APPLICATIONS OF MATHEMATICS IN THE REAL WORLD: TERRITORY AND LANDSCAPE

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Abstract: The year 2002 has been dedicated to the Mountains by United Nations Organization. The aim of this paper is to present some relationship between mathematics, territory and landscape. Territory is a geographic entity, which we can divide in two parts: political and natural. The environment (urban and extra urban) consists of biological and cultural evolution of the Territory. The landscape is an aesthetic concept strictly connected with the relation: Man and Nature. In this paper we will refer of our attempt to describe Natural territory in the extra-urban environment using a mathematics approach.

1. INTRODUCTION

Starting point for our investigation is the following question: "Is it possible to describe the territory where we live using the mathematics?"

To answer this question we have defined a mathematics and geometric approach. In fact, to describe the territory we have identified the following subjects:

- Numbers (e.g., to count the peaks present in a mountain chain);
- Symmetry (e.g., the symmetry presents in the natural crystals);
- Geometric shapes (e.g, curves, and meanders present in the rivers);
- Euclidean geometry (e.g., to describe the Nature using a simple geometry);
- Stereographic geometry (e.g., to compare a mountain shape and a geometric shape);
- Fractal geometry (e.g., to describe the irregular shapes present in the Nature or to create the mountains, and the dendritic structures using the computer graphics techniques).

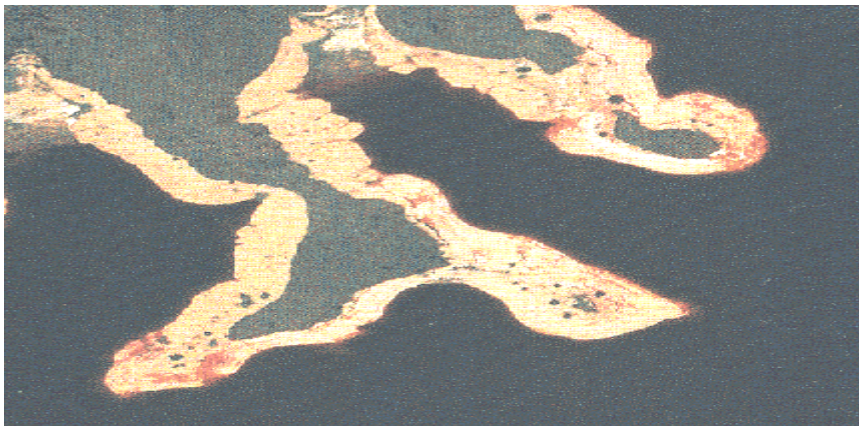Figure 1 shows an island we can note the presence of an irregular shape in the coastline.

Figure 1 An example of irregular shape which is present in a coastline.

In this paper we will describe an approach to model territories and landscapes.

The paper is organized as follows: the section 2 presents a geometric approach to describe the Territory and Landscape; section 3 describes the fractal geometry to model territory and landscapes, section 4 contains   some fractal models to realize some virtual terrain with the computer graphics; finally, in the section 4, we have our conclusions.

2. A GEOMETRIC APPROACH TO DESCRIBE TERRITORY AND LANDSCAPE

A geometric approach to represent territory and landscapes is to determine 3D objects which consist in finding a model that represent a set of data point:

$$(x_i, y_i, z_i) \in R^3, \quad \forall i = 0,..., n$$

A wide variety of representation methods have been studied for model these surfaces [5]. For example, figure 2 illustrates the visualization of the 3D function named "Lorentz hat". It is defined by the equation: $y(x,z) \sim 1/(x^2 + z^2 + 1)$.

Unfortunately, these models do not recover rough surfaces, i.e. surfaces defined by continuous functions that are nowhere differentiable. Models that are able to produce rough surfaces are mostly based on random processes, and for this reason these models are not suitable for approximation. To propose an
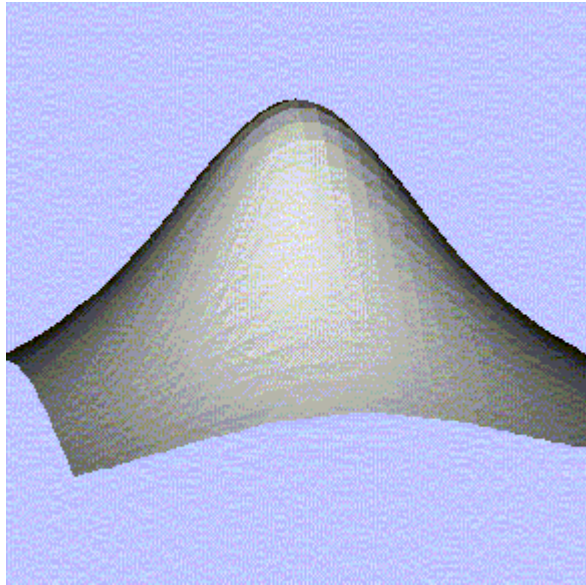
Figure 2 An example of 3D function: "Lorentz hat".

efficient solution to the problem of rough surface approximation, the current study presents different models based on fractal geometry [10, 11, 16, 19]. Before the development of fractal geometry, typically Nature was regarded as "noisy" Euclidean geometry. A mountain is primarily a roughened cone, for example. Indeed, this view was codified by Paul Cezanne's statement about painting: "*Everything in Nature can be viewed in terms of cones, cylinders, and spheres*." In contrast to this, Benoit Mandelbrot asserts, "*Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.*"

3. FRACTAL GEOMETRY TO MODEL TERRITORY AND LANDSCAPES

The fractal geometry is a young discipline (the first studies are the works of Gaston Julia (1893 - 1978) at the beginning of last century) but, only with the mathematical power of computers it is become possible to obtain the beautiful and colourful images derived by the complex formulas.

The word "fractal" was coined less than thirty years ago by one of history's most creative mathematicians, Benoit Mandelbrot, father of fractal geometry. In his book *The Fractal Geometry of Nature* introduced and explained concepts underlying this new vision.   A fractal object is self - similar in that subsections of the object are similar in some sense to the whole object. No matter how small a subdivision is taken, the subsection contains no less detail than the whole. Mandelbrot, has defined fractal models to describe natural phenomena [14].

To realize a "fractal mountain" we can take an elastic string, then a random vertical displacement is applied to its middle point. The process is repeated recursively to the middle point of every new segment.
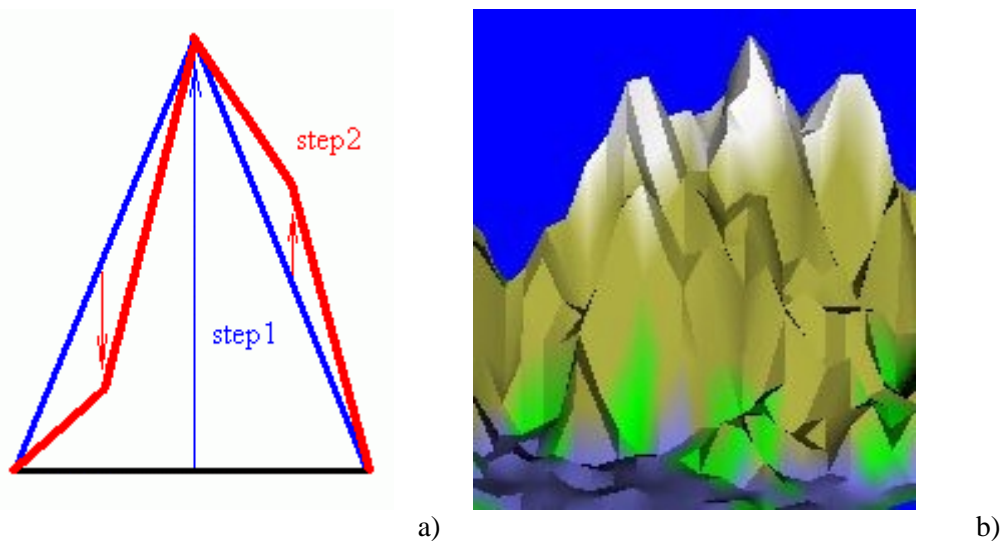

a) b)

Figure 3 Examples of fractal mountains.

The random displacement decreases two times each iteration. Figure 3 shows the generation process (3 a) and the mountain realized using this procedure (3b).

To realize 3 D mountains is more complicated than 2D. Some approaches are based on the midpoint displacement method which can work with triangle, square and hexagonal grids [19]. Another method which uses random midpoint displacement has been introduced by Fournier, Fussel and Carpenter (1982). They developed a mechanism for generating a class of fractal mountains based on recursive subdivision [10].

For example, an initial square is subdivided into four smaller squares (see figure 4). We obtain four points: $[x_0, y_0, f(x_0, y_0)]$, $[x_1, y_0, f(x_1, y_0)]$, $[x_0, y_1, f(x_0, y_1)]$, $[x_1, y_1, f(x_1, y_1)]$. In the first step we add one vertex into the middle. The vertex is denoted by $[x_{1/2}, y_{1/2}, f(x_{1/2}, y_{1/2})]$.

The added vertex is shifted in *z*-coordinate direction by random value denoted by δ_l.

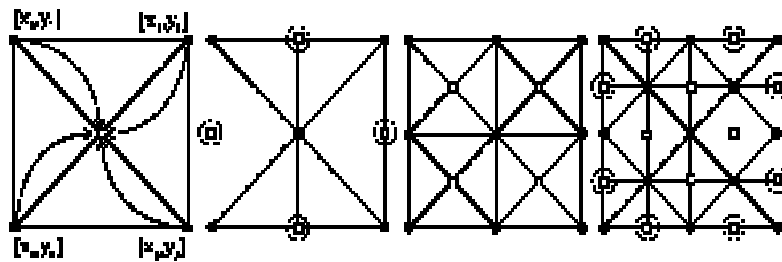This procedure is recursively repeated for each sub-square, then for every their descendants, and so on.

Figure 4 First four steps in random midpoint displacement method.

In order to be resulting fractional Brownian surface, the random number δ_l must be generated with Gaussian distribution ($\mu = 0$ and $\sigma = 1$) and in the *i-th* iteration step the variation σ_i have to be modified according to the following equation:

$$\sigma^2_i = (1/2^{\,2H(i+1)}) \, \sigma^2 \qquad (1)$$

where *H* is Hurst exponent ($1 \leq H \leq 2$) [14]. From equation (1) we can note, that the first iteration has the biggest influence to the resulting shape of the surface and influence of the others decreases. In the second step the points on the edges of initial square can be calculate. We can virtually rotate square by $45°$ and calculate the values as in the previous step. The problem is in the cases when the new point has just three neighbours (the encircled points in Figure 4). In this case we calculate the average of three neighbours only. The error produced on the border could be neglected.

In the next step we can virtually rotate the square back by $45°$ and we recursively apply the first two steps on the four new squares as is mentioned above. This recursive process ends after given number of iteration. Fractal dimension *D* of this surface is obtained by $D = 3 - H$.

An example of fractal terrain obtained with random midpoint displacement algorithm is illustrated in Figure 5. The fractal dimension of this surface is $D=2.5$.
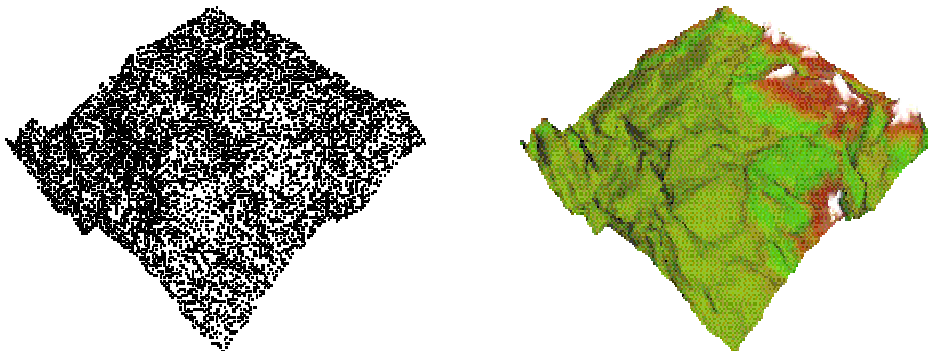

Figure 5 A fractal terrain generated with random midpoint displacement algorithm

In the this example, we have considered only a terrain but not its erosion. Prusinkiewicz and Hammel (1993) used context sensitive rewriting processes based on a random midpoint displacement method [19] on a triangular grid. Their method creates one non-branching river as result of context sensitive L-system [22] operating on a set of triangles. Musgrave et al. (1989) introduced techniques which are independent of the terrain creation algorithm and can be applied to already generated data represented as regular

height fields [18]. They introduce two methods: hydraulic erosion and thermal weathering. Hydraulic erosion is caused by the presence of water in the form of rain. Water is dropped on each vertex in a high field and transfers the material. Thermal weathering is caused by temperature changes causing small portions of the material to crumble and pile up on the bottom of an incline [16, 18]. These models can be applied in the Computer Graphics to generate virtual environments.

4. FRACTAL MODELS IN COMPUTER GRAPHICS

Computer graphics concerns the pictorial synthesis of real or imaginary objects from their computer-based models, whereas the related field of image processing treats the converse process the analysis of scenes, or the reconstruction of models of 2D or 3D objects from their pictures [9]. Computer graphics also uses fractal geometry to realize mountains, trees, and textures. Using a technique called spatial subdivision we can generate Fractal terrain. These results in surfaces are similar in appearance to the earth terrain.

The idea behind spatial subdivision is quite simple. We consider a square on the x-y plane,
(1) split the square up into a 2x2 grid
(2) vertically perturb each of the 5 new vertices by a random amount
(3) repeat this process for each new square decreasing the perturbations each iteration.
Figure 6 shows three iterations of this process.

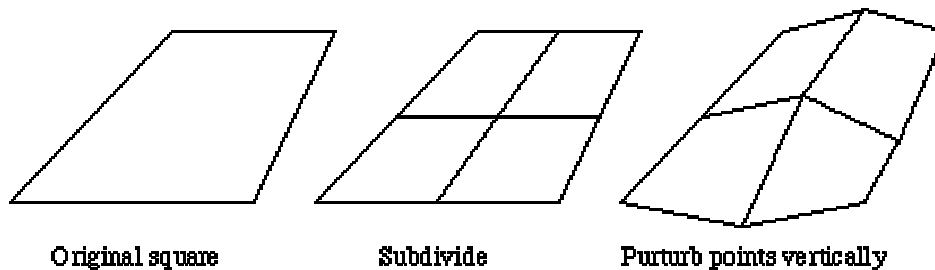Original square   Subdivide   Purturb points vertically

Figure 6 Three iterations in the generation process of fractal terrain.

The controls normally available when generating such landscapes are:

- **A seed for the random number generator**. This starts the random number generator and means that the same landscape can be recreated by remembering only one number.
- **A roughness parameter**. This is normally the factor by which the perturbations are reduced on each iteration. A factor of 2 is the usual default, lower values result in a rougher terrain, higher values result in a smoother surface.
- **The initial perturbation amount.** This set the overall height of the landscape.
- **Initial points**. It is often desirable to specify some initial points, normally on the corners of the initial rectangles. This provides some degree of control over the macro appearance of the landscape.
- **Sea level**. This "flood" the terrain to a particular level simulating the water level.
- **Colour ramp**. This is used for shading of the terrain surface based on the height. Normally two or three colours are defined for particular heights, the surface at other heights is linearly interpolated from these points.
- **Number of iterations**. This results in the density of the mesh that results from the iteration process.

The following figure 7 illustrates a terrain surface at various grid resolutions from 2x2 to 32x32.

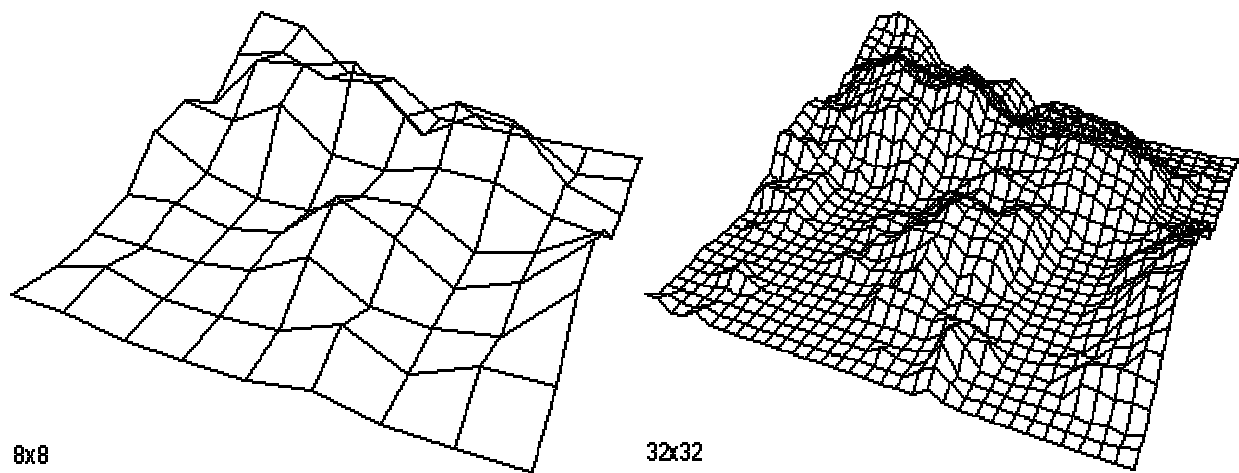2x2   4x4
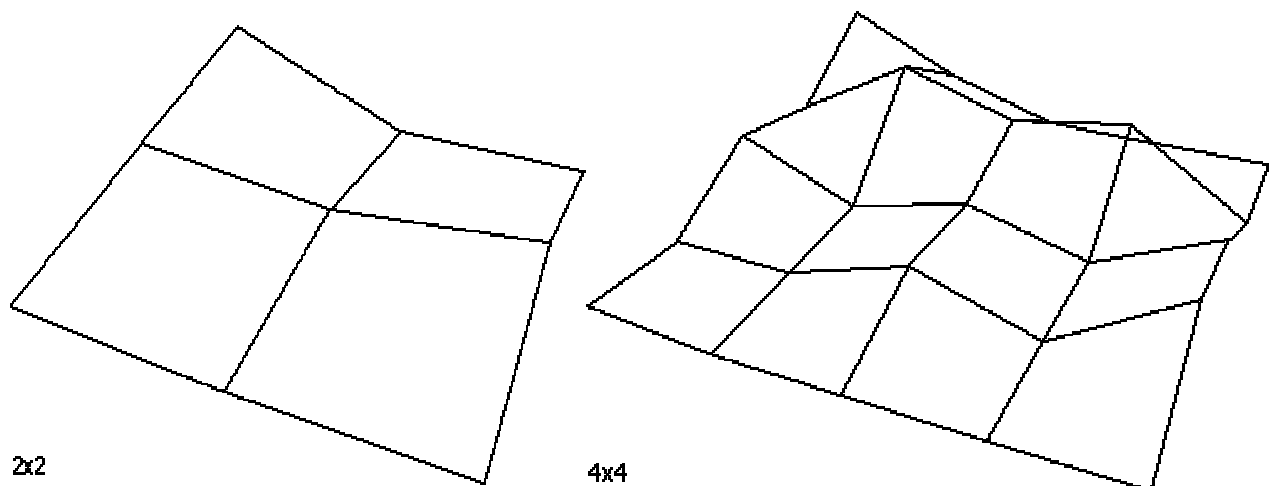
Figure 7 A terrain surface at various grid resolutions from 2x2 to 32x32.

After the wire frame views, we can realize the rendering phase which includes: hidden line, coloured, and shaded. Next figure 8 illustrates two steps of this process.
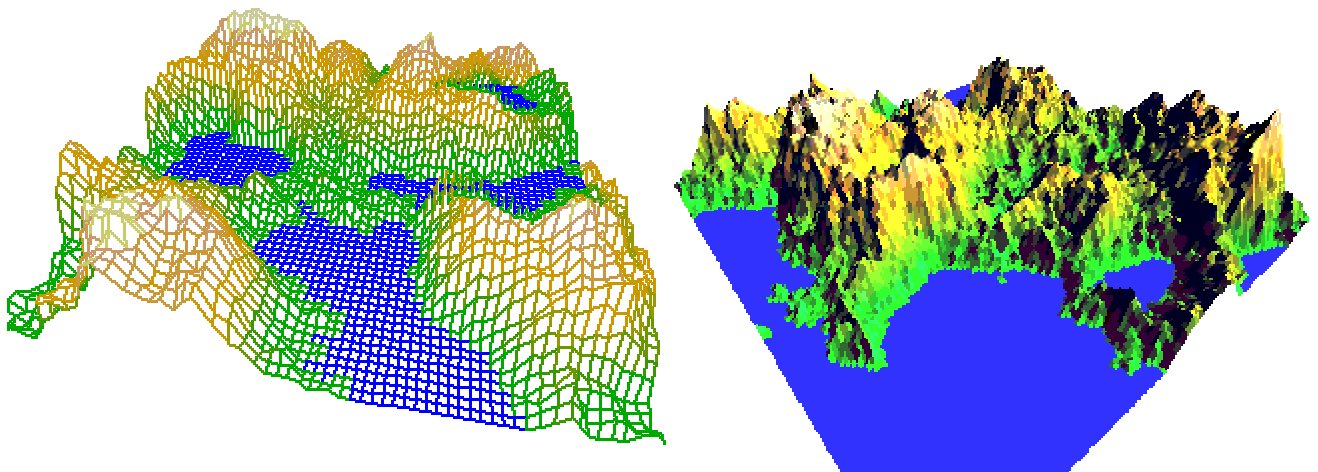

Figure 8 The phase of rendering to generate a landscape.

Zair and Tosan (1996; 1997) proposed an interesting model for fractal curves and surfaces. This method combines two classical models: an fractal model (IFS attractors) and Computer Aided Geometric Design model (CAGD) [24; 25].

The IFS (Iterated Function Systems) model generates a geometrical shape or an image with an iterative process [4].

An IFS – based modeling system is defined by a triple (*X, d, S*) where:

- (*X, d*) is a complete metric space, *X* is called iteration space;
- *S* is a semigroup acting on points of *X* such that: $\lambda \in X \rightarrow T \lambda \in X$ where T is a contractive operator, *S* is called iteration semigroup.

This method permit to reconstruct smooth surfaces, and not only rough surfaces. Figure 10 shows the experiment realized by Guérin at al. (2002) on a natural surface [11]. The original surface is has been extracted from a geological database (found at the United States Geological Survey Home page, http: // www. usgs.org). Terrains and mountains are only a part of the landscape. To generate a complete virtual landscape it is necessary to create the plants and the trees. We can use L-Systems (studied by Aristid Lindenmayer). L-System is a set of string rewriting rules which takes an initial string of characters called he axiom and on every iteration replaces each of the characters in the string by other strings called production rules. For example consider the axiom string: F+F+F+F and the single production rule F-->F+F-F-FF+F+F-F. Now

if some characters are giving geometric meaning then the string can be drawn. Using these geometric meanings for the symbols an example of an axiom, production rule, and the resulting string after a few iterations , is a well known fractal curve which has a fractal dimension
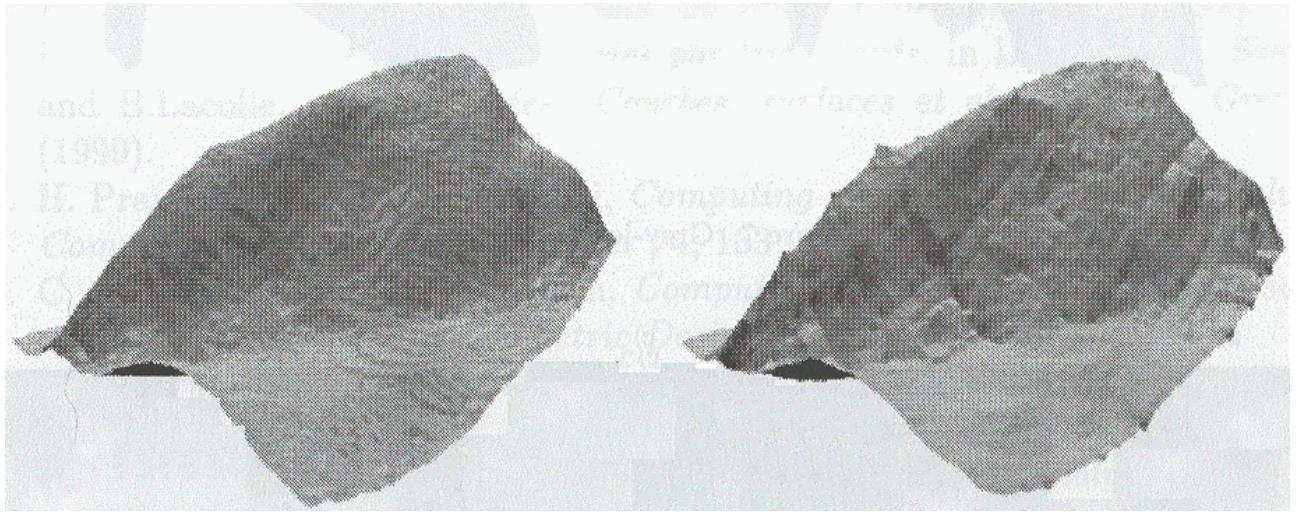
Original   Approximation
Figure 10 Geological surface

between 1 and 2 (for example the van Koch snowflake, space filling curves such as the Hilbert and Peano curves, the dragon curve, as well as kolam patterns).
Recent usage of L-Systems is for the creation of realistic looking objects that occur in nature and in particular the branching structure of plants. One of the important characteristics of L-Systems is that only a small amount of information is required to represent very complex objects. Using suitably designed L-System algorithms it is possible to design the L-System production rule that will create a particular class of plant. Figure 11 shows some plants generated using   L-Systems.

Figure 11 Some plants generated using a L-System.

Another way to realize, in Computer Graphics, the plants and the trees is to use Iterated Function Systems. This approach, applying all the transformations to the entire picture, is called the Deterministic Algorithm for generating fractals. Iterated function systems are described by repeatedly computing terms in two series, one series describes the x coordinate and the other series the y coordinate. The equations describe translation, scaling, rotation, and shearing of points in a plane with the restriction that the transformations are "affine". Figure 12 shows a fractals fern generated using Iterated Function Systems.

Figure 12 A fractal fern.

5. CONCLUSIONS

We can apply the mathematics point of views in different disciplines. For example, in the arts, in architecture, in design [1, 6, 7, 8, 12, 15, 23]. In our investigation we have presented an approach where mathematics and fractal geometry play a central role to describe the territory and the landscapes.

Mandelbrot presented one of the curiosities of fractal terrain: fractal terrain look like fractal terrain even when you turn them upside down [14]. Statistical parameters are equal in both cases. This property is typical for geologically new landscapes in nature. Moreover; the fractal techniques [2, 3, 10, 14, 19], provide only visual approximation of real terrains. This is almost never the case in nature, where depressions in the landscape fill up with all manners of detriment. This causes the landscape to be smoother over the ages of geologic time. Terrain is eroded by water, by particles of sand and crannied from influence of temperature amplitudes. Moreover; most of the terrains are also influenced by human factors. Majority of the natural influences are reflected as a kind of erosion. This problem has been presented for example by Fournier et al. (1982), Musgrave et al. (1989), Peitgen & Saupe (1988), and Maràk et al. (1997). Mandelbrot also pointed out in 1988 [19]: "*The most basic defect of fractal landscapes - the fact that this landscapes do not include the river network*". The rivers represent another kind of erosion in the same way as rain, temperature, wind, and time. This problem has been addressed for instance in Kelley et al. (1988), and Prusinkiewicz & Hammel (1993) [13, 21].

Considering the criterion of terrain erosion, techniques for generation of eroded terrain models can be divided into two classes. The first class of algorithms generates eroded terrain [10, 13, 19, 20, 21] The second group describes erosion of any terrain [16, 18], regardless whether the data representing the terrain was obtained with some artificial technique or real data was used. The erosion algorithm run on real data (e.g. obtained from satellite photos) can be considered as a simulation and therefore provides practical results useful in the Geographical Information Systems (GIS), and in ecology.

Most frequently terrain model is defined as height field. Regular height field is defined as two dimensional array of altitude values where the distance between rows and columns is constant.

We have also observed that fractal geometry can help to explain the Natural shapes and the Landscapes (e.g., coastlines, stones, trees) but we have to define a limit between natural and virtual environment [17].

This paper is only an attempt to describe some applications of Mathematics in the studies of territory and landscapes. We are in agreement with Galileo that has emphasized the powerful of mathematics (1623): "*Philosophy is written in this grand book - I mean universe - which stands continuously open to our gaze, but which cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.*"

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