Developing mental abilities through structured teaching methodology

Mihaela Singer

From informational islands to structured knowledge

In the traditional common teaching, the goal is to teach so that students assimilate information. The total amount of information considered necessary and useful to be taught within mathematics from the first school year to the last one is divided into chunks fitting each school stage. Then, within each school stage, the information specific to each school year is selected through the curricula. Further on, in the textbooks, information is split into chapters and lessons. The reverse path, from the first lesson in the first school year to the last lesson in the secondary school, is supposed to be followed by the pupil's mind with the purpose of "knowing" the mathematics assigned to the pre-university education by the experts in curriculum development. Thus, the pupil faces a mixture of rigorously detailed informational islands. The pupil must go through these. In the end, the pupil must know all he/she has learned, or at least the essential of each mathematical concept. Moreover, the pupil must also possess a mathematical thinking, which has a special human and social value; otherwise, the whole teaching would be pointless. Beyond the barren instruction, there is a need to develop mathematical thinking of a much greater value, therefore transcending the boundaries of the formal content. The logical rigorousness, the ability to do quantitative analyses, the quality of the professional activity are tightly related to the quality of the mathematical instruction of each person.

The need for developing the mathematical thinking has been permanently emphasized in didactics for a long time. The problem is not whether the development of the mathematical thinking represents the essential goal of mathematical instruction in school, but how to pass from the intention to its implementation. Nowadays, the dominant idea is that the mathematical competence, or mathematical thinking, is a spontaneous result of instruction. The person learns and *automatically* becomes competent, according to the following rule: the one who gets information is also able to think on the appropriate level. As we see, according to the traditional philosophy of learning, which is still very strong everywhere in the world, information represent the essential in the mathematics teaching, the competence (mathematical thinking) being just a mechanical consequence of their assimilation.

The results of this philosophy of learning are expressed in the present-day fact: the average pupil's failure to learn mathematics. This average pupil possesses, at best, just "informational islands". Researches carried out on secondary-school students reveal major flaws exactly on the level of mathematical thinking; but these flaws have their roots precisely at the beginning of the informational stairway, that is as early as the first grade. The teaching that is focused on developing mental capacities implies an extremely structured *knowledge organization*.

From this point of view, the primary school, being the most stable part of the system, is in a paradoxical situation: though it has not been preparing graduates for decades, primary school still remains, in very many countries, within the same paradigm: i.e. end a cycle instead of giving the necessary roots for the following stage. More specifically, primary education still offers a restricted area of contents, in which learning "reading, writing and computing" remains the fundamental objective, despite the intentions stated in the curricula's rationales. The superabundance of difficult problems in textbooks is not liable to break the deadlock, but to deepen, in the child's mind, the confusions produced by the lack of perspective. It follows from the above that the change must start with the primary school, and that this change is substantial on this level, both in content (*what* is learnt), and in methodology (*how* it is learnt). This shift in the targets of learning in the early ages has been done by Great Britain in the nineties and by other English languages countries, as Singapore, Australia, and New Zealand in successive steps.

From problem solving to generators learning

It is not enough to teach the child how to solve the problem. It is equally important that the child should be able to **communicate** the problem's solution. To communicate the problem's solution means to present a series of items of reasoning or operations accepted by *consensus* as explaining the way to get to the result. This so-called consensus represents the accumulation of an historical experience, so it is a cultural acquisition. Hence, teaching the pupils how to solve problems and teaching them to communicate the

solving are two different things that must be understood and practiced as such during the problem-solving lessons.
To develop strategies for efficient learning in problem-solving is important to face some questions. There are a huge variety of problems. What kind of problems should the child solve with mathematics? And more, how numerous must these problems be in order to ensure learning? How should these problems be sequenced in a textbook so that understanding might occur without an exaggerated effort? As answers to these questions, our research was oriented to dispose of an inventory of the **generator-types** of problems. This article is concentrated on primary education. "Generator" means, in this case, that on its basis, *by combination, substitution, and enlarging-narrowing the domain, by varying the actions, changing the topic*, etc., a great variety of problems can be created.
Learning the generators has the advantage that it structures the understanding of the mathematical phenomena hidden behind particular statements. If, together with the ability to solve problems, the pupil gets the ability to understand and use the generators of a class of problems, then the child's cognitive acquisition is definitely superior and it refers to the arising of an over-learning phenomenon. The school has as a final scope to accelerate learning, in other words, to create shortcuts for simplifying learning. It is not only about the necessity of bringing into the mass school things that time ago were known only by the experts (for example, elements of the set theory, or elements of the functions theory), but it is also about the rapid rhythm of information evolution. To succeed in developing shortcuts in a mastery learning is also necessary to help children to show confidence and initiative in handling mathematical subjects, in describing, orally or in writing their own work and the obtained results, and in supporting all these with intuitive arguments. It is also important to conduct children to use mathematical ideas, rules and models in tasking practical problems and everyday situations and to understand the advantages offered by mathematics in tackling, clarifying, and following such problems or situations.
Keeping narrow separated zones of formal instruction is more and more in opposition to the globalization which is rapidly growing even it is politically correct or not. Today the multitude of external stimuli (media, ICT, etc) show that the process of information globalization could not be stopped. Consequently, the solution for mathematics learning could no longer be learning tables by heart. This results in transforming concepts understanding in an unproductive ballast, which is handicapping children; it is perhaps a sign of healthiness of human being the refuse of learning in that case. Maybe here there is the explanation for a "global" experience of the lack of interest manifested nowadays in many European and American countries for mathematics learning. That is way in order to organize a good training for children it is necessary to make use of an inventory of possible situations and to offer to children as many opportunities as possible to interfere with various learning specific environments. The solution is not to transform mathematics in a joke or in a game; mathematics itself has enough resources to create learning motivation in children. It is challenging and stimulating and the brain needs this type of stimulus, because it is self-generating.
In the following, an example of the scale of transition from concrete to abstract will be given regarding the addition learning in early ages. From the mathematics point of view, addition is a binary operation defined on a Cartesian product of identical sets with values in that set:
$+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. Particularly, in primary grades, $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$. How is it possible to internalize such a model? No matter what type of numbers is about, to understand the process of addition supposes a high ability of abstractness. This happens because it is not about learning an algorithm, it is about internalizing the concept of addition as deep as other concepts in daily life are internalized in a natural way when the child "learns" all the complex meaning of common words in his/her mother tongue. To understand *the concept* of addition at an early age it is necessary to develop *thinking abilities* and *learning thinking abilities* in the child.
Addition learning could not be separated from practical problems in which union is appearing in a natural way. To classify the elements of addition problems, we took into account the following criteria: type of theme, type of terms, type of action, and type of connection terms-action. The theme is offering the context for practicing union when using various categories of objects. The action is characterized by direction, sense and rhythm. Here the word "term" is used with a very large meaning; for a child, it is

about living beings that are moving by themselves - active, or about objects which are put together by somebody else - passive. The action could be explicit (external, visible, evident) or implicit (internal, non-visible). The connection terms-action can have different degrees of mobility from static to dynamic.
What are the situations that, systematically followed, conduct to the internalization of the union concept?

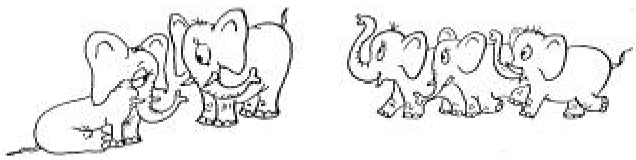
1. The first is the moment of "**drama**", or of the role-play when the children become characters in a "story" like this: "*Two children are writing on the blackboard. Three children are coming. How many children are writing on the blackboard now?*" (They act the whole scene.)

2. Next comes the **role-play having objects as characters**: *"I have two pencils and I get four more. How many pencils do I have now?"* (The teacher mimics the action while speaking, and the children do the same.)

The next steps consist in the gradual representations given by suggestive images:

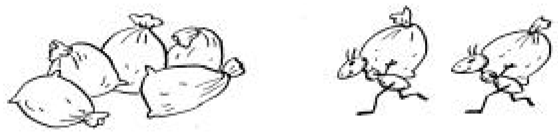
3. Explicit active dynamic union

A concrete meaning of this category is given by representing living beings that are moving in a visible way. Example: *Describe the action and compute:* 2 + 3 =



4. Explicit passive dynamic union

A concrete meaning of this category is given by representing objects that are moved in a visible way. Example: *Describe the action and compute*: 5 + 2 =



5. Explicit static union

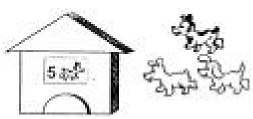
A concrete meaning of this category is given by representing objects that are connected together in a visible way. The representation gives a post-action image. Example: *Write the result*: 4 + 4 =

6. Implicit static union

A design convention using countable schematized objects is used to represent union. Example: *Write the result*: 3 + 4 =

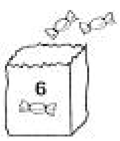
7. Active dynamic union in which one of the terms is abstract

It is similar to **3** but to solve this task category, the child is obliged to begin the counting from a given start, different from 1. Example: *Compute, counting further*: 5 + 3 =



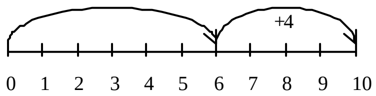
8. Passive dynamic union in which one of the terms is abstract

This is a more abstractized version of **4**, as well as **7** is for **3**. Example: *Compute, counting further*: 6 + 2 =



9. Icon representation (number line)

On the one hand, this is offering the support of counting "sticks" or "bricks" represented in a simplified way and, on the other hand, this is underlying the ordinal characteristics of natural numbers. Example: *Use the number line to compute*: 6 + 4 =



10. Horizontal symbolic writing

11. Vertical symbolic writing

12. Mental computing with no support.

In the sequence above, the material support of computing gradually acquires a simplified (iconic) representation and then disappears. The scale of transition from concrete to abstract grows, being increasingly refined.

From "drill and practice" to "practice and structure"

This type of training is practiced firstly on a systematic basis and then on a randomly one. Further, the training is focused on *development of understanding word problems* containing addition or subtraction. The teacher gives groups of objects and then images, and asks for the formulation of problems. A simple problem of addition ($a+b=x$, where x is unknown) is written on the blackboard. The problem is reformulated keeping the same numbers (changing the question position, for example). The problem is extended as far as to contain three or four operations of addition, then operations of subtraction or mixed ones. The initial problem is compared with other ones in order to determine the similarities and dissimilarities. Other problems are devised starting from series with missing or incomplete elements. The child is stimulated to create word problems starting with composing and decomposing a number, or starting from a given exercise. The initial exercise is compared with another one, concerning the number of terms, operations, etc. The number of terms and number of operations are increased. The objective is to make students create as many problems as possible by keeping the numbers, but varying the context of the problem. These are practiced orally, silently, in written form, maintaining the interest in exercising as many capacities as possible.

Understanding word problems is closely connected with *analyzing and transforming the problems*. The teacher gives the schema $a+b=x$ and asks for the development of problems within its limits. Then the teacher asks for the formulation of varied word problems. The position of the unknown is changed ($a+x=c$; $x=a+b$; $x+b=c$; etc.) with the same requirements (creating different exercises and problems). The same procedure is carried on starting from one of the schemes: $a-b=x$, $a+b+c=x$, $a-b-c=x$, etc., or from graphical models, diagrams, tables. All these are practiced in written form, orally or silently and the stress is laid on practicing the passage from one type of task to another.

In extra, there is a systematic training of *becoming aware of errors*. The teacher "hides" some errors in the exercises or problems, in series of numbers, in comparisons, etc. These errors are analyzed with the pupils. The goal is to eliminate the pupils' typical errors, as well as the improvement of the analysis' ability. *Estimations and approximations* are also systematically taken into consideration. The objective is to understand the significance of a number's order size and to verify the computations validity. The children estimate the number of objects in a transparent vessel, the number of pieces in a construction, the size of various objects, the possible computing results before computing, etc. All these activities are practiced orally, silently, in written form (without or with minimum verbalizing, and the result is required for checking).

The targets are: to raise the internalized structure to the level of representation and then to the level of primary notion of natural numbers and of operation with these (i.e. the cardinal of a set of objects and the cardinal of the reunion of disjoint sets) and to transform the internalized structure into a dynamic one (i.e. able to solve various problems in various contexts by identifying the invariants).

The first task is accomplished mostly through the exercises present at the beginning of each lesson. These exercises become gradually complicated, thus requiring "the movement of thinking" according to the initial model, in the most various ways: with direct support from objects or concrete schemes, or without this support, inclusively by operating in an internal language, and later by operating with the literal symbols. However, to obtain a dynamic structure is much more complicated and this is realized by "shifting" each element of the model, by passing it and confronting it with different thinking operations (which also become intellectual capacities, that must be practiced and learnt as such): comparing, developing patterns, generating new word problems, mental calculation, symbolizing, composing and decomposing numbers, etc. Some of these intellectual operations are mathematical operations, but the stress is laid on training the intellect, and not only on learning - memorizing the mathematics involved in here.

This training leads to the evolvement of a dynamic mental structure, able to get mobilized in various situations and to find creative solutions for complex problems. Therefore, in serial arrangement, each element of the initial model is considered either starting point, part or end of the series; in creation, exercises and problems are constructed with each of the elements of the initial model; in computing, all operations are done by various passages from concrete to abstract (using objects, patterns, schemes, etc.); in symbolizing, all the elements of the model are restored.

To evaluate the results of this kind of training we used the following sources: examining the behavior of the pupils from the experimental classes; examining the pupils behavior not involved in the experiment; systematically testing the pupils participating in the experiment; testing, at intervals, the experimental classes and the control ones; collecting the opinions of the teachers involved in those experiments. Classroom observation was considered as the essential element for drawing the conclusions, as this permits, in addition, to ascertain the children's individual and group reaction, and to evaluate the motivation level, the interest, the spontaneity, the atmosphere in the class, etc.

The implementation of this methodology does not require a very special teacher training, since most of the tasks have already been accidentally applied by the teachers. What is supposed to happen is a shift from the so called "drill and practice" technique to the "practice and structure" strategy. In this new approach common tasks are restructured and are used as methods of *systematically developing intellectual capacities*.

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