborate and discuss with their workmate.

During the activity the teacher avoided giving suggestions. At a later time, came the discussion, a fundamental time for the acquisition and sharing of knowledge. In this phase all of the different answers were taken into consideration, compared and commented upon. The teacher tried to make everyone participate acting as “moderator”, re-proposing to the class the ideas which emerged, in such a way that they could be clarified, possibly corrected or shared.

The analyses of the activities that is, in part, reported here is, as already mentioned, of the qualitative type and was carried out both by using the compiled work cards, and by means of the observations of a university student in the process of getting her degree who participated as observer both of laboratory activities and of discussion times in class.

At the beginning of the work unit, the pupils knew how to work with natural numbers, in particular they knew their operations and their properties, including raising to power, and the relation of divisibility. Moreover, they had already encountered situations in which it was necessary to translate a situation expressed in verbal language into a symbolic language (the most common exercise which is usually proposed to the pupils at different age levels is the translation of the situation described by the text of a problem into an expression) and they had already encountered situations in which letters were used to represent numbers.

The activity required a total of 20 hours of work and consisted of three parts that will be described in the next paragraphs, each with its own specific objectives.

Algebraic writing and writing with a symbolic manipulator

The activities of the first group of work cards aimed to get the pupils to reflect on the conventions of arithmetic and algebra language, with particular attention to the use of parentheses and to the priority of the operations. This reflection was solicited by means of the translation into algebraic language of several situations, proposed in natural language, and by means of the use of the “algebra window” in the *Derive* software, which requires writing the expressions in one line to be able to enter them in that window in such a way that they could be elaborated with the help of the software. The activity, proposed here as a necessary exercise for familiarizing the pupils with the software, has as its objective consolidating the knowledge of the arithmetic-algebraic linguistic conventions by means of the passage from normal symbolism to that required by the software.

In this cycle of activities, the pupils are asked to think about the operations which intervene in an expression, about their priority and about how they can be codified so that *Derive* transcribes them according to the normal conventions. The pupils generally know and accept quite easily the normal conventional priority of the operations and follow it with a certain ease when they are requested to execute written calculations. Different, however, is the awareness required when it has to do with translating the same expression from one code to another as happens precisely in this case, in which, as in programming languages, the conventions are somewhat different and at times more rigid than those usually adopted in algebra. The pupils, in general, know the rules of use of the parentheses, but it is difficult for them to utilise them correctly when they have to translate a text into an expression or in any case they have to insert parentheses. Often, it is observed that the parentheses exist in the mind of the pupil, who carries out the calculations as if they were there, but they don’t appear on the paper. To avoid this problem, decoding writings elaborated by others can be useful. Nevertheless, the situation proposed here (quite analogous to that which one has when using some kinds of calculators or some programming languages), in which the parentheses are necessary to have the desired expression visualised on the screen, can be particularly useful. The pupil in this way is motivated to accept the necessity of the correct use of the conventions.

We remark that to write an expression in a symbolic manipulator it is necessary to handle two types of partially different conventions: the algebraic ones and the ones which are typical of software language that, being an "in line" language, require a greater number of parentheses. This fact creates further difficulties for the pupils, even constituting, as already observed, a stimulus for the acquisition of greater awareness and mastery of the writing conventions.

One of the activities proposed requires, given some expressions written in one line, to conjecture how they will appear on the screen and then to check with the software. This activity has brought with it some difficulties both in its execution and in the discussion because it requires an attentive examination of different formal codes. The children interpret the proposed expressions, translate them into normal symbolism, then compare their conjecture to the formula that appears in the “algebra window” of *Derive*. The undeniable difficulties notwithstanding, it is observed that with the discussion and the comparison of the typologies of expression proposed, the pupils manage to identify

the correct expressions and the wrong ones and to explain the "why" of the errors. It should be noted that at the time of the discussion the pupils have already performed checks with *Derive* and therefore already know which is the correct form. The discussion has the aim of comparing and justifying the various typologies proposed by the pupils, in such a way that all of them can understand the difference between the various proposed ways of writing. In the discussion phase it was found to be interesting, in the presence of mistaken conjectures, asking which should have been the expression entered to obtain that writing. It was thus possible to observe that some pupils at this point re-proposed sometimes the same expression or expressions already previously discarded, thus demonstrating, that exercises of this type are not simple for many pupils and that often the results which seem to be shared by the class are not effectively understood by all.
We note also that this type of exercise helps to highlight some difficulties that often do not show themselves easily, but that have evident consequences for the correct use of algebra. For example, some pupils demonstrated their uncertainty about the equivalence between the two following formulas:

$$3 \cdot \frac{b-1}{b-5} \quad \text{e} \quad \frac{3 \cdot (b-1)}{b-5}$$

supplying the teacher, in this way, an interesting opportunity for revision and deepening, suggesting simpler and more graduated situations.

Study of some properties of natural numbers

The cards for the second part of the work asked the pupils to look for and investigate some properties of natural numbers: evenness, unevenness, divisibility. This kind of activity is frequently carried out in class with pen and paper. Here the role of the mediation of the software, that allows carrying out many numeric tests and writing and transforming expressions, concentrating oneself on the meanings rather than on the calculation, is fundamental. Our aim is to bring the students to use algebraic language as an instrument of generalisation and verification, in such a way as to recognise, in the symbolic language, an efficient means of expression. The generalisation of the situations also constitutes a fundamental step toward the beginning of demonstration; the proposed situations in this set of activities are, however, quite simple and can be generalised using verbal language. The request to utilise the software, to examine numeric examples and to elaborate the formulas which translate the situations presented, induce the pupils to propose solutions that are, in general, at an intermediate level between argumentation and demonstration, an important premise for successive developments.
It is helpful if the teacher asks the pupils to use the software to carry out various tests, in that on the basis of our experience, the pupils often say they are able to do with less, generalising starting from a few particular cases. The discussion will then have to, with appropriate counter-examples, highlight the incorrect conjectures, in a way to help the pupils understand how generalisation beginning with a few cases can bring one to an erroneous conclusion.
Some cards of this set present situations which are a bit more complex than the others in that they require a sort of "inverse procedure". For example, knowing a property of the product of two numbers (for example knowing that it is an even number), it asks them to make conjectures about its factors. In the experience examined here, most of the children sensed that it would be easy to start from factors to examine the various cases. Many turned to formal expressions but did not always arrive at exact conclusions because they had committed writing errors in the formalisation. For example, in examining the possible cases for the product of two numbers, some correctly write $2a\cdot 2b = 4ab$, but some also write $2a\cdot 2b+1$, which obviously represents the product of two even numbers to which 1 is added, while in their understanding it is the product of an even number and an uneven number, and $2a+1\cdot 2b+1$, which is the sum of two even numbers with the adding of 1, while for them it is the product of two uneven numbers. In these cases the pupils often trust the software to transform the expressions written by them, but it is evident that if the formal expression used is not correct, the elaboration can do nothing but give erroneous results. The possibility of effectuating various tests with numbers could help the teacher in soliciting the pupils to find counter-examples to possible incorrect conclusions obtained by a mistaken use of the conventions of writing or also for demonstrating the incorrectness of the formalisation used even if, as in the case of the product of two uneven numbers, the conclusion was correct although starting with mistaken ways of writing.
We note that even if the software, in this set of cards, is proposed overall as an instrument for carrying out tests with numbers before elaborating a conjecture, in general however, the pupils, at this point in the work, do few numeric tests with *Derive*, only those necessary for formulating a hypothesis of

solution. More meaningful, instead, in this activity is the "new use" of the software as facilitator for algebraic calculation. The pupils write their formalisation and use the software to develop and transform it. We think it is important to underline that in this way the pupils understand spontaneously that, by way of the transformation of the algebraic form of an expression, they can have information about the properties of the result. The fact that it is the software to supply the transformation allows them to make passages that they would not always know how to carry out correctly "by hand" and to concentrate their attention on the "reading" of the result.

The teacher must encourage the pupils' reflection on their results, inviting them to compare their conclusions to the numeric examples, and encouraging a positive use of the software. It is necessary, as a matter of fact, to pay attention that the pupils do not accept, in an a-critical way, the results of the elaborations carried out with *Derive*.

Toward literal calculation with Derive

The last part of the didactic unit proposes more involving activities both from the point of view of the comprehension of the text and from that of the study of the situation, overall in relation to the skills and age of the children. The cards still regard the properties of natural numbers, but they confront more complex situations. Formulating conjectures is required starting from examples and examining them by means of the way of writing and of the transformation of symbolic expressions. In these problems, the phase of decoding verbal language and its translation into algebraic language assume an important role. It is an essential moment of many problematic situations, which is found at different levels in all the phases of this work, but which becomes more meaningful for the greater difficulty of the problems proposed.

The children discover the utility of doing many tests before elaborating a conjecture. They feel the need to make other tests suitably chosen (with large or small numbers or particulars for the situation) to confirm it or refute it and in the end they are asked to pass from the argumentative level to the demonstrative level. This passage is extremely delicate and it is not always shared by the whole class. The times of approach to the demonstration make strong use of the help of the software and require, at times, the intervention of the teacher to refute conclusions, repropose or reformulate questions.

It is important to note that, in the performance of this activity, the pupils, with *Derive*, carry out, arriving at generally correct results, elaborations of formulas that, in many cases, they do not know how to do with pen and paper. This use of the software is proposed to allow the pupils to concentrate their attention on the interpretation of the results instead of the calculations, but presents the risk that the pupils a-critically consider the transformations carried out or they interpret them in an incorrect way. This is a key point which in our opinion should not be seen as a limit, but as an important didactic opportunity. The discussion phase in which the teacher's mediation is central, was, in the experience studied by us, fundamental. As a matter of fact, it is his/her task to operate in such a way that beginning from the answers given, by means of a debate which involves all the class, it is possible to reach the comprehension of the elaboration of the formulas done by the software. Let's examine briefly some of the problems proposed:

Subtracting 1 from the square of a natural number, can you obtain a prime number? Is the natural number, by which this fact happens, unique? Why?

The pupils respond quite easily to the first question and using many numeric attempts they convince themselves that the solution is unique. They factorise, utilising the proper instructions, the results obtained to check if they are prime numbers. Supplying an explanation of the uniqueness of the solution, however, is more complex in that it requires the mastery of the concept of prime numbers and the correct and aware use of algebraic formalism. The most appropriate way to reach a correct conclusion is that of utilising symbolic language to translate and elaborate the situation analysed.

Another problem proposed is the following:

Take a natural number, cube it and subtract, from the result obtained, the same number. What do you get? Which properties does the number obtained have?

Many pupils, given the difficulty of the card, started again to do many numeric trials with *Derive*. There were many also who represented the proposed situation in formal language, even if at a very different level:

- some pupils limited themselves to writing the expression n^3-n without transforming it in any way
- others formalise in n^3-n and then try to factor without the help of the software arriving at wrong expressions or at the correct expression $n(n^2-1)$, that they cannot further elaborate
- still others arrive at the expression $n^3-1 = n(n+1)(n-1)$ in which the factorisation is obviously obtained using *Derive*

It is then, however, difficult for the children to utilise the factorisation obtained to "read" properties of the number obtained.
A third situation proposed is the following:
Take a natural number, multiply it by the next number after the number which follows it and add 1 to the result. What do you get?
This activity also turned out to be rather difficult both for the correct interpretation of the text and for its translation into formal language.
The pupils, having worked with even and uneven numbers, in general differentiated the two cases and observed that beginning from an even number you obtain an uneven number and vice versa. Some pupils observed also that when the result is even it is also a multiple of 4. Some moreover, showed that the result is the square of a number, but, in our experience, no-one observed that it is the square of the number after the one considered.
Having noted that the pupils encountered difficulty in formalisation, but overall in elaboration and interpretation, it was thought, the problems posed being particularly difficult, to facilitate them suggesting some questions, only after the pupils had carried out numeric trials and had formulated their conjectures, that would lead them to arrive at the expected conclusions.
We note that in this last part of the work the role of the software was clearer and more important. The pupils used it to do many numeric trials and the number of them who used the result of the symbolic transformation to check their own conjectures grew. Moreover, all of those who formalised then used *Derive* to elaborate their expressions.

Closing observations

We think that the study described here has supplied some important indications overall about the use that the pupils make of the software if they can freely choose to utilise it in the solution of problems and about the type of check that they do on the results produced in a C.A.S. environment. The pupils studied did not use the possibility of making many examples and counter-examples very much, overall when the situation proposed did not appear to them to be very difficult or stimulating. Furthermore, they tended to "trust" algebraic elaborations not always constructed beginning from correct expressions. Nevertheless, the exercises proposed at the end of the work on problems which required formalising simple situations, performing some algebraic steps and interpreting the results gave a positive outcome.
It is considered necessary, however, to study in depth many of the problems which emerge from work such as that described and in particular:

- the relationship between verbal and symbolic language in C.A.S. and pen and paper environments both in the modelling phase of the problem situation and in the phase of the interpretation of the results.
- the role of the teacher and organisation of the class in relation to the mediation role of the software
- the appropriate modalities of observation and analysis which keep track of the specificity of the activity carried out with the mediation of a software

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