Understanding of three dimensional arrays of cubes - Children in transition

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This presentation is an attempt to look into specific details of the learning behaviour of children in transition from one level of volume understanding to another, and suggests ways in which the use of physical material could facilitate this transition. The cases of specific children were selected from observations of performance on volume measurement and conservation tasks obtained for a larger sample. The selection of the specific cases was based on the apparent inconsistencies observed in their responses showing a shift in understanding as they moved from one task to another. Focusing on their attempts to solve an intermediate task which involved construction of physical arrangements of unit cubes, we argue that similar activities could produce discovery learning that would facilitate the transition to a more operational understanding of the volume concept.

Framework

The tasks and responses presented here, were selected from the results of a larger field study contacted with the aim of evaluating 5th and 6th grade Cypriot children learning of the concepts of measurement and conservation of volume. In this larger study each child was presented with tasks on measurement and conservation of volume in a structured interview setting and in a written task in the form of a questionnaire. The tasks included the original Piagetian (Piaget et al., 1960) transformation task of "building a house of the same room on an island of different size", and tasks of measurement and conservation similar to those used in the CSMS project 1975-1989 (Brown et al., 1984).

The responses to volume measurement tasks involving rectangular constructions made out of unit cubes were categorised according to the SOLO-Taxonomy Theory observed levels of response (Campbell, Watson& Collis, 1992) as follows: (a) Successful Strategies (Relational and Multistructural Levels of Solo): (i) The child uses multiplication of the tree dimensions in a meaningful way (in the sense that they are able to provide an adequate explanation), or a layer strategy (addition or multiplication of layers, columns or rows) or counts the visible cubes of a construction and then adds the invisible. (b) Unsuccessful Strategies (Unistructural Solo Level): (ii) The child counts visible and invisible cubes in an organised but structurally incorrect manner and fails to include the correct number of invisible cubes, (iii) The child counts area (i.e. squares and not cubes) on the visible faces of the rectangular construction producing an incorrect response.

The responses of three specific children were discussed for the purposes of this presentation because they appeared to be inconsistent in the methods they used to approach different volume measurement tasks throughout the testing sequence. It was observed that while they used unsuccessful strategies (b) (i) and (ii) as described above at the beginning of the test (Question1 below), towards the end of the test they were using successful strategies (a) as above. It was concluded that a form of learning has taken place through the testing sequence to produce a shift from an incorrect response to a correct one at the final task. We look at their responses in two consecutive tasks of conservation and measurement of volume (Questions 5 and 6) and argue that there was a transition from one level of understanding the volume concept to a higher one involving realization of the structural organization of cubes into layers, columns or rows leading to the use of consecutive addition or multiplication to produce the final correct response.

Children's Responses

Question 1

Look at this construction. (The interviewer shows a physical 3x4x5 inches construction as in Fig.1.) Can you calculate the number of cubes that make it?

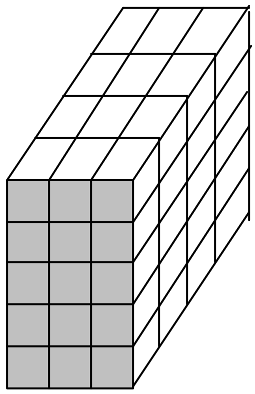


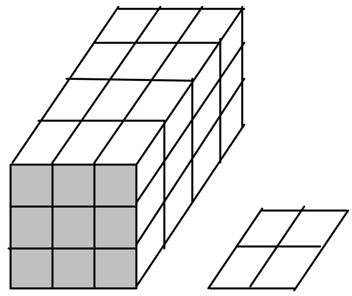
Figure 1
3x4x5 physical construction

Child A	Child B	Child C
M: I think 30 cubes. **Int:** How did you calculate it? **M.:** I counted the cubes on the sides. **Int.:** Show me. **M.:** 4 times 5 is 20 and 3 times 5, 15. I was wrong, the cubes are 35. **Int.:** So did I use 35 cubes to make this construction? **M.:** Yes. **Int.:** O.K.	**K.:**150 cubes. **Int.:** How did you calculate it? **K.:**There are 4 and 5 cubes on this side (showing the side face). And 20 here (showing between the first and second layer) and 20 here (showing between the second and third layer) and 20 here (showing the other side face). So 80 until now. And 3 times 15 on this side (showing the front face) and 15 here (showing the face between the first and second layer) and 15 here, (showing the face between the second and third layer) and 15 here (showing the face between the third and forth layer) and 15 here (showing the back face). So 75 for these sides. 75 plus 80, 155. I did a small mistake on the addition before. All the cubes are 155. **Int.:** O.K. So were 155 used to make the construction? **K.:**Yes. **Int.:** O.K.	**G.:**I cannot make it. **Int.:**We want to know how many cubes were used to make the construction. **G.:**55 cubes. **Int.:**How did you calculate it? **G.:**There are 20 cubes on each side of the construction, 20 plus 20, 40. And there are 5 here (showing the middle column of the front face) and 5 on the back (showing the middle column of the back face), 50. And there are 2 on the top that I did not count. So 52 not 55. **Int.:**So does this tells us how many are all the cubes that make the construction? **G.:**Yes. **Int.:**O.K.

In this task Child A does not seem to realise volume as space feeling or the unit cube as unit of measurement of volume and calculates number of squares on the top and the side of the block. Child B tries to include invisible cubes but again being preoccupied by the visible aspects of the arrangement it measures number of faces instead of unit cubes. Child C actually measures cubes and avoids to double count the cubes in the corners, but fails to grasp the structural organization of the arrangement it does not account for the invisible cubes.

Question 5

<u>This is a house</u> (showing the construction with dimensions 3x4x3 inches, Fig.2).<u>The house is build on an island.</u>(She puts the construction on a white card base separated by lines in to 3x4 square inches then onto a blue cardboard sheet representing the sea). <u>But the inhabitants of the house have to leave it. So they decide to build a new one on another island. This island here.</u>(She shows another white card base separated by lines into 2x2 square inches (Fig.2) and places it on the blue cardboard). <u>They want their new house to have as much room as their old one. What will the new house look like?</u>



House on the island

Figure 2

First we should notice that it was necessary for the interviewer to provide the explanation that each of the inhabitants had a room of his own in the old house and wanted to have a room of his own in the new house as well. This explanation was not provided to children that immediately responded - after they were presented with the task - that the new house would be taller. The three children whose answers we are presenting, initially responded that either they did not understand or that the height of the new house would be the same. After they were given the explanation they seemed to realise the internal structure of the block and actually state that the room in the new house would be the same and that if one dimension decreases the other would have to increase.

Child A although, still not able to measure the number of unit cubes that make the new and the old house she clearly states that the new house would have to be taller and makes a rough estimation of the height of the new house.

Child B uses two different ways to calculate the number of cubes of a block. For the block she has build she uses a layer strategy but for the old ready made construction she counts faces of cubes visible and invisible as with the first task. When she is asked if both ways are correct she seems to realise that she has been counting faces rather than cubes and changes here approach to a layer strategy.

Child C is still measuring only visible cubes of the ready made block but she thinks of a rearrangement of the cubes on the new smaller island and this way she manages to calculate the correct height. This shows that she realised that there are cubes inside the construction, but when it comes to measuring them she is distracted by the visible aspects of the block.

Child A	Child B	Child C
M.: We will have to put only 4 on each row and it will be until here. (She shows the old house's height). **Int.:** Will they have the same room then? **M.:** I think so. **Int.:** O.K. Would you like to try building it? Maria takes cubes and starts arranging them into layers of 4. She stops after she stuck 3 layers. **Int.:** So do you think that they have the same room now? **M.:** No. They have more room in the old house. **Int.:** How can they manage to have the same room? **M.:** They will have to put more floors. **Int.:** How may floors will the new house have? **M.:** The new house will be about this tall. (She shows about the double height of the old house) **Int.:** Can you tell me how many cubes tall it will be? **M.:** I do not know. **Int.:** O.K. Would you like to complete your building? Maria stucks three more layers so that the height is now 6 cubes and stops. **M.:** Here it is. **Int.:** So does this one have the same room as the old one? **M.:** Yes. **Int.:** How do you know? **M.:** Because the island is smaller now, so the new house has to go taller in order to have the same room. Int.: O.K., but how can you be sure about the height? **M.:** I do not know. It looks all right to me. **Int.:** Well, each one of these cubes is a room. Every inhabitant had a room of his own in the old house. They want to have a room of their own in the new house as well. **M.:** I think they will have a room of their own. **Int.:** Why do you think so? **M.:** Because the room in the house is the same. **Int.:** O.K.	**K.:** Mmmm, I do not know, I do not understand very well this question. **Int.:** Well, each one of these cubes is a room. Every inhabitant had a room of his own in the old house. They want to have a room of their own in the new house as well. **K.:** **Oh, now I understand. They could build the new house with the same number of cubes as the old one.** If we could use the cubes from the old house we would take these here (showing 4 cubes on the top face of the construction) and put them on the island. Then we would take 4 more and put them on the top of them. And then I would put the remaining on their top. **Int.:** Do you know how tall the new house would be? **K.:** Do you mean how many cubes would it have in height? **Int.:** Yes. **K.:** It will be about double the old house. But then some cubes remain. So 6 for these 8 columns. And then one more for the remaining. So it will be 7 cubes tall. **Int.:** Would you like to build the new house? **K.:** Yes. Korina builds a 2x2x7 construction. **K.:** This house has 24 cubes. **Int.:** How do you count them? **K.:** I put 4 on each layer. There are 4 layers so 7 times 4, 28. **Int.:** So does everybody have a room of their own now? **K.:** I do not know. It seems that the old house has more. **Int.:** How many rooms are there in the old house? **K.:** There are 12 on the front and 12 , 24 and 24 for the sides between them, 48, and 9 and 9 and 9 and 9 and 9 , 45. It is like the one I counted before. 48 plus 45, 93 cubes. **Int.:** To measure the cubes of the new house you are using another way. Are both ways correct or is one of them wrong? **K.:** I think that I am counting the same cubes twice if I count the cubes inside the construction. **Int.:** O.K. Would you like to try and calculate the number of cubes again? **K.:** There are 12 on the front. So 12 times 3 is 36. So I need to put 36 cubes on the new house and I have only 28. I need to put two more layers. **Int.:** O.K. So how tall will the new house be? **K.:** It will have 7 and 2, 9 layers in total. **Int.:** Well done.	**G.:** Can they build the new house smaller? **Int.:** What they want is the new house to have the same room as their old one. **G.:** I do not know. There is a problem here. **Int.:** What is the problem? **G.:** How are they going to build such a big house on a small island. The base of the old house is 12 and the new island has only 4 squares. Can they enlarge the island? **Int.:** No they cannot. **G.:** I do not know. **Int.:** Well, each one of these cubes is a room. Every inhabitant had a room of his own in the old house. They want to have a room of their own in the new house as well. **G.:** Can they make the house bigger? **Int.:** What do you mean bigger? **G.:** They can build it taller. **Int.:** Yes they can build it taller. Can you find out how tall the new house will be? **G.:** I need to count the cubes of the old house and then use the same number of cubes for the new house. There are 4, 8, 12 on the side face. And 12 and 12 and 6, 30. **Int.:** Why are you adding 12 and 12 and 6? **G.:** Because there are 12 on the side, 12 in the middle and 3 on each side that were not counted. **Int.:** OK, so can you now calculate how tall is the new house going to be? **G.:** About this tall. (showing with her hand about three times the height of the old house) **Int.:** Good. How many cubes will it have along the height? **G.:** 9 cubes. **Int.:** How did you calculate it? **G.:** If we take a piece of the old house and put it in the new island, and another piece and another piece then the new house will be about three times taller than the old house. **Int.:** Good.

Question 6

Now look at these two boxes. Can you tell me if the two boxes have the same capacity or if one of them has bigger capacity than the other? (She shows the two boxes with dimensions 3x4x5 inches and 6x5x2 inches Fig. 3).

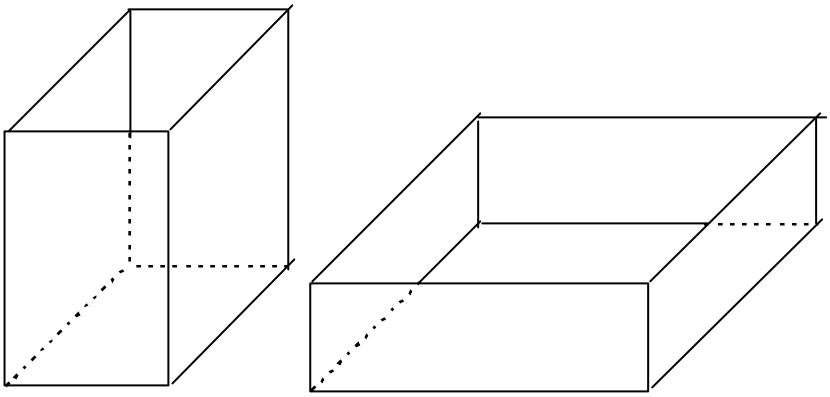


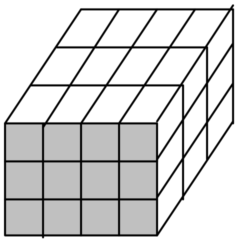
Figure 3
3x4x5 and 6x5x2 empty boxes

Child A	Child B	Child C
M.: I think that this box has bigger. (showing the 6x5x2 inches box) **Int.:** Why do you think so? **M.:** It looks bigger. **Int.:** Can you be sure about it? **M.:** Shall I count with cubes? **Int.:** If you want. Maria fill in the bottom of the 6x5x2 inches with 30 cubes. **M.:** So 60 cubes can fit in this box. **Int.:** Why do you say so? **M.: Because I have put 30 until now, and 30 more can fit on the top. so 30 plus 30, 60 in total .** **Int.:** How about the other box? Maria fills in the bottom of the box with 12 cubes. **M.:** I think that it will take 36 cubes to fill this box. **Int.:** Why do you think so? **M.:** Because if we put 12 and 12 on the top, and another twelve, it will be full. **Int.:** Are you sure that 12 and 12 and 12 cubes will fill the box completely? Maria takes 5 cubes and puts them along the height on the outside of the box. **M.: They are going to be 12 times 5.** So 60 cubes as well. **Int.:** Good. **M.:** So they have the same capacity. **Int.:** Good.	**K.:** I think that they have equal capacity. **Int.:** Why do you think so? **K.: Because the pink box is more open. If we put the blue box in the pink about half the box will remain out. But the blue box is much taller. So it must be about the same.** **Int.:** How can you make sure? Korina takes cubes and puts them along the dimensions outside of the 3x4x5 cm box. **K.:** This box can take 3 times 4, 12. 12 times 5, 60 cubes. Korina builds a wall of 6x2 cm on the outside of the pink box and puts 5 cubes along the its width. **K.:** This box can take 12 times 5, that is 60. So the capacity is 60. It is equal. **Int.:** Good.	**G.:** Can I use cubes to measure? **Int.:** Yes you can. Georgia puts some cubes on the bottom of the 6x5x2 cm box. **G.:** The pink box can take 30 on the bottom so 60 cubes in total. She puts 12 cubes on the bottom of the 4x5x6 cm box and 5 along the height on the outside of the box. **Int.:** 12 times 5 is 60. So 60 cubes can fit in her as well. **G.:** So what do you have to say about the capacity of the boxes. **Int.:** It is the same. **G.:** O.K.

In the last task the children have the chance to handle the cubes and arrange them into the empty boxes. This gives them the opportunity to construct a mental model of a set of cubes as organised arrays into layers, columns or rows and use multiplication or sequential addition to obtain the correct number of cubes.

The responses of the children to tasks on calculation of volume included in the written part indicated that some form of learning has taken place during the interview tasks. The children were asked to provide a numerical answer and also provide an explanation of how they obtained this answer in words or numbers. For example Question 2(a) of the written part and the responses of the children were as follows:

(a) How many cubes make the block (there are no gaps inside)?



The block is made out of **36** cubes.

Child A 12+12+12=36

Child B 12x3=36

Child C 12+12+12=36

Conclusions

The responses presented seem to support the finding of Battista and Clements (1996) that students initially conceive a 3-D rectangular array of cubes as an uncoordinated set of faces and gradually move to a conceptualization of the set of cubes in terms of rectangular arrays organised into layers and therefore, become able to measure the volume of a block made out of unit cubes correctly. Handling the physical material during the two consecutive tasks involving enumeration of number of cubes, conservation of volume and calculation of the capacity of empty boxes seemed to facilitate the "structuring" (realization of the structural arrangement of cubes in the rectangular constructions). As they reflected on experience of counting or building cube configurations, they gradually became capable of co-ordinating the separate views of the arrays. This was followed by "integration" (construction of one coherent and global model of the array) and correct calculation of the number of cubes in the rectangular arrays. These tasks appeared to have helped the children to construct the structure of volume and coordinate perspectives of it. By covering the bottom layer of the empty box with cubes helped them to construct a mental model of how the cubes are organized into arrays and thus they were able to repeat this method to answer to the tasks of the second part of the test.

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