International Conference on
The Humanistic Renaissance in Mathematics Education
September 20-25, 2002
Hotel Città del Mare, Terrasini, Palermo, Italy (www.cittadelmare.it)

Following Goldbach's tracks

Aldo Scimone[1]

Pour un esprit scientifique, toute connaissance est réponse à une question. S'il n'y a pas eu de question, il ne pout y avoir de connaissance scientifique. Rien ne va de soi. Rien n'est donné. Tout est construit.

[For the scientific mind, all knowledge is a response to a question. If there had not been any questions, it would not have been possible to have scientific knowledge. Nothing comes of itself. Nothing is given. Everything is constructed.]

G. Bachelard, *La formation de l'esprit scientifique*, 5th ed. Paris, 1967, p. 14.

Undoubtedly, the intersection of mathematics teaching and the history of mathematics has had a long, fruitful tradition; so, nowadays one of the most frequent questions is: in what way may the history of mathematics influence mathematics education? If it is true that most knowledge is a response to a question, it is as true that the history of mathematics shows that mathematical concepts are constructed, modified, and extended in order to solve problems, so an alternative way of writing a history of mathematics is that of a history of problem solving.
The pedagogical value of open problems and conjectures for mathematics teaching is in general remarkable, expecially in the educational methodology of problem-solving.
In fact, such a methodology is important above all for the following reasons:
- it allows pupils to use their acquired knowledge to solve problems;
- it improves their logical-deductive abilities;
- it contributes in consolidating knowledge already mastered in a consistent fashion;

Moreover it allows :
- the acquisition of a scientific approach in facing mathematical problems;
- the working out of personal strategies in modeling;
- encourages teamwork;
- forms an antidogmatic mentality by which one can always move ahead.

Thus the aim of this paper is that of analyzing the educational value of mathematical conjectures to improve some pupil's abilities when confronting unsolved questions.

[1] Component of G.R.I.M. (Gruppo di Ricerca sull'Insegnamento delle Matematiche), Dipartimento di Matematica ed Applicazioni, Palermo. E-mail: aldo.scimone@libero.it

Through facing a conjecture a pupil may be stimulated in acquiring his own ways of reasoning by either following his particular mathematical background or individual intuitive approach in order to solve a question.
We are interested in the following kind of conjecture according to Balacheff (Balacheff, 1994):

a conjecture is a statement strictly connected with an argumentation and a set of conceptions wherein the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it.

The relationship between argumentation and proof, strictly connected to the relantionship between conjecture and valid statement, has been recently analized (Pedemonte, 2000) supposing that, during a solving process, which leads to a theorem, an argumentation activity is developed in order to produce a conjecture.
Instead, in the present case we want to analyze the gradual passage of pupils' attempts from an argumentation to a proof, while they are facing a known unsolved conjecture. We have choosen a historical conjecture like Goldbach's one essentially for the simplicity of its statement and its fascinating empirical evidence.
Goldbach's conjecture states that:

"Every even number greater than 2 can be represented as the sum of two primes."

This conjecture belongs to number theory which has a greater number of conjectures than other mathematical fields. So, Goldbach's conjecture seems to be useful in order to point out the following points:

- pupils' conceptions in relation to a conjecture faced during the historical development of mathematics;

- pupils' attempts proving a conjecture reclaimed from history and compared with their argumentative processes;

- to what extent the history of mathematics can favour the study of pupils' conceptions about arguing, conjecturing and proving;

- their reaction to a conjecture's terms seemingly simple to solve;

- their approach in the solving of a conjecture;

- their abilities in carrying out non-standard solving strategies (lateral thinking);

As we know a conjecture can be transformed into a theorem if a proof justifying it is produced; namely, if it is possible to use a mathematical theory allowing the construction of a proof of it.
The basic reason why we have decided to propose an unsolved conjecture like Goldbach's one is, as we have said, to emphasise the role of problems in the historical development of mathematical knowledge. As we know a branch of mathematics maintains mathematicians' interest alive as long as there are always new problems to be solved, because it is only in this way that mathematical knowledge can progress, giving new lymph for the growth of other collateral branches.
It is impossibile to do mathematics without asking oneself problems and trying to solve them; or rather the main activity of a mathematician is the solving of problems posed by others or which he puts himself, according to his own tastes and choices. It is in this

manner that one can encounter with a really important theorem which enlightens an entire branch of mathematics and through which other trunds of search trends are set in motion. Doubtless, there are really a lot of open problems and unsolved conjectures in number theory, and their number grows yearly, giving continous inspiration to mathematicians.

The experimental work

The experimental work was carried out by two phases: the first statistical survey was made using a champion of 88 pupils of a high school in Palermo (Sicily) into two final classrooms, precisely into a third and fourth classroom. The students worked in pairs for the part relating to interviews and individually for the production of solution protocols related to the proposed conjecture[2].
The surveyed data were analyzed by the software of inferential statistics CHIC 2000 (*Classification Hiérarchique Implicative et Cohésitive*) and the factorial statistical survey S.P.S.S. (*Statistical Package for Social Sciences*). The variables used were 15 and they were the basis for the analysis a-priori of possible answers by pupils.
As an example, here is the analysis a-priori of the problem explained by the following steps:

1) He/she verifies the conjecture by natural number taken at random.(N-random)

2) He/she sums two prime numbers at random and checks if the result is an even number. (Pr-random)

3) He/she factorizes the even number and sums its factors, trying to obtain two primes. (Factor)

4) *Golbach's method 1*
He/she considers odd prime numbers lesser than an even number, summing each of them with successive primes. (Gold1)

5) *Golbach's method* 2 (*letter to Euler*)
He/she writes an even number as a sum of more units, combining these in order to get two primes. (Gold2)

6 *Cantor's method*
Given the even number $2n$, by subtracting from it the prime numbers $x \leq 2n$ one by one, by a table of primes one tempts if the obtained difference $2n - x$ is a prime. If it is, then $2n$ is a sum of two primes. (Cant)

7) *The strategy for Cantor's method*
He/she considers the primes lower then the given number and calculates the difference between the given number and each of primes. (S-Cant)

8) *Euler*
He/she is uneasy to prove the conjecture because one has to consider the additive properties of numbers. (Euler)

9) *Chen Jing-run's method (1966)*
He/she expresses an even number as a sum of a prime and of a number which is the product of two primes. (Chen)

[2] The author is grateful to Proff. Marilina Ajello, Carmelo Arena, Egle Cerrone and Emanuele Perez for their helpfulness in carrying out the experimentations into their classrooms.

l0) He/she subtracts a prime number from an any even number (lower then the given even number) and he/she ascertains if he/she obtains a prime, so the condition is verified. (Spa-pr)

11) He/she looks for a counter-example which invalidates the statement. (C-exam)

12) He/she considers the final digits of a prime to ascertain the truth of the statement. (Cifre)

13) He/she thinks that a verification of the statement by some numerical examples needs to prove the statement. (V-prova)

14) He/she does not argue anything for the second question. (Nulla)

15*)* He/she thinks the conjecture is a postulate. (Post)

Hypothesis of search

The two experimentation were based essentially on the following hypothesis of search, which could be either validated or falsificated:

I. Pupils are not able to go beyond the empirical evidence of the conjecture because they do not know how to represent mentally any general method useful for a demonstration.

II. Pupils can reach only intuitive conclusions about the validity of Goldbach's conjecture.

The text for the individual work

The pupils working individually had two hours for answering the following two questions:

Answer the following questions arguing about or motivating every answer:

***a*) Using the enclosed table of primes, the following even numbers can be written as a sum of two primes (in an alone or in a manner more)?**
248; 356; 1278; 3896

***b*) If you have answered the previous question, are you able to prove that it occours for every even number?**

The text for the interview for pairs

The interviewes for pairs were made to two pairs of pupils, respectively 16 and 17 aged. We shall name the pupils of the first pair by the letters L, G and the others by the letters R, S. In both cases the interview lasted 30 minutes, and it was audio recorded. Here is the text of the interview:

Answer the following question writing only what you have agreed on:

- Is it always true that every even natural number greater than 2 is a sum of two prime numbers?

Let argue about the demonstrative processes motivating them.

The second experimentation

The second experimentation about Goldbach's Conjecture was made thanks to a group of teachers, coordinated by the author, in some of their classrooms of primary, middle and high school in Piazza Armerina, a provincial town of Enna[3].
The general subject of the experimentation has been about *arguing, conjecturing and proving*.

An overview about achieved results

Writing this work we asked ourselves some questions answering some of them, not others. Our first task was to point up the following points:

- pupils' conceptions in relation to a conjecture faced during the historical development of mathematics;

- pupils' attempts proving a conjecture reclaimed from history and compared with their argumentative processes;

- to what extent the history of mathematics can favour the study of pupils' conceptions about arguing, conjecturing and proving;

- their reaction to a conjecture's terms seemingly simple to solve;

- their approach in the solving of a conjecture;

- their abilities in carrying out non-standard solving strategies (lateral thinking).

Well, throughout the analysis of results one can argue that all of the trials of pupils were implicated by the historical attempt of Golbach. So, this validates the abovewritten two hypothesis.
Moreover, we want to point up that the second experimentation pointed out a characteristic behaviour of pupils from primary to high school while facing Goldbach's conjecture.
Most pupils of middle and high school faced the conjecture by a solving methodology based on random choice and on sequential thinking.
There was a difference between methods of facing the conjecture by pupils of middle and high school. Pupils of middle school in general faced the conjecture basing on an empirical approach, also arguing their choices; but their task went on until a certain point of verification and not beyond.
On the contrary, there was the presence either of argumentation and attempt of proving in high school pupils' approach. Really many of them tried to infer a prove by their argumentation, and some of them reached also to Chen strategy, wondering all of us. We came down to the same conclusions by the results of the two interviews made during the first experimentation. Well, in these cases our initial hypothesis were falsificated, and this was a fine surprise.
The results of the experimentation realizes some questions which would be deeped:
- how do pupils get consciousness of a demonstrative process?

[3] The author is grateful to Proff. Gabriella Termini, Salvatore Marotta, Salvatrice Sorte, Angela Milazzo, Lina Carini, Carmela Buscemi and Fabio Lo Iacona for their helpfulness in carrying out the experimentations into their classrooms.

- how do pupils get consciousness of the necessity of a demonstrative process?
- how pupils are able to pass from an argumentation to a demonstration?
- are pupils fully conscious of the difference between a verification and a proof?

These and similar questions can give rise to significant experimentations in order to comprise even better metacognitive processes which are basic for the learning phase of pupils and their cultural growth.

But the fundamental kernel of this experimentation about the interplay between history of mathematics and mathematics education is that such results could not be pointed out if the analysis a-priori had not been made by the historical-epistemogical remarks which have inspired it.

Essential Bibliografy

Ajello M., *L'analisi a-priori come strumento per la strutturazione dell'insegnamento per moduli*, GRIM di Palermo, Quaderni di ricerca in Didattica, n. 9, 2000.

Balacheff N., *Conception, connaissance et concept*, Didactique et tecnologies cognitives en mathématiques, séminaires, Université Joseph Fourier (Grenoble 1), 1994-1995, pp. 219-244.

Bologna V., *Esperienze francesi sulla dimostrazione*, Quaderni di Ricerca in Didattica del GRIM, Palermo, 2 (1992), 5-23.

Brousseau G. , *Theory of didactical situations in mathematics*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997, Edited and translated by N. Balacheff, M. Cooper, R. Sutherland, and V. Warfield.

Brousseau G., *Fondaments et méthodes de la didactique des mathématiques*, Recherchesen didactique des mathématiques, Paris, 1986.

D'Amore B., Elementi di Didattica della Matematica, Pitagora Editrice, 1999.

Gras R., *Les fondaments de l'analyse statistique implicative*, Quaderni di Ricerca in Didattica, n. 9, Palermo, 1999.

Pedemonte B., *Some cognitive aspects of the relationship between argumentation and proof in mathematics*, Proceedings of the 25th conference of the international group for the Psychology of Mathematics Education Utrecht, Editor: van den Heuvel-Panhuizen M., 1999.

Spagnolo F., Insegnare le matematiche nella scuola secondaria, La Nuova Italia Editrice, Ith Ed. 1998, 1999 (the first reprint).

Spagnolo F., *L'analisi a-priori e l'indice di implicazione statistica di Gras*, Quaderni di ricerca in didattica GRIM, n. 7, Palermo, 1997.

Spagnolo F., *La recherche en didactique des mathématiques: un paradigme de référence*, Zbornik, Bratislavskeho seminara z teorie vyucovania matematiky, Bratislava, 1999.

Spagnolo F., La ricerca in didattica nel lavoro dell'aula, Convegno nazionale organizzato dall'IRRSAE-Sicilia sul tema Ricerca e d Autonomia, Palermo, Dicembre 1999.

Spagnolo F., *Obstacles Epistèmologiques: Le Postulat d'Eudoxe-Archimede, Tesi di Dottorato di Ricerca,* Università di Bordeaux I (Francia), Luglio 1995, Pubblicata dall'Atelier National de Réproduction des thèses Microphiches (BP - 38040 Grenoble Cedex 9, Francia) e dai Quaderni GRIM di Palermo per la diffusione in Italia (Supplemento al n.5, 1995).

Spagnolo F., *The role of history of Mathematics in Research in Mathematics Education*, Amman (Giordania), 18-23 Novembre 2000, International Conference on Mathematics Education into the 21st Century.