The Solution is Just the Beginning: Using Rich Learning Tasks to Develop Mathematical Creativity

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Frequently, students are satisfied with getting an answer to a problem and not looking at it any further. They often do not progress to the fourth "looking back" stage that Polya (1945) discussed in his seminal book, *How to Solve It*. In this book, Polya discusses a four phase problem-solving method:
1. Understand the problem 2. Plan 3. Carry out the plan 4. Look back at the completed solution

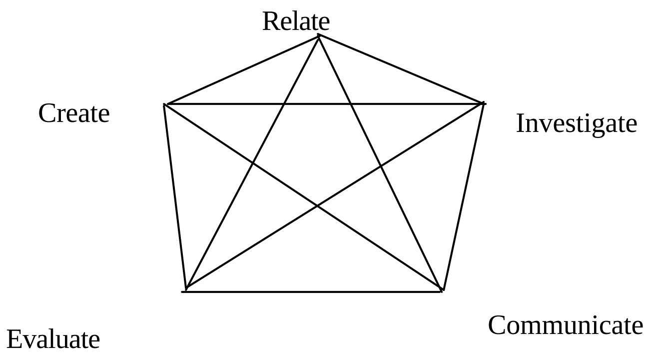
In this fourth stage, Polya recommends that students review the completed solution and the path taken to obtain it. He suggests that students should check their results and try to obtain the same result in another fashion.

For years, the National Council of Teachers of Mathematics has listed problem solving both as a major goal and a means of learning mathematics. The NCTM Principles and Standards for School Mathematics (2000, p. 52) state "By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom." These habits of mind should go beyond merely finding a solution to a given problem.

Mathematicians will tell you that the solution to a problem is often just the beginning of real mathematics. We might think of learners of mathematics on a continuum that goes beyond problem solving to problem creating in a manner such as the following:

Following this continuum, students may experience the excitement of thinking deeply about mathematical ideas and discovering new concepts. Students may learn to explore problems to find that the fun has only just begun when the original problem has been solved.

Using the following open-ended heuristic is one way to get students to become more creative in the exploration of rich mathematical tasks.



Students might start anywhere on this model and proceed in a non-linear fashion to creatively investigate a problem. For example, a student might relate ideas about solving this problem to previous problems that have been solved, investigate those ideas, create new problems to work on, evaluate solutions, communicate the results, and think of other related problems to work on.

Questions to Encourage Thinking and Reasoning

In order to encourage students to investigate mathematical concepts on a deeper, more creative level, you should use rich, interesting problems that can be explored on a variety of levels. Expect problems to be solved in a variety of ways and give students a chance to explain their reasoning to each other. Use one problem as a springboard for several others. Work on these problems with colleagues before trying them with children and see how many solutions, patterns, generalizations, and related problems you can find. Ask questions that help students explore the big ideas, such as:

Comparisons and Relationships

1. How is this like other mathematical problems or patterns that I have seen? How does it differ?
2. How does this relate to "real-life" situations or models?
3. How are two factors or variables related?

Structure, Organization and Representation

1. How can I represent, simulate, model, or visualize these ideas in various ways?
2. How might I sort, organize, and present this information?
3. What are the essential elements of this problem?

Rules and Procedures

1. What steps might I follow to solve that? Are they reversible? Is there an easier or better way?
2. Do I have enough information? too much information? conflicting information?
3. What if I change one or more parts of the problem? How does that affect the outcomes?

Patterns and Generalizations

1. What patterns do I see in this data?
2. Can I generalize these patterns?

Reasoning and Verification

1. Why does that work? If it does not work, why not?
2. Will that always work? Will that ever work?
3. Is that reasonable? Can you prove that? Are you sure?

Optimization and Measurement

1. How big is it? What is the largest possible answer? the smallest?
2. How many solutions are possible? Which is the best?
3. What are the chances? What is the best chance?

Assessment Criteria

If you wish students to develop deeper understanding of mathematical concepts, you should use criteria for assessment that encourage depth and creativity such as:

- **Depth of understanding** - the extent to which core concepts are explored and developed
- **Fluency** - the number of different correct answers, methods of solution, or new questions formulated
- **Flexibility** - the number of different categories of answers, methods, or questions.
- **Originality** - solutions, methods or questions that are unique and show insight
- **Elaboration or elegance** - quality of expression of thinking, including charts, graphs, drawings, models, and words
- **Generalizations** - patterns that are noted, hypothesized, and verified for larger categories
- **Extensions** - related questions that are asked and explored, especially those involving why and what if

Using a rubric such as the following, rather than grading students simply on whether their responses are correct may encourage students to mine the rich depths of problems.

Sample Scoring Rubric to Encourage Depth and Complexity in Mathematical Reasoning

Scores

Assessment Criteria	**1**	**2**	**3**	**4**
Depth of Understanding	Little or no understanding	Partial understanding; minor mathematical errors	Good understanding; mathematically correct	In-depth understanding; well-developed ideas
Fluency	On incomplete or unworkable app	At least one appropriate approac or related question	At least two appropriate approaches or good related questions	Several appropriate approaches or new relat questions

Flexibility		All approaches use the same method (e. g., all graphs, all algebraic equations and so on)	At least two metho solution (e. g., geometric, graphica algebraic, physical modeling)	Several methods of solution (e. g., geometric graphical, algebraic, physical modeling)
Originality	Method may be different but does not lead to a solution	Method will lead to a solution but is fairly common	Unusual, workable method used by only a few students	Unique, insightful method used only by one or two students
Elaboration or Elegance	Little or no appropriate explanation given	Explanation is understandable but may be unclear in some places	Clear explanation using correct mathematical terms	Clear, concise, precise explanations making good use of graphs, char models, or equations
Generalizations and Reasoning	No generalizations made, or they are incorrect and reasoning is unclear	At least one correc generalizations made; may not be well-supported with clear reasoning	At least one well-made, support generalization, or more than one correct but unsupported generalization	Several well-supported generalizations; clear reasoning
Extensions	None included, or extensions are not mathematical	At least one related mathematical question appropriately explored	One related questio explored in depth, o than one appropriately explored	More than one related question explored in dept

If you use a rubric such as this one to score a students' mathematical work, you will need to choose rich problems for students to explore. Good tasks should meet the following criteria:

1. Tasks should ask questions that make students think, not questions that make them guess what the teacher is thinking.
2. Tasks should enable children to build on previous knowledge and to discover previously unknown mathematical principles and concepts.
3. Tasks should be connected to core standards and benchmarks in mathematics.
4. Tasks should be rich, with a wide range of opportunities for children to explore, reflect, extend, and branch out into new related areas.
5. Tasks should give children the opportunity to demonstrate abilities in a variety of ways, verbally, geometrically, graphically, algebraically, numerically, etc.
6. Tasks should allow children to use their abilities to question, reason, communicate, solve problems, and make connections to other areas of mathematics as well as to other subject areas and "real world" problems.
7. Tasks should make full use of technology such as calculators and computers as well as mathematical manipulatives and models.
8. Tasks should give time for individual reflection and problems solving as well as time for group exploration and discovery.
8. Tasks should be interesting and should actively involve the child.
9. Tasks should be open with more than one right answer and/or more than one path to solution.

These open-ended and open-middle problems should move from:

- Straightforward skill and drill problems with one right answer
- To puzzle problems requiring reasoning and justification. These problems should encourage students to think deeply about simple things.
- Or exploration problems with several solutions and extensions. These problems should encourage students to question the answers, not just answer the questions.

A Few Examples of Adding Depth and Complexity to Mathematical Questions

Skill and Drill – If you add two consecutive whole numbers, is the sum odd or even? Explain.

Puzzles- Find five consecutive whole numbers that add to 15. Can you do this with 2 consecutive numbers? 3? 4? Why or why not?

Exploration – (From the Ross Program 2002 application problems - http://www.math.ohio-state.edu/ross/app/problems02.html)

Let's call a number n "nice" if it can be expressed as a sum of two or more consecutive positive integers. For example, the expressions 5 = 2+3 and 6 = 1+2+3 show that 5 and 6 are nice numbers.

(a) Which numbers are nice? Justify your answer.

Some numbers are "very nice", in the sense that they are nice in more than one way. For example, 15 is very nice because 15 = 1+2+3+4+5 = 4+5+6 = 7+8.

(b) Which numbers are very nice? Explain.

Consecutive Addend Sums

Number of Addends

Sums	2	3	4	5	6	***a***
1						
2						
………..						
34						
35						
36						
n						

II. Skill and Drill: Fill in the next three lines on the chart below and explain your method.

1

3 5

7 9 11

13 15 17 19

Puzzle: Without completing the chart, tell where the number 289 will appear. Explain your reasoning.

Explorations: List all the patterns you can find on the chart above. Make and validate generalizations about your patterns.

III. Skill and Drill: Dan has decided to buy tops, toy planes, and/or stuffed animals for the school fair, and he must spend according to the following prices:

Tops $ .50 apiece Toy planes $ 1.00 apiece Stuffed Animals $10.00 apiece

If he buys 15 tops, 35 planes, and 50 stuffed animals, how much will he spend?

Puzzle: Latisha has exactly $100 to buy toys for the local children's charity and wants to buy exactly 100 toys. She has decided to buy tops, toy planes, and/or stuffed animals, and she must spend according to the following prices:

Tops $ .50 apiece Toy planes $ 1.00 apiece Stuffed Animals $10.00 apiece

What combinations of toys might she buy?

Exploration: Jerome has exactly $100 to buy toys for the local children's charity and wants to buy at least 60 toys. He has decided to buy tops, toy planes, and/or stuffed animals, and he must spend according to the following prices:

Tops $ .50 apiece Toy planes $ 1.00 apiece Stuffed Animals $10.00 apiece

What combinations of toys might he buy? Make a chart to display your findings. What patterns do you notice?

References

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Polya, George. (1945). *How to Solve It: A New Aspect of Mathematical Method.* Princeton, NJ: Princeton University Press.