Rabbit Ears to Slope to Derivatives: Longitudinal Development of an Algebraic Concept

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Abstract: From their earliest arithmetic experiences, students develop mental models conducive to algebraic thinking. Teachers can nourish this development by providing well-structured activities that use visual and tactile clues and cues to promote thinking. Experiences that allow students to recognize patterns and to represent and communicate their thinking through a variety of forms—verbal, concrete, pictorial, virtual, and symbolic—set the stage for later, more sophisticated representations such as tables, graphs, and equations. Using the algebraic concept of direct variation, this paper (a) describes how developmentally appropriate activities can be used across the grades to help students arrive at and communicate mental models, (b) provides specific examples of these investigations along a developmental continuum, and (c) discusses how this concept translates among various representational forms.

Developing Algebraic Thinking through Representation

From students' earliest experiences in arithmetic to their study of advanced topics such as trigonometry and calculus, algebraic concepts are born, developed, enhanced, deepened, and broadened. Algebraic thinking takes on different forms at different stages in students' development as their understanding moves from concrete to more abstract. Ideally, the activities and representations that students encounter along the way match their developmental readiness.

Younger students need physical materials in order to experience concepts in a more direct, concrete way. Experiences of this sort develop students' intuitive understanding of mathematics, providing them with a rich internal pool from which to generate hypotheses worthy of testing. Starting from a physical experiential base, students can represent their experiences in what Bruner calls an enactive mode, "a mode of representing past events through appropriate motor responses" (1964, p. 2). As they move towards abstraction, students can produce images that represent what they observe, beginning, perhaps, by drawing pictures and then becoming increasingly more abstract by creating tables, diagrams, and graphs. These iconic representations enable the perceiver to "summarize events by the selective organization of percepts and of images" (p. 2). Eventually, students can invent symbol systems to summarize and present their observations, or simply adopt the symbol systems that are conventionally used in mathematics. Students achieve symbolic representation when they acquire "a symbol system [which] represents things by design features that include remoteness and arbitrariness" (p. 2). Kaput (1995) describes two kernels of mathematical activity — generalization and formalization. Throughout this developmental continuum, students move from action to conceptualization to formalization (Ginsburg & Opper, 1979) towards the ultimate goal: accurate and rich internalization of concepts.

To enhance their development of conceptual and abstract thinking, students need opportunities to express their understanding in multiple modes of representation. They must describe things in their own words, discern and illustrate relationships and patterns pictorially, and represent concepts symbolically in the shorthand of written mathematics. When students achieve the ability to move flexibly among various mathematical representations and forms, they have arrived at understanding. Therefore, instructional activities should be rich and varied and should employ and encourage multiple modes of representation (Greeno & Hall, 1997).

Mathematics educators and cognitive researchers suggest that representation involves much more than translating a situation using symbolic notation (Goldin & Kaput, 1996). Beyond being the physical product of abstraction, representation is an internal, cognitive *process*: in order to represent, one must interpret a problem or concept actively and internally, giving it meaning. Although we cannot directly observe this internal process of representation in students, we can infer mental configurations from what they say or do. The teacher's challenge is to sequence and scaffold experiences in order to ensure that students are progressively developing concepts. According to Thompson (1994), educators' ultimate goal is not to have students physically represent functions but rather to have students sense the connections among representations. While forms of representation contribute to the acquisition of understanding and to the communication thereof, they should not be taught as ends in themselves. Through a combination of experiences, inquiry, and rich discussion that unifies their developing thoughts, students acquire fluency in moving between various representations.

For this paper, we have selected the algebraic concept of direct variation to illustrate how a concept can be developed longitudinally over students' mathematical experiences, in synchrony with their cognitive development. To better prepare students for the formal study of slope in algebra and derivatives in calculus, elementary and middle-grade teachers must provide mediated learning experiences that help students develop the language and use the tools needed to record and communicate their representations of concepts and to discuss the varying relationships among them. We will provide visual examples to convey the interconnectedness of various representations. The examples focus on simple direct variation with the variable that will result in constant slope, even though output can vary directly with the square or cube of the variable. The concept of direct variation is often a major focus of first-year algebra courses, and these examples consider elementary foundations that can lead to the elaborations that develop later in advanced mathematics.

Early Elementary Years

When teachers have profound understanding of fundamental mathematics and teach from this base, they provide opportunities for students to discover patterns that help establish the foundations of algebraic concepts. One such opportunity is having students count a number of rabbits and their ears and discover the relationships between these numbers. For example, one rabbit has two ears, two rabbits have four ears, and so on. When allowed to complete, describe, continue, and generalize growing patterns, students may discover that the number of rabbit ears varies directly with the number of rabbits (Cuoco & Curcio, 2001). This early introduction to direct variation underlies the algebraic concept of slope. One way to make this concept more concrete for young children is to use pattern blocks to create a visual representation.

Once students are able to recognize and create algebraic patterns, they need the opportunity to communicate these relationships. Because some students at this age have difficulty verbalizing the given pattern, a visual organizer is a key representation that can help facilitate the next level of reasoning and understanding. Students can record their discoveries in a picture diagram, and the pictures they create enable them to identify relationships among the elements (Cuoco & Curcio, 2001).

When neither rabbits nor pictures of rabbits are physically present, abstraction has begun. For instance, students might tabulate their counts in a table organized horizontally or vertically (Fig. 1 and 2). The table allows students to step back from the necessary but distracting task of counting so that they can see their data in an organized and informative way. A table succinctly summarizes the relationship between the number of rabbits and the number of ears. It is a means through which students can make the connection between the rabbits, the data, and the diagrams, thus facilitating their reasoning and their communication thereof.

Rabbits	1	2	3	4	5
Ears	2	4	6	8	10

Figure 1.

Rabbits	Ears
1	2
2	4
3	6
4	8
5	10

Figure 2.

Upper Elementary Years

In the upper-elementary grades, teachers can continue to build a foundation for function, direct variation, and slope by expanding the vocabulary they use to include terms like covariation (Thompson, 1994) and proportion. They can also move from the table format to a spreadsheet and from a diagram to a bar graph, drawn either by hand or by using the graphing feature in spreadsheet programs (Fig. 3).

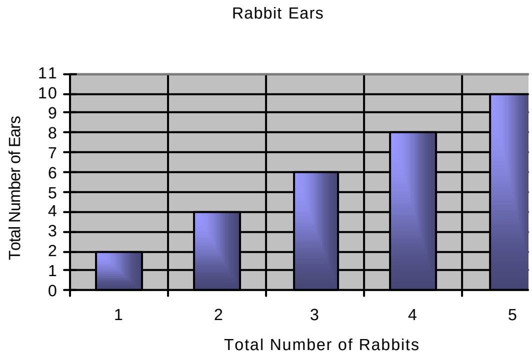

Figure 3.

Rabbits	1	2	3	…	10
Ears	2	4	6	…	20

Figure 4.

Students can explore the data for patterns, theorize why the pattern occurs, and go beyond the data to make predictions for the number of ears on a quantity of rabbits they have not directly observed, posing questions such as, "If one rabbit has two ears, how many ears do ten rabbits have?" Because some students will be able to solve this problem if they hear it but not if they read it, reading the problem aloud to students would provide appropriate scaffolding.

Through questioning, teachers can guide students to the multiplicative relationship that represents the covariation within the rows (Fig. 4). This relationship can be verbally expressed as 8 ears per 4 rabbits, 6 ears per 3 rabbits, or as the fundamental ratio that unites both of these, 2 ears per rabbit. The ratio can be written as ears:rabbits = 2:1, or $\frac{\text{ears}}{\text{rabbits}} = \frac{2}{1}$. Teachers who provide these experiences in thinking about the constant multiplicative relationship, ratio, and proportion are laying the foundation for students' thinking about the linear function, *f(x)=mx*, where *m*, the constant of proportion, is the slope—a critical ratio for middle school students to learn (Smith, 2002).

Middle School Years

In the middle grades, students focus their study of patterns and relations on those that relate to linear functions, preferably within a context that is meaningful to them (Van de Walle, 1998). They examine functions in various representations including tables, graphs, symbols, and words. New representations at the middle school level include graphs on the Cartesian plane (Fig. 5), as well as formulas and equations.

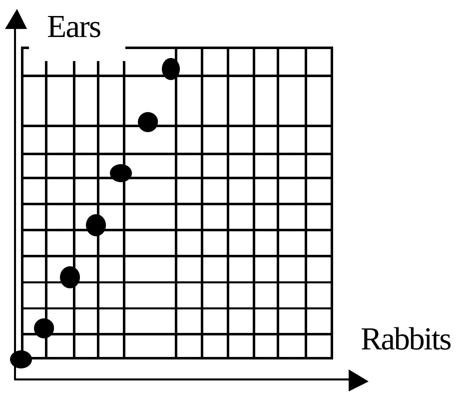

Figure 5.

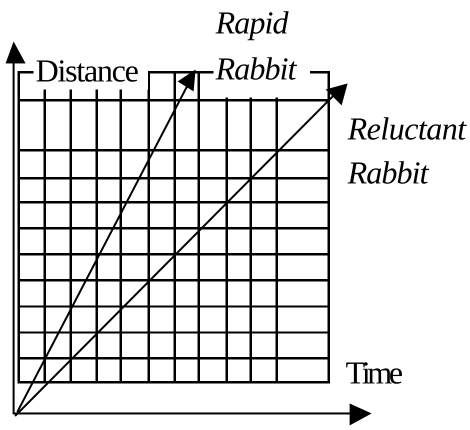

Figure 6.

Discrete values give way to continuous values as the problem at hand changes. Situations in which the values of the quantities compared belong to the set of real numbers rather than the integers give rise to the linear graph. Examples of these types of situations might include the problem, "When two rabbits race and Rapid Rabbit hops twice as fast as Reluctant Rabbit, how do the distances they have hopped compare as the race goes on?" The graph in Figure 6 represents the accumulated distance hopped by each rabbit over the time of the race. Students can compare the different ratios of change between the two racing rabbits' distances and then describe verbally how they computed the ratio. This thinking can lead them to derive a formula for the relationship: $f(x) = mx$, in which m is the ratio of change. By observing and discussing the effects of different ratios on the appearance of the graph, students will begin to develop the necessary skill needed to interpret slope computed from the classic slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$.

High School Years

When students reach high school, the generalization of patterns as well as formalization into conventional notation increase significantly. It is through continued study of algebraic topics in courses like Algebra 2, College Algebra, and PreCalculus that students can further develop their understanding of linear functions and the concepts of direct variation and slope. At this point, students must draw upon the foundations laid in earlier years and it is critical that their understandings be solid enough to withstand the transition.

A thorough understanding of slope prepares students for their encounter with the concept of the derivative in calculus. It is no leap to find the slope of a secant that goes through two distinct points on a curve. What is new is finding the slope of a tangent that touches the curve at a single point. The computer applet, Secant Lines and Tangent Lines, provides a virtual representation of secant and tangent lines that helps students connect the pictorial and symbolic representations of slope (http://www.ies.co.jp/math/products/calc/applets/limsec/limsec.html). The model illustrates that when two points are very close together, the slope between them is equal to the slope of the tangent line at a point between them on the curve.

Symbolically, the leap from the slope through two points to the slope at a single point requires the concept of the limit. To make the leap, students extend their use of the classic slope formula. If $(c, f(c))$ is a point and $(c + \Delta x, f(c + \Delta x))$ is a second point on the graph, then the slope of the secant line through the two points is given by substitution into the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$ becomes

$m_{\sec} = \frac{f(c+\Delta x) - f(c)}{(c+\Delta x) - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$ (Larson, 1998). To find the limit, the interval between the points is made smaller and smaller:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m \quad \text{(Larson 1998).}$$

The value of the limit is the slope of the tangent, or the slope at a single point.

Ultimately, as the interval becomes infinitesimally small, students arrive at the derivative, $\frac{dy}{dx}$. Surfing Derivatives, a computer applet, provides a virtual representation illustrating that derivatives are slopes (http://www.ies.co.jp/math/products/calc/applets/doukan/doukan.html).

Students in calculus courses may no longer need concrete representations of slope to inform their understanding, but the skill of transferring among various representations should continue to be developed. The use of verbal, symbolic, and especially virtual representations plays a vital role in helping students make connections between the concepts of slope and derivative.

Conclusion

In order to successfully internalize mathematical concepts and build foundational knowledge, students need experiences that help move their thinking from concrete to abstract and that allow them to represent and communicate their thinking in a variety of ways. Most educators teach students over a narrow span of this developmental continuum, yet it is imperative that they recognize how the material they are teaching at their particular level fits into students' overall mathematical development. Using various representations—verbal, concrete, pictorial, virtual, and symbolic—throughout students' mathematical experiences provides opportunities for students to come to their own understanding of and to make connections among various concepts. Teachers who artfully encourage their students to recall their prior related math experiences and skillfully foreshadow those that are yet to come facilitate the weaving of knowledge into a robust and sturdy conceptual fabric.

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