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PUPILS' CONCEPTIONS ABOUT AN OPEN HISTORICAL QUESTION: GOLDBACH'S CONJECTURE. THE IMPROVEMENT OF MATHEMATICAL EDUCATION FROM A HISTORICAL VIEWPOINT.

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Introduction

Pour un esprit scientifique, toute connaissance est réponse à une question. S'il n'y a pas eu de question, il ne pout y avoir de connaissance scientifique. Rien ne va de soi. Rien n'est donné. Tout est construit.

[*For the scientific mind, all knowledge is a response to a question. If there had not been any questions, it would not have been possible to have scientific knowledge. Nothing comes of itself. Nothing is given. Everything is constructed*.]

G. Bachelard, *La formation de l'esprit scientifique*, 5th ed. Paris, 1967, p. 14.

Undoubtedly, the intersection of mathematics teaching and the history of mathematics has had a long, fruitful tradition; so, nowadays one of the most frequent questions is: in which way may the history of mathematics influence mathematics education? As E. Barbin has pointed out recently (Barbin, 1996):

> Maths teachers who at some point in their career become interested in the history of their subject often report that the understanding they gain influences their teaching or, at the very least, the way in which they perceive mathematics education. These teachers may or may not choose to introduce an historical perspective into their teaching, and they may or may not give their pupils historical texts to read. Nevertheless they say that they have a different view about the errors their pupils make, and they have a better interpretation of certain remarks their pupils make, and are better able to respond to them. They also pay attention to the different stages that have to be passed through in acquiring mathematical knowledge and, in particular, to those obstacles that must be overcome on the way.

If it is true that most knowledge is a response to a question, it is as true that the history of mathematics shows that mathematical concepts are constructed, modified, and extended in order to solve problems, so an alternative way of writing a history of mathematics is that of a history of problem solving.

The pedagogical value of open problems and conjectures for mathematics teaching is in general remarkable, expecially in the educational methodology of problem-solving.

In fact, such a methodology is important above all for the following reasons:
- it allows pupils to use their acquired knowledge to solve problems;
- it improves their logical-deductive abilities;
- it contributes in consolidating knowledge already mastered in a consistent fashion;

Moreover it allows :
- the acquisition of a scientific approach in facing mathematical problems;
- the working out of personal strategies in modeling;
- encourages teamwork;
- forms an antidogmatic mentality by which one can always move ahead.

Thus the aim of this paper is that of analyzing the educational value of mathematical conjectures to improve some pupil's abilities when confronting unsolved questions.

Through facing a conjecture a pupil may be stimulated in acquiring his own ways of reasoning by either following his particular mathematical background or individual intuitive approach in order to solve a question.

We are interested in the following kind of conjecture according to Balacheff (Balacheff, 1994):

a conjecture is a statement strictly connected with an argumentation and a set of conceptions wherein the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it.

The relationship between argumentation and proof, strictly connected to the relantionship between conjecture and valid statement, has been recently analized (Pedemonte, 2000) supposing that, during a solving process, which leads to a theorem, an argumentation activity is developed in order to produce a conjecture.

Instead, in the present case we want to analyze the gradual passage of pupils' attempts from an argumentation to a proof, while they are facing a known unsolved conjecture. We have choosen a historical conjecture like Goldbach's one essentially for the simplicity of its statement and its fascinating empirical evidence.

Goldbach's conjecture states that:

“Every even number greater than 2 can be represented as the sum of two primes.”

This conjecture belongs to number theory which has a greater number of conjectures than other mathematical fields. On this subject, in the sixth Josiah Willard Gibbs Lecture presented in New York in 1928, the eminent mathematician G.H. Hardy (1877-1947), who was one of the XXXth century's most famous number-theorist, said (Hardy, 1929):

> The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subjectmatter is tangible and familiar, the processes of reasoning which it employs are simple, general and few, and it is unique among the other sciences in its appeal to natural human curiosity. A month's intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and at least ten times as entertaining as the same account of “calculus for engineers.”

In the same way, H: Davenport (1907-1969) wrote (Davenport, 1983):

> It [number theory] certainly has very few direct applications to other sciences, but it has one feature in common with them, namely the inspiration which it derives from experiment, which takes the form of testing poccible general theorems by numerical examples.

So, Goldbach's conjecture seems to be useful in order to point out the following points:

- pupils' conceptions in relation to a conjecture faced during the historical development of mathematics;

- pupils' attempts proving a conjecture reclaimed from history and compared with their argumentative processes;

- to what extent the history of mathematics can favour the study of pupils' conceptions about arguing, conjecturing and proving;

- their reaction to a conjecture's terms seemingly simple to solve;

- their approach in the solving of a conjecture;

- their abilities in carrying out non-standard solving strategies (lateral thinking);

As we know a conjecture can be transformed into a theorem if a proof justifying it is produced; namely, if it is possible to use a mathematical theory allowing the construction of a proof of it.

The basic reason why we have decided to propose an unsolved conjecture like Goldbach's one is, as we have said, to emphasise the role of problems in the historical development of mathematical knowledge. As we know a branch of mathematics maintains mathematicians’ interest alive as long as there are always new problems to be solved, because it is only in this way that mathematical knowledge can progress, giving new lymph for the growth of other collateral branches.

It is impossibile to do mathematics without asking oneself problems and trying to solve them; or rather the main activity of a mathematician is the solving of problems posed by others or which he puts himself, according to his own tastes and choices. It is in this manner that one can encounter with a really important theorem which enlightens an entire branch of mathematics and through which other trunds of search trends are set in motion.

Doubtless, there are really a lot of open problems and unsolved conjectures in number theory, and their number grows yearly, giving continous inspiration to mathematicians.

Such a paradigmatic example, even though exceptional, is the scientific production of Paul Erdös (1913-1996), one of the keenest mathematical minds of our times. Voluntarily stateless, he crossed the whole world continuously suggesting problems to be solved or solving problems suggested to him. The contributions made by Erdös to mathematics were numerous and diverse. The problems which attracted him most were problems in combinatorics, graph theory, and number theory. However he did not just want to solve problems, he wanted to solve them in an elegant and simple fashion. According to Erdös a proof had to provide insight into the reason by which the result was true, but it had not only to be a complicated sequence of steps constituting a formal proof without any understanding. His mathematical creativity is a perfect counter-example to the statement according to which doing mathematics is an activity only for young men. In the past other counter-examples were H. Poincaré (1854-1912), D. Hilbert (1862-1942), and J. Von Neumann (1903-1957); and in past times L. Euler (1707-1783), J.L. Lagrange (1736-1813), C.F. Gauss (1777-1855) and A.L. Cauchy (1789-1857).

Unsolved problems such as conjectures are the source of extraordinary developments for mathematics. One could give many examples, but to be brief it is enough to remember the case of E. E. Kummer (1810-1893) who trying to prove Fermat's Last Theorem, introduced the so-called *ideal numbers.*

The following list suggests an educational activity for other experiments in the classroom.

1] The Odd Goldbach Problem: Every odd n > 5 is the sum of three primes.

There has been substantial progress on this, the easier case of Goldbach's conjecture. In 1937 Vinogradov proved that this is true for *sufficiently* large odd integers *n* (but without specifying the term "sufficient"). In 1956 Borodzin showed $n > 3^{14348907}$ is sufficient (the exponent is 3^{15}). In 1989 Chen and Wang reduced this bound to 10^{43000}. The exponent must be reduced dramatically before we can use computers to take care of all the small cases.

2] Is every even number the difference of two primes in infinitely many ways?

For example: 12=19-7=29-17=23-11. Chen's work mentioned in the discussion of the Goldbach conjecture also showed that every even number is the difference between a prime and a P_2, that is a number with no more than two prime factors

3] For every even number 2*n* are there infinitely many pairs of *consecutive* primes which differ by 2*n*?

It was conjectured by Polignac in 1849, and it is clear that if $n = 1$ this is the *twin primes conjecture*. It is easy to demonstrate that for every positive integer *m* there is an even number 2*n* such that there are more than *m* pairs of consecutive primes with difference 2*n*.

4] Twin Primes Conjecture: Are there infinitely many twin primes?

This conjecture can be formulated also in another way: are there infinitely many integers *n* such that *n*-1 and *n*+1 are both primes? Today we know almost 100.000 twin primes for *n* = 4, 6, 12, 18, 30, … 1.000.000.000.062, 1.000.000.000.332, …, 140.737.488.353.508, 1.140.737.488.353.700, … and it is probably true that the following conjecture is even stronger: if *z*(N) is the number of the pairs of twin primes, $n - 1$ and $n + 1$, $5 \le n + 1 \le N$, then:

$$z(N) \approx 1,3203236 \int_2^N \frac{dn}{(\log n)^2}$$

The constant term is not empirical but it is given by an infinite product:

$$1,3203236... = 2 \prod_3^\infty \left\{ 1 - \frac{1}{(p-1)^2} \right\}$$

taken over all odd primes.

A partially known result was demonstrated in 1919 by Viggo Brun in a famous paper[1] that introduced the *sieve method*: the sum of reciprocals of twin primes converges, and so the sum B = (1/3 + 1/5) + (1/5 + 1/7) + (1/11 + 1/13) + (1/17 + 1/19) + ... is called *Brun's* constant, and according to the valuation made by Richard P. Brent[2] its value is:

$$B = 1.9021604 \pm 5 \cdot 10^{-7}$$

5] Are there infinitely many primes of the form n^2+1?

Some examples of such numbers are: 1^2+1, 2^2+1, 4^2+1, 6^2+1, 10^2+1. There are infinitely many numbers of the forms n^2+m^2 and n^2+m^2+1. A more general form of this conjecture is: if *a, b, c* are relatively prime, *a* is positive, $a+b$ and *c* cannot be both even, and $b^2- 4ac$ is not a perfect square, so there are infinitely many primes of the form an^2+bn+c.

6] Is the number of Fermat primes finite?

Hardy and Wright, in their well known footnote[3] in their classic book, *An Introduction to the Theory of Numbers*, argue in favour of this conjecture which is roughly as follows. By the prime number theorem the probability that a random number *n* is prime is at most $a/\log(n)$ for

some choice of *a*. So the expected number of Fermat primes is at most

[1] V. Brun, *La série 1/5+1/7+1/11+1/13+1/17+1/19+1/29+1/31+1/41+1/43+1/59+1/61+ …, où les dénominaturs sont 'primes jumeaux' est convergente ou finie*, Bull. Sci. Math., 43, 1919, pp. 101-104, 124-128.

[2] R.P. Brent, *Irregularities in the distribution of primes and twin primes*, Math. Comp., 29, 1975, pp. 43-56.

[3] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, Oxford, Clarendon Press, 5th ed., 1979, p. 15.

$$A \sum \frac{1}{\log\left(2^{2^n}+1\right)} < A \sum \frac{1}{2^n} < A$$

However, as Hardy and Wright noted, the Fermat numbers do not behave "randomly" in that they are relatively prime pairwise .

7] Is there always a prime between n^2 and $(n+1)^2$?

For example, if we choose $n = 5$, then between $5^2 = 25$ and $(5+1)^2 = 36$ there are two prime numbers: 29 and 31; but is it always true, whenever one can choose n?

8] Are there infinitely long arithmetic progressions of consecutive primes?

For example,the progression 251, 257, 263, 269 has length 4. The largest example known has length 7.

9] Are there infinitely many primes of the form *n*# + 1?

(where *n*# is the product of all primes *n*.)

10] Are there infinitely many primes of the form *n*! – 1?

Some examples of primes of this form are: 5 = 3! - 1, 23 = 4! - 1, 119 = 5! - 1.

11] Does the Fibonacci sequence contain an infinite number of primes?

As one knows, the Fibonacci sequence is:

1, 1, **2**, **3**, **5**, 8, **13**, 21, 34, 55, **89**, 144, **233**, **377**, 610, …

and we quickly note some prime numbers (in bold), but we do not know anything about the way to prove their infinity or not.

12] If p is prime, $(2^p – 1)$ isn't always indivisible by the square of a prime?

13] Does a prime at least exist among triangular numbers or consecutive squares?

Chapter One

1.1 History of Goldbach's conjecture

There are many unsolved problems in mathematics, but one can be sure that some of the most famous belong to number theory, essentially for the simplicity of their statements, difficulty in confrontation, and for intrinsic interest.

Conjectures and open problems are important for the development of mathematics, because they keep the searchers's interest going, by inciting them to turn to other ways as regards those already known, so that they can often make discoveries not only shedding light on initial problem, but also producing further mathematical problems.

Number theory is a mathematical field full of conjectures, some of which obstinately refuse any attempt at proof, both in a positive sense and a negative one, so that they, if only for their pecularity, have rightly become famous.

So, one can say that the so-called *Goldbach's conjecture*, among all the open number theoretic questions, is not surpassed by anyone else, because it is unproved to this day, for almost three centuries.

It is true that Riemann's hyphothesis on non-trivial zeros of the so-called zeta-function is another celebrated number theoretic conjecture, but it has not such a genuine impact as that of Goldbach's, because, in order to understand it, it is necessary to enter a more specialistic field of number theory, i.e. the analytic one.

Briefly, Goldbach's conjecture states that every even number is the sum of two primes, but, before giving modern terms of it, it will be useful to sketch some aspects of Goldbach's life.

Goldbach was born in Königsberg in 1690; a man of versatile tastes and talents, obviously well educated and well-to-do, he travelled extensively in his younger days, seeking out learned men and scientists everywhere.

He was a minister's son who studied law at university but cultivated wide-ranging interests in many other fields, most importantly in languages and mathematics, particularly in the theory of numbers, differential calculus and series theory. He formed acquaintances with many of the leading thinkers of his time, including Nicolas Bernoulli (1687-1748), his younger brother Daniel (1700-1782), Euler (1707-1785), and Leibniz (1646-1716), whom he met during his extended travels in Europe, and he maintained an active correspondence that lasted through his lifetime, especially with Euler. He seems to have been regarded by the Bernoullis, and also by Euler, as an influential friend and patron.

Even though it is probable that Goldbach started on his journey to Russia out of mere curiosity, he stayed there until his death, living at times in Petersburg and at times in Moscow. His knowledge and his papers gave him the opportunity to serve as Secretary of the Academy of Sciences in St Petersburg, and in 1742 he was also in charge of decode dispatches at the Ministry of Foreign Affairs. He was an invaluable corrrespondent for Euler from 1729 until his death, in 1764. At first, the correspondence was conducted enterely in Latin and continued so even after Goldbach's return to Petersburg in 1732, at a time when he and Euler must have seen each other almost daily. Then around 1740 they dropped into German, and when Euler, in the following year, left for Berlin, their correspondence continued generally in German, but with a sprinkling of French words tending to revert into Latin, especially in the mathematical passages.

The conjecture was first formulated in a letter, dated Moscow, June 7, 1742 from Goldbach to Euler but it was not published[4], however, until 1843.

Perhaps, by chance, perhaps by the passionate reading of Fermat's number theoretic notes, Goldbach had been attracted by the apparent regularity of the following partitions:

6 = 3 + 3 8 = 3 + 5 10 = 5 + 5 12 = 5 + 7…

It seemed that *every even number not lesser than 6 could be expressed as a sum of two odd prime numbers*.

Euler claimed to be convinced of the truth of the conjecture, and so all it could do was adjust the proof, or a strategy for it.

For example, few tests are hardly sufficient to deduce that the choice of a pair of primes by which a given even number is partioned is not generally unique:

6 = 3 + 3 without any alternative

8 = 3 + 5 " "

[4] The letter was published for the first time by P.H. Fuss, *Correspondance mathématique et physique de quelques cèlébres gèométres du XVIIIème siècle*, tome I, St. Pétersbourg, 1843. However, the correspondence between Euler and Goldbach has been quite published by A.P. Juskevich and E. Winter, *Leonhard Euler und Christian Goldbach*, Berlin, Akademie-Verlag, 1965.

10 = 3 + 7 but also: 10 = 5 + 5
12 = 5 + 7 without any alternative
14 = 3 + 11 but also: 14 = 7 + 7
16 = 3 + 13 but also: 16 = 5 + 11
18 = 5 + 13 but also: 18 = 7 + 11
20 = 3 + 17 but also: 20 = 7 + 13
22 = 3 + 19 but also: 22 = 11 + 11 and 22 = 5 + 17
24 = 5 + 19 but also: 24 = 7 + 17 and 24 = 11 + 13
26 = 3 + 23 but also: 26 = 7 + 19 and 26 = 13 + 13
28 = 5 + 23 but also: 28 = 11 + 17
30 = 7 + 23 but also: 30 = 11 + 19 and 30 = 13 + 17

Moreover, if we consider some more larger even numbers, the possible choices seem to increase. Why? According to which law? Is there some regularity in this? One can suppose that Goldbach, during his countless proof attempts,could work on tables of the following type:

n	**3+...**	**5+…**	**7+...**	**11+...**	**13+...**	**17+...**	**19+…**	**.23+...**
6	3+3							
8	3+5	5+3						
10	3+7	5+5	7+3					
12		5+7	7+5					
14	3+11		7+7	11+3				
16	3+13	5+11		11+5	13+3			
18		5+13	7+11	11+7	13+5			
20	3+17		7+13		13+7	17+3		
22	3+19	5+17		11+11		17+5	19+3	
24		5+19	7+17	11+13	13+11	17+7	19+5	
26	3+23		7+19		13+13		19+7	23+3
28		5+23		11+17		17+11		23+5
30			7+23	11+19	13+17	17+13	19+11	23+7
32	3+29				13+19		19+13	

....

What could have Goldbach thought? Simply that in the above table every row has to be occupied at least by a sum of prime numbers. So, it was necessary to understand how prime numbers are distributed among odd numbers. Undoubtedly, Goldbach will have wondered whether his great friend Euler would have wanted to dedicate some time to that simple question. Then Goldbach wrote to Euler[5]:

I do not believe it useless also to pay attention to those propositions which are very likely, although there does not exist a real demonstration. Even in case they turn out at a later time to be false, yet they may have given occasion for the discovery of a new truth. The idea of Fermat, that every number $2^{2^{n-1}} + 1$ gives a sequence of prime numbers, cannot be correct, as you have already shown[6]- but it would be a remarkable fact if the series were to give only numbers which can be divided into two squares in only one way. Similarly, I also shall hazard a conjecture: that every number which is composed of two prime numbers is an aggregate of as many numbers as we like (including unity), till the combination of all unities [is reached].[7] [Goldbach added in the margin]: After rereading this I find that the conjecture can be demonstrated in full rigor for the case n+1 if it succeeds in the case for n and if n+1 can be divided into two prime numbers. The demonstration is very easy. It seems at any rate that every number greater than 2 is an aggregate of three prime numbers.[8] [The text of Goldbach's letter continues]: For example:

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5 See D.J. Struik, A Source Book in Mathematics, 1200-1800, Cambridge (Mass.), Harvard Univ. Press, 1969, pp. 47-48.

6 Goldbach refers to the fact that Euler had yet proved, in 1732, that for n=5 the Fermat number $F_5 = 2^{32}+1$ is not prime.

7 Goldbach points out that every number n which is a sum of two primes is a sum of as many primes as one wishes up to n. One can note that for Euler and Goldbach, as we have alreadyt said, 1 is a prime number.

8 This is the first formulation of Goldbach's conjecture. When one begins the series of primes from 2, the conjecture can be formulated as follows: every even number is the sum of two numbers that are either primes or 1. A more general formulation is that every even number >2 is the sum of two primes. Therefore every odd number >5 is the sum of three primes. For the history of the conjecture see L.E. Dickson, *History of the Theory of Numbers*, Carnegie Institution, Washington, D.C., 2nd ed., 1934, I, pp. 421-424; and R.C. Archibald, "Goldbach's theorem", *Scripta Mathematica* 3 (1935), pp. 44-50, 153-161.

$$4 = \begin{cases} 1+1+1+1 \\ 1+1+2 \\ 1+3 \end{cases} \qquad 5 = \begin{cases} 2+3 \\ 1+1+3 \\ 1+1+1+2 \\ 1+1+1+1+1 \end{cases} \qquad 6 = \begin{cases} 1+5 \\ 1+2+3 \\ 1+1+1+3 \\ 1+1+1+1+2 \\ 1+1+1+1+1+1 \end{cases}$$

Goldbach's letter to Euler June 7, 1742

Euler’s reply (from Berlin) to Goldbach, June 30, 1742, was no less interesting:

When all numbers included in this expression $2^{2^{n-1}} + 1$ can be divided into two squares in only one way, these numbers must also be prime, which is not the case, for all these numbers are contained in the form 4m+1, which, whenever it is prime, can certainly be resolved into two squares in only one way, but when 4m+1 is not prime, it is either not resolvable into two squares, or is resolvable in more ways than one. That $2^{32} + 1$, which is not prime, can be divided into two squares in at least two ways I can show in the following way: I. When *a* and *b* are resolvable into two squares, then also the product *ab* will be resolvable into two squares. II. If the product ab and one of the factors a were numbers resolvable into two squares, then also the other factor b would be resolvable into two squares. These theorems can be demonstrated rigorously. Now $2^{32} + 1$, which is divisible into two squares, namely 2^{32} and 1, is divisible by $641 = 25^2 + 4^2$. Hence the other factor, which I call b for short, must also be a sum of two squares. Let b = pp + qq, so that $2^{32} + 1 = (25^2 + 4^2)(pp + qq)$; then

$$2^{32} + 1 = (25p + 4q)^2 + (25q - 4p)^2$$

and at the same time

$$2^{32} + 1 = (25p - 4q)^2 + (25q + 4p)^2$$

hence $2^{32} + 1$ is divisible into a sum of two squares in at least two ways. From this, the double reduction can be found a-priori, since p = 2556 and q = 409, hence

$$2^{32} + 1 = 65536^2 + 1^2 = 622664^2 + 20449^2.$$

That every number which is resolvable into two prime numbers can be resolved into as many prime numbers as you like, can be illustrated and confirmed by an observation which you have formerly communicated to me, namely that every even number isa sum of two prime numbers. Indeed, let the proposed number n be even; then it is a sum of three, and also four prime numbers, and so on. If, however, n is an odd number, then it is certainly a sum of three prime numbers, since n-1 is a sum of two prime numbers, and can therefore also be resolved into as many primes as you like. However, that every number is a sum of t wo primes, I consider a theorem which is quite true, although I cannot demonstrate.

It seems Euler never attempted to demonstrate the conjecture, but in a letter in December 1752 he stated the further conjecture (probable suggested by Goldbach too) that every even number of the form 4n+2 was the sum of two prime numbers

of the form 4m+1. So, for example: 14=1+13; 22=5+17; 30=1+29=13+17. Independently of Goldbach, also the English mathematician Edward Waring (1734-1798), in his work *Meditationes Analyticae* (Cambridge, 1776), had stated the same conjecture in the following form: "*Every even number is the sum of two primes and every odd number is either prime or the sum of three prime numbers.*"
On the other hand, also René Descartes (1596-1650), although with some doubt[9], is said to have stated in a posthumous manuscript that every even number is a sum of 1, 2, or 3 primes. Even if the empirical evidence that has been accumulated in support of Goldbach's conjecture is overwhelming, there had been some dissent[10] on the validity of the conjecture; and in addition the great Indian mathematician Srinivasa Ramanujan (1887-1920) expressed some perplexity about the validity of the conjecture regarding very large even numbers.
Certainly, until a complete and rigorous proof is forthcoming, Goldbach's conjecture remains for the mathematicians a challenging, outstanding and open problem in the theory of numbers, and, as G.H. Hardy (1877-1847) said, during a lecture[11] given on October 6, 1921 to the Mathematical Society of Copenhagen, this famous problem is probably as difficult as any of the unsolved problems of mathematics.
Now we will consider the empirical evidence that has been accumulated in support of the conjecture.
In 1855, A. Desboves verified that every even number between 2 and 10000 is a sum of two primes in at least two ways. Moreover he stated that when this even number is the double of an odd number, it is at the same time a sum of two primes of the form 4k+1 and a sum of two primes of the form 4k+3.
Much more interesting however was a table prepared by G. Cantor (1845-1918) in 1894[12], in which he verified the conjecture up 1000. A few entries chosen for this table are reproduced below. Consider $n= 2N=x+y$, where x and y are primes such that $x \leq y$ and where n(n) denotes the number of such decomposition of n:

9 See R:G. Archibald, quoted, p.p. 46-47.
10 See F.J.E. Lionnet, *"Note sur la question 'Tout nombre pair est-il la somme de deux impairs premiers?' ", Nouvelles Annales de Mathématiques*, tome 18, 2° série, 1879, pp. 356-360.
11 See G.H. Hardy, Collected Works, Oxford,
12 See G. Cantor, *Vérification jusqu'a 1000 du théorème empirique de Goldbach*, Association française pour l'avancement des scieces, *Comptes rendus de la XXIII session,* Caen (1894), pp.117-134.

n=2N	x	ν(n)
10	3, 5	2
22	3, 5, 11	3
34	3, 5, 11, 17	4
40	3, 11, 17	3
78	5, 7 11, 17, 19, 31, 37	7
86	3, 7, 13, 19, 43	5
100	3, 11, 17, 29, 41, 47	6
1000	3, 17, 23, 29, 47, 53, 59, 71, 89, 113, 137,173, 179, 191, 227, 239, 257, 281, 317, 347, 353, 359, 383, 401, 431, 443, 479, 491	28

Thus, for example, 22=3+19=5+17=11+11; 40=3+37=11+29=17+23. As we have said, Cantor's table is interesting because it seems to indicate that not only Goldbach's conjecture is probably true but that the value of ν(n), except for the inevitable oscillations in value which are peculiar to arithmetic functions of this nature, indicates a continually increasing value with its increasing values of n. From 1896 to 1903 A. Aubry verified too Goldbach's conjecture for numbers from 1002 to 2000, and always in 1896 R. Haussner verified it up to 10000, by prepearing a set of tables. In his first table he followed the lines of Cantor's table by furnishing the number ν(n) of representations of every even number n up to 3000 as a sum of two primes, also giving the values of the smaller of the two primes involved in each representation. In the second table he gave the value of ν(n) for each value of n=2N lesser than 5000, and by employing this he was able to affirm, by additional computation, that Goldbach's conjecture was true for all even numbers up to 10000. But even if these tables were interesting because they furnished a fascinating empirical evidence for the truth of the conjecture, on the other hand these direct procedures were very undesirable and much too laborious in order to obtain extensive tables which could really suggest a way to demonstrate the conjecture. In fact, this direct method would mean the subtracting from every even number 2N of all the prime numbers $x \leq$ 2N and then by means of a table of primes determine which of the 2N-x differences are also prime numbers, until one could obtaine a 2N-x difference which was a prime, and so this meant 2N represented as a sum of two primes. However, it consumed a great deal

of time because of the fact that the 2N-*x* differences had to be done for *every* prime $x \leq$ N, whether the result incurred a prime or not, without considering moreover that the method was open to the objection that it gave no means of checking on errors which could very readily occur, except by direct reevaluation of the results. Nevertheless, Cantor as well as Haussner observed from their tables an apparent law regarding just the value of ν(2N). In fact, it appeared that for all 2N numbers which were divisible by 3, ν(2N) had a relative maximum with respect to the two preceding and the two following numbers; that is, the number ν(2N). seemed to depend essentially upon how many different odd prime factors the number N contained and not, generally speaking, upon how often a particular odd prime occurred as a 2N factor. It appeared a-priori to be impossible to determine an expression for ν(2N) in terms of 2N. Nevertheless if it could have been shown that ν(2N) was always positive for every N, Goldbach's conjecture would have been established. In order to get approximate results for the value of ν(2N), particularly for large values of N, considerable researche was done. Thus, J.J. Sylvester (1814-1897) stated[13] that the number of ways of representing a very large even number *n*=2N as a sum of two primes was approximately equal to the ratio of the square of the number of primes lesser than *n* to *n* itself, and hence was in a finite ratio to the quotient of *n* by the square of the natural logarithm of *n*. In a paper of 1897 Sylvester[14] expressed the hope of being able to prove the conjecture by an original method and stated a stricter conjecture, namely, that every even 2N number was a sum of two primes, one greater than N/2 and the other lesser than 3N/2, claiming that he had verified it for even numbers from 2 to 1000. In 1896 P. Stäckel denoted[15] by G_{2N} the total number of all decompositions of 2N into a sum of two primes, but without the restriction $x \leq y$.
Hence: $G_{2N} = 2\,\nu\,(2N) - \varepsilon$ where ε was 1 or 0 according to whether N was a prime or not. He observed that the value of G_{2N} depended upon the multiplicative structure of 2N but not upon the number of times a prime factor divides 2N.

[13] Unfortunately, this first paper was not published in the mathematical papers of Sylvester, and there is only an abstract, without proofs, in the *Proceedings of the London Mathematical Society*, v. 4, (1871-73), pp. 4-6.
[14] J. J. Sylvester, *On the Goldbach-Euler Theorem regarding prime numbers*, The Mathematical Papers, v. IV, Chelsea Publishing Company, New York, N.Y., pp. 734-737.
[15] P. Stäckel, *"Ueber Goldbach's empirisches Theorem: Jede grade Zahl kann als Summe von zwei Primzahlen dargestellt werden"*, Nachrichten vonder Königl. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, 1896, pp. 292-299.

Consequently, he defined P_k as the number of all odd primes from 1 to k (considering 1 a prime), obtaining approximations to G_{2N} for large values of N, in terms of Euler's ϕ-function, which gives the number of the numbers lesser than N and prime to N. However, he noted that his results did not agree with those of Sylvester. In 1900 the German mathematician E. Landau (1877-1938) considered Stäckel's approximation formula for G_{2N} and showed that, for large x values had the true approximation: but that, if one used Stäckel's approximation formula to form the sum, one did not obtain a result of the correct order of magnitude. Goldbach's conjecture was also verified by A. Cunningham[16] for all numbers of a special type up to 200 million, and he also gave a summary of the evidence for the conjecture. In particular, he used numbers of the form:2^n; $2^n \cdot w$; $(4\,w)^n$; $2 \cdot (2\,w)^n$; $(2\,w)^n$; $2^n \cdot (2^n \pm 1)$; $2 \cdot w^n$; $2 \cdot (2^n \pm w)$where w denoted a small number. A turning point in the approachs to Goldbach's conjecture was pointed out by Jean Merlin who in 1915 was the first to draw attention[17] to the fact that prime numbers pairs, each pair differing by 2, and the prime numbers used in the conjecture could be determined by a method analogous to the classic Sieve of Eratosthenes. Unfortunately, his general methods were not brought to fruition due to his untimely death. In 1919, employing Merlin's methods and the Sieve of Eratosthenes, Norwegian mathematician Viggo Brun (1885-1978) published[18] an important paper about the series formed by the reciprocals of the twin prime numbers, and in 1920 he proved[19] by elementary methods, namely without making use of the notion of an analytic function, that every "sufficiently" large even number could be represented as the sum of two numbers, each of which was a product of no more than 9 (equal or distinct) primes. This result was a great achievement and enabled him to show that there existed an infinity of pairs of numbers having a difference of 2, each number of the pair being a product of not more than 9 primes.Brun's method and his result were improved by several mathematicians, namely H. Rademacher, T. Estermann and G. Ricci. In fact, Hans

16 A. Cunningham, *"Evidence of Goldbach's Theorem", The Messenger of Mathematics*, New Series, v. 36 (1906), pp. 17-30.
17 The paper was entitled: "Un travail de Jean Merlin sur les nombres premiers", published by "J. H." in *Bulletin des sciences mathématiques*, (2), 1re partie, t. 39 (1915), pp. 121-136.
18 V. Brun, *"La série 1/5+1/7+1/11+1/13+1/17+1/19+1/29+1/31+1/41+1/43+1/59+1/61+ … où les dénominateurs sont 'nombres premiers jumeaux' est convergente ou finie"*, quoted.
19 V. Brun, *"Le crible d'Eratosthène et le théorème de Goldbach", Skrifter utgit av Videnskapsselskapet i Kristiania,* 1920, I. *Mathematik Naturvidenskabelig Klasse*, v. I, No. 3, pp. 1-36.

Rademacher (1892-1969), in 1923, obtained[20] a better result than Brun, by proving that every sufficiently large even number could be represented as a sum of two numbers each of which contained at most 7 prime factors, multiplicity being counted. But he also proved that there existed infinitely many pairs of odd numbers whose difference was 2, which were composed of at most 7 prime factors, and moreover none of these factors could be too small, because each prime factor of such a number might be greater than the eighth root of the given number. Rademacher's result was improved in 1932 by T. Estermann, who proved[21] that every sufficientely large even number was representable as a sum of two numbers each of which was a product of at most 6 (equal or distinct) primes. G. Ricci (1901-1973) in 1937, using Brun's method, proved[22] that every sufficiently large integer can be represented as a sum of at most 67 prime numbers. In a memorable lecture given on 6, October 1921 to the Mathematical Society of Copenhagen the great English mathematician G.H. Hardy said[23]:

> And the question that I wish to put to you is this: is it reasonable, in the present state of mathematical knowledge, to hope to obtain an elementary proof of Goldbach's theorem? If I reply to this question in the negative, as I must and shall, if I say that I am compelled to regard all such efforts as foredoomed to failure, I trust that you will not misunderstand me. I cannot believe that the methods of Merlin and Brun are sufficiently powerful or suifficiently profound to lead to a solution of the problem. But I am very far from meaning that I regard their work as devoid of interest and value. There is much in Brun's work in particular that seems to me very beautiful, and some of his theorems ought, I think, to find their way into every book on the theory of numbers.We have however to take acconut both of the histiory and the logical structure of our subject. Let us turn back then for a moment to its central theorem, the 'Primzahlsatz' or 'prime number theorem' expressed by tyhe equation 849. It seems plain that this must be at any

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[20] H. Rademacher, "Beiträge zu Viggo Brunschen Methode in der Zahlentheorie", *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität*, III_1 (1923), pp. 12-30.

[21] T. Estermann, *Eine neue Darstellung und neue Anwendungen der Viggo Brunschen Methode*, Journal für die reine und angewandte Mathematik, v. 168 (1932), pp. 106-116.

[22] G. Ricci, *Su la congettura di Goldbach e la costante di Schrinelmann*, Prima memoria, Annali della R. Scuola Normale Superiore di Pisa, serie II, vol. VI, 1937 (XV), pp.71-76; idem, *Su la congettura di Goldbach e la costante di Schrinelmann*, Seconda Memoria, Annali della R. Scuola Normale Superiore di Pisa, serie II, vol. VI, 1937 (XV), pp. 91-116.

[23] See G.H. Hardy, *Goldbach's Theorem*, Collected Papers of G.H. Hardy, v. I, Oxford at the Clarendon Press, 1966, pp.549.

rate an easier theorem than Goldbach's theorem. No elementary proof is known, and one may ask whether it is reasonable to expect one. Now we know that the theorem is roughly equivalent to a theorem about an anlytic function, the theorem that Riemann's Zeta-function has no zeros on a certain line. A proof of such a theorem, not dependent upon the ideas of the theory of functions, seems to me extraordinary unlikely.

In 1923, the same Hardy and Littlewood, assuming an unproved hyphothesis, proved[24] that every sufficiently large odd number (namely, from a certain point onwards) is the sum of three odd primes, and moreover they also obtained an asymptotic expression for the number of such representations. In 1930, a Russian mathematician L.G. Schnirelmann (1905-1938) proved[25] a theorem of existence, according to which every integer ≥2 is the sum of at most c primes.

After seven years, another Russian mathematician I.M. Vinogradov (1891-1983), showed that every *sufficiently large* odd integer can be written as the sum of at most three primes, and so every sufficiently large integer is the sum of at most four primes. One result of Vinogradov's work is that we know Goldbach's theorem holds for almost all even integers. However, Vinogradov was not able to define rigorously the concept of *sufficiently large numbers*, but in 1956, one of his students, K.V. Borodzin, has found that $33^{3^{15}}$ (a number having more than six million digits) is an upper bound. If all odd numbers lesser than $33^{3^{15}}$ were the sum of three primes, then the so-called *Goldbach's weak conjecture* would be proved.

In 1947, Atle Selberg produced another sieve method which leads to a more precise result than Brun's method in every known case, when it can be applied, and moreover, as Wang Yuan has pointed out, it is surprisingly simple.

After a series of important improvements on Brun's method and his result, in 1966 Chinese mathematician Chen Jing Run (1933-1996) established[26] that every large even number is the sum of a prime and a product of at most two primes.

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[24] G.H. Hardy, J.E. Littlewood, *Some problems of 'Partitio Numerorum' III: On the expression of a number as a sum of primes*, Acta Mathematica, v. 44 (1923), pp. 1-70.

[25] L.G. Schnirelmann, *Ob additiwnich swoistwach tschisel (Concerning additive properties of numbers)*, Iswestija Donskowo Polytechnitscheskovo Instituta (Nowotscherkask), v. 14, (1930), pp. 3-27.

[26] Chen Jing Run, *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Kexue Tongbao, 17, 1966, pp. 385-386.

Today, the list of mathematicians who have worked (and who are working) upon Goldbach's conjecture is considerable, but it seems that the most effective attempts of their search are based upon improvements of the so-called sieve method after Bombieri's[27] and Chen Jing Run's results.

Recently, in 1997, Jean-Marc Deshouillers, Yannik Saouter and Hermann J.J. te Riele have proved that the conjecture is true for all positive integers lesser than 10^{14}.

To end, Douglas R. Hofstadter was based on Goldbach's conjecture to sketch out a delightful dialogue between Achilles and the Turtle who are two characters of his classic book[28] *Gödel, Escher, Bach: an Eternal Golden Braid*.

The dialogue is in Chapter XII (Part II) of the book, and its characters are Achilles and the Turtle. The form of the dialogue is based *upon Goldberg's Variations* and its content concerns some problems of number theory as Goldbach's conjecture and Collatz's one. In brief, Hofstadter shows how there are many variations about the search in number theory, taking into consideration only its investigative range of natural numbers. So, there can be many variations, some of which led to finite investigations, others to infinite investigations and still others to investigations fluctuating between the finite and infinite ones.

So, Goldbach's conjecture leads to a finite investigation, because if one wants to verify if an even 2n number is a sum of two odd primes, the procedure for making such a verification will end surely, because primes have to be sought into the finite set of prime numbers lesser than 2n. Consider, on the other hand, the singular property of even numbers which Hofstadter calls *Turtle's property*, namely that an even number can be expressed as a difference of two odd primes:

$$2 = 5 - 3 = 7 - 5 = 13 - 11 = 19 - 17 = \ldots$$
$$4 = 7 - 3 = 11 - 7 = 17 - 13 = 23 - 19 = \ldots$$
$$6 = 11 - 5 = 13 - 7 = 17 - 11 = 19 - 13 = \ldots$$
$$8 = 11 - 3 = 13 - 5 = 19 - 11 = 31 - 23 = \ldots$$

Well, if we want to verify, given an even 2n number, with or without such a property, the procedure to be adopted is *potentially infinite*, because one has to extend the search upon all the infinite set of prime numbers.

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[27] E. Bombieri, *On the large sieve*, Mathematika, 12, 1965, pp. 201-225.
[28] D.R. Hofstadter, Gödel, Escher, Bach: an Eternal Golden Braid, Basi Books, Inc., 1979.

If an even number did not have the turtle's property, it would have -as Hofstadter says- *Achilles's property*, namely it could not be expressed as a difference of two odd prime numbers.

There is a third type of investigations, namely those which could be infinite or finite, as Collatz problem, which was posed by L. Collatz[29] in 1937, also called the 3*x*+1 mapping, 3*x*+1 problem, Hasse's algorithm, Kakutani's problem, Syracuse algorithm, Syracuse problem, and Ulam's problem. Consider the following function (called the "3x + 1" function), which takes positive integers to positive integers:

$$f(x) = \begin{cases} \dfrac{x}{2} & if\ x\, is even \\ 3x+1 & if\ x is\ odd \end{cases}$$

The problem asks what happens when *f* is applied repeatedly, starting with an arbitrary positive integer. So, given any number, it is very difficult to know in advance how much "one will have to get on" before arriving at the final 4-2-1 series. For example, if *x*=15, by applying the procedure, one gets the series:

15 - 46 -23 - 70 - 35 - 106 - 53 - 160 - 80 - 40 - 20 - 10 - 5 - 16 - 8 - 4 - 2 - 1

So, the greatest number achieved will be 160, but it is quite plausible to think that by choosing other numbers one can get on more and more without getting down, even if, however, this question has been tested and found to be true for all numbers $\leq 3 \cdot 2^{53}$. The members of the sequence produced by the Collatz problem are sometimes known as *hailstone numbers*, and because of the difficulty in solving this problem, P. Erdos commented that "mathematics is not yet ready for such problems".

As far as Goldberg's variations are concerned, Hofstadter refers to 30 variations written by J.S. Bach (1685-1750) for Saxon Lord Kaiserling and played by a young harpischordist named Goldberg, to whom they were attributed, because the lord wished to listen them every evening, since he suffered from insomnia.

A charming representation of Goldbach's conjecture can be obtained by a cartesian diagram, by representing the even numbers on the x- axis and on the y-axis the number of sums of two prime numbers into which an even number can be

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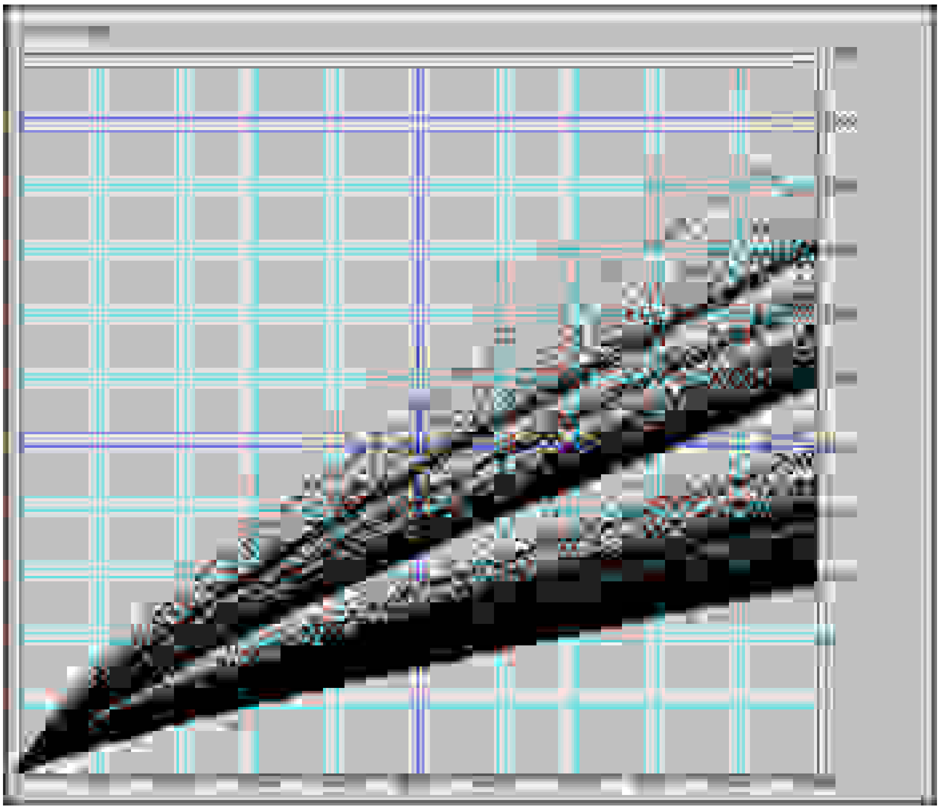
[29] See J. C. Lagarias, *"The Problem 3x+1 and Its Generalizations." Amer. Math. Monthly* **92**, 1985, pp. 3-23.

partitioned. Such a representation, with computer processing, for even numbers 2k, $4 \leq 2k \leq 100000$, produces a very fine form of a comet[30], named *Goldbach's comet.*



Goldbach's Comet

But surely, a way for proving Goldbach's conjecture is not that of a simple empirical verification but a demonstration of it, talking of which H.S. Vandiver (1882-1973) jested[31] that if he came back to life after death and was told that the problem had been solved he would immediately drop dead again. Futhermore, it seems today that a real advance towards a proof of this celebrated conjecture could be possible after having proved the other famous conjecture about the distribution of prime numbers, namely the so-called Riemann's hypothesis. In fact, as the great German mathematician D. Hilbert (1862-1943) pointed out in the 8th problem among his 23 famous Paris problems of 1900:

[30] See Jean-Paul Delahaye, *Merveilleux nombres premiers. Voyage au cœur de l'arithmétique*, Paris, Belin, 2000, p. 156.
[31] Quoted in D MacHale, *Comic Sections* (Dublin 1993).

After an exhaustive discussion of Riemann's prime number formula, perhaps we may sometime be in a position to attempt the rigorous solution to Goldbach's problem, viz., whether every integer is expressible as the sum of two prime numbers; and further to attack the well-known question, whether there are an infinite number of pairs of prime numbers with the difference 2, or even the more general problem, whether the linear diophantine equation $ax + by + c = 0$ (with given integral coefficients each prime to the others) is always solvable in prime numbers x and y.

1.2 Goldbach's generalized conjecture

We do not want to finish this historical survey on Goldbach's conjecture without quoting a recent generalization of it, namely, the so-called Smarandache conjectures on primes' summation concerning odd and even numbers. Other interesting experimentations could be made through these generalizations, but this will be a task for future works. The generalizations are the following:

Odd Numbers

A) Any odd integer *n* can be expressed as a combination of three primes as follows:

1) As a sum of two primes minus another prime:

$$n = p + q - r$$

where p, q, r are all prime numbers (the trivial solution is not to be included: $p = p + q - q$ when p is prime).

For example: 1 = 3 + 5 - 7 = 5 + 7 - 11 = 7 + 11 - 17 = 11 + 13 - 23 = ... ;

3 = 5 + 5 - 7 = 7 + 19 - 23 = 17 + 23 - 37 = ... ;

5 = 3 + 13 - 11 = ... ;

7 = 11 + 13 - 17 = ... ;

9 = 5 + 7 - 3 = ... ;

11 = 7 + 17 - 13 = ...

The following questions arise:

a) Is this conjecture equivalent with Goldbach's Conjecture (any odd integer ≥ 9 is the sum of three primes)?

b) Is the conjecture true when all three prime numbers are different?

c) In how many ways can each odd integer be expressed as above?

2) As a prime minus another prime and minus again another prime:

$$n = p - q - r$$

where *p*, *q*, *r* are all prime numbers.

For example: 1 = 13 - 5 - 7 = 17 - 5 - 11 = 19 - 5 - 13 = ... ;
3 = 13 - 3 - 7 = 23 - 7 - 13 = ... ;
5 = 13 - 3 - 5 = ... ;
7 = 17 - 3 - 7 = ... ;
9 = 17 - 3 - 5 = ... ;
11 = 19 - 3 - 5 = ...

a) Is this conjecture equivalent with Goldbach's Conjecture (any odd integer ≥ 9 is the sum of three primes)?

b) Is the conjecture true when all three prime numbers are different?

c) In how many ways can each odd integer be expressed as above?

B) Any odd integer n can be expressed as a combination of five primes as follows:

3) $n = p + q + r + t - u$, where *p*, *q*, *r*, *t*, *u* are all prime numbers, and $t \neq u$.

For example: 1 = 3 + 3 + 3 + 5 - 13 = 3 + 5 + 5 + 17 - 29 = ... ;

3 = 3 + 5 + 11 + 13 - 29 = ... ;

5 = 3 + 7 + 11 + 13 - 29 = ... ;

$7 = 5 + 7 + 11 + 13 - 29 = ...$;

$9 = 7 + 7 + 11 + 13 - 29 = ...$;

$11 = 5 + 7 + 11 + 17 - 29 = ...$.

a) Is the conjecture true when all five prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

4) $n = p + q + r - t - u$, where p, q, r, t, u are all prime numbers, and $t, u \neq p, q, r$.

For example: $1 = 3+7+17-13-13 = 3+7+23-13-19 = ...$;

$3 = 5+7+17-13-13 = ...$;

$5 = 7+7+17-13-13 = ...$;

$7 = 5+11+17-13-13 = ...$;

$9 = 7+11+17-13-13 = ...$;

$11 = 7+11+19-13-13 = ...$.

a) Is the conjecture true when all five prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

5) $n = p + q - r - t - u$, where p, q, r, t, u are all prime numbers, and $r, t, u \neq p, q$.

For example: $1 = 11 + 13 - 3 - 3 - 17 = ...$;

$3 = 13 + 13 - 3 - 3 - 17 = ...$;

$5 = 3 + 29 - 5 - 5 - 17 = ...$;

$7 = 3 + 31 - 5 - 5 - 17 = ...$;

$9 = 3 + 37 - 7 - 7 - 17 = ...$;

$11 = 5 + 37 - 7 - 7 - 17 = ...$.

a) Is the conjecture true when all five prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

6) $n = p - q - r - t - u$, where p, q, r, t, u are all prime numbers, and $q, r, t, u \neq p$.

For example: 1 = 13 - 3 - 3 - 3 - 3 = ... ;

3 = 17 - 3 - 3 - 3 - 5 = ... ;

5 = 19 - 3 - 3 - 3 - 5 = ... ;

7 = 23 - 3 - 3 - 5 - 5 = ... ;

9 = 29 - 3 - 5 - 5 - 7 = ... ;

11 = 31 - 3 - 5 - 5 - 7 = ...

a) Is the conjecture true when all five prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

Even Numbers

A) Any even integer *n* can be expressed as a combination of two primes as follows:

1) $n = p - q$, where p, q are both primes.

For example: 2 = 7 - 5 = 13 - 11 = ... ;

4 = 11 - 7 = ... ;

6 = 13 - 7 = ... ;

8 = 13 - 5 = ...

a) In how many ways can each odd integer be expressed as above?

B) Any even integer n can be expressed as a combination of four primes as follows:

2) $n = p + q + r - t$, where all p, q, r, t are primes.

For example: 2 = 3 + 3 + 3 - 7 = 3 + 5 + 5 - 11 = ... ;

4 = 3 + 3 + 5 - 7 = ... ;

6 = 3 + 5 + 5 - 7 = ... ;

8 = 11 + 5 + 5 - 13 = ...

a) Is the conjecture true when all four prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

3) $n = p + q - r - t$, where all p, q, r, t are primes.

For example: 2 = 11 + 11 - 3 - 17 = 11 + 11 - 13 - 7 = ... ;

4 = 11 + 13 - 3 - 17 = ... ;

6 = 13 + 13 - 3 - 17 = ... ;

8 = 11 + 17 - 7 - 13 = ...

a) Is the conjecture true when all four prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

4) $n = p - q - r - t$, where all p, q, r, t are primes.

For example: 2 = 11 - 3 - 3 - 3 = 13 - 3 - 3 - 5 = ... ;

4 = 13 - 3 - 3 - 3 = ... ;

6 = 17 - 3 - 3 - 5 = ... ;

8 = 23 - 3 - 5 - 7 = ...

a) Is the conjecture true when all four prime numbers are different?

b) In how many ways can each odd integer be expressed as above?

Etc.

GENERAL CONJECTURE:

Let $k \geq 3$, and $1 \leq s < k$, be integers. Then:

i) **If *k* is odd, any odd integer can be expressed as a sum of *k* - *s* primes (first set) minus a sum of *s* prime (second set) [such that the primes of the first set is different from the primes of the second set]**.

a) Is the conjecture true when all *k* prime numbers are different?
b) In how many ways can each odd integer be expressed as above?

ii) **If *k* is even, any even integer can be expressed as a sum of *k* - *s* primes (first set) minus a sum of *s* primes (second set) [such that the primes of the first set is different from the primes of the second set]**.

a) Is the conjecture true when all *k* prime numbers are different?
b) In how many ways can each even integer be expressed as above?

Bibliography

Archibald R. G., *Goldbach's theorem*, Scripta Mathematica, 3 (1935), 44-50, 153-161.

Aubry V., *Vérification du théorème de Goldbach*, L'intermédiaire de math. 3 (1896) 75., 4 (1897), 60, 10 (1903), 61+ 62, (errata, 283).

Barbin E. *The role of Problems in the History and Teaching of Mathematics*, Vita Mathematica, Editor Ronald Calinger, MAA Notes n. 40, 1996.

Bohman J., Fröberg C. E., *Numerical Results on the Goldbach Conjecture*, BIT 15 (1975), 239-243.

Bottazzini U., *Il flauto di Hilbert. Storia della matematica moderna e contemporanea*, Torino, Utet, 1990.

Bottazzini U., Freguglia P., Toti Rigatelli L., *Fonti per la storia della matematica*, Firenze, Sansoni, 1992.

Brun V., *Le crible d'Eratosthène et le théorème de Goldbach*, C. R. Acad. Sci. Paris Sér. I Math. 168 (1919), 544-546.

Brun V., *Le crible d'Eratosthène et le théorème de Goldbach*, Christiania Vidensk. Selsk. Skr. 3 (1920).

Brun V., *Über das Goldbachsche Gesetz und die Anzahl der Primzahlpaare*, Archiv for Math. og Naturvidenskab 34 Nr. 8 (1915), 8-19.

Brun V., *Untersuchungen über das Siebverfahren des Eratosthenes*, Jahresber. Deutsch. Math.-Verein. 33 (1924), 81-96.

Cantor G., *Vérification jusqu'à 1000 du théorème de Goldbach*, Association Française pour l'Avancement des Sciences, Congrès de Caen (1894), 117-134.

Chen J. R., *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica, I 16 (1973), 157-176, II 21 (1978), 421-430.

Chen J. R., Wang Y., *On the odd Goldbach problem*, Sci. Sinica, 32 (1989), 702-718.

Chengdong P., *Goldbach conjecture*, Science Press, Beijing (1992).

Corput J. G. v. d., *Sur la Démonstration de l'Hypothése de Goldbach Pour les Nombres Impairs*, Paris: Gauthier-Villars (1938).

Cudakov N. G., *Über das Goldbachsche Problem*, Uspekhi mat. nauk IV (1938), 14-33.

Davenport H., *The Higher Arithmetic*, Dover Publications, Inc., New York, 1983.

Delone B. N., *Die Petersburger Schule der Zahlentheorie* Moskow, Leningrad: Izdatel'stvo AN SSSR (1947), 365-395.

Deshoulliers J. M., te Riele H. J. J., Saouter Y., *New experimental results concerning the Goldbach conjecture*, Proc. 3rd Int. Symp. on Algorithmic Number Theory, LNCS 1423 (1998), 204-215.

Deshoulliers J.-M., Granville A., Narkiewicz W., Pomerance C., *An upper bound in Goldbach's problem*, Math. Comp. 61 (1993), 209-213.

Dickson L. E., *Goldbach's Theorem* in: History of the theory of numbers, Washington (1919), 421-424.

Dickson L.E., *History of the Theory of Numbers*, New York, Chelsea, 1923 (Dover, 1966), v. I, pp. 421-424.

Fliegel, H. F. and Robertson, D. S., *Goldbach's comet: the numbers related to Goldbach's conjecture*, J. Recreational Math., 21:1 (1989), pp. 1-7.

Fuss P. H., *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIe siècle*, tome I, St. Pétersbourg (1843), 127+135.

Granville A., te Riele H. J. J., Lune J. v. d., *Checking the Goldbach Conjecture on a Vector Computer*, Number Theory and Applications (R. A. Mollin, ed.), Kluwer, Dordrecht (1989), 423-433.

Halberstam H., Richert H.-E., *Sieve Methods*, Academic Press (1974).

Hardy G. H, Littlewood J. E., *Some problems of `partitio numerorum': III: on the expression of a number as a sum of primes*, *Acta Math.*, 44 (1923) 1-70, Reprinted in *Collected Papers of G. H. Hardy*, Vol. I, pp. 561-630, Clarendon Press, Oxford, 1966.

Hardy G. H., *Goldbach's Theorem*, Mat. Tidsskrift B (1922), 1-16.

Hardy G. H., Littlewood J. E., *Some problems of 'partitio numerorum'; III: On the expression of a number as a sum of primes*, Acta. Math. 44 (1922), 1-70.

Hardy G. H., Littlewood J. E., *Some problems of 'partitio numerorum'; V: A further contribution to the study of Goldbach's problem*, Proc. London Math. Soc. ser. 2, 22, part 1 (1924), 46-56.

Hardy G. H., *An Introduction to the Theory of Numbers*, Bulletin of the American Mathematical Society, 35 (1929).

Hardy G. H., Wrigh E. M., *An Introduction to the Theory of Numbers*, Oxford, Clarendon Press, 1975.

Juskevic A. P., *Christian Goldbach: 1690-1764*, Vita mathematica, Birkhäuser Basel (1994).

Juskevic A. P., Winter E., *Leonhard Euler und Christian Goldbach - Briefwechsel 1729-1764*, Abhandlungen der Deutschen Akademie der Wissenschaften zu Berlin 1 (1965).

Kapoor S. K., *Goldbach Theorem*, M/s. Arya Book Depot, 30, Nailwala, Karol Bagh, New Delhi 110005 (INDIA)

Kline M., *Storia del pensiero matematico*, 2 voll., Torino, Einaudi, 1991.

Landau E., *Über die zahlentheoretische Funktion phi(n) und ihre Beziehung zum Goldbachschen Satz*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, (1900) 177-186.

Light W. A., Forrest J., Hammond N., Roe S., *A Note on Goldbach's Conjecture*, BIT 20 (1980), 525.

Lionnet F. J. E., Nouv. Ann. Math. Sér. 2 (1879), 356-360.

Maillet E., *Sur le théorème de Goldbach*, L'intermédiaire de math. 26 (1919), 81.

Merlin J., *Sur quelques théorèmes d'Arithmétique et un énoncé qui les contient*, C. R. Acad. Sci. Paris Sér. I Math. 153 (1911), 516-518.

Merlin J., *Un travail de Jean Merlin sur les nombres premiers*, Bull. sci. math. 2 39 (1915), 121-136.

Montgomery H. L., Vaughan R. C., *The exceptional set in Goldbach's problem*, Acta. Arith. 27 (1975), 353-370.

Nastasi P., Scimone A., Da Euclide a Goldbach. Storie di Uomini e Numeri, Sigma Edizioni, 2001.

Oblath M. R., *Sur le problème de Goldbach*, Mathesis 61 (1952), 179-183.

Raitzin C., *New exact formulae for and the number of Goldbach decompositions of an even positive integer*, Bull. Number Theory Relat. Top. 7 No. 2 (1982), 13-17.

Ramaré O., *On Schnirelman's constant*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 22:4 (1995) 645-706.

Ribenboim P., *The new book of prime number records*, 3rd edition, Springer-Verlag, New York, 1995.

Ricci G., *Su la congettura di Goldbach e la costante di Schrinelmann*, Prima memoria, Annali della R. Scuola Normale Superiore di Pisa, serie II, vol. VI, 1937 (XV), pp.71-76; idem, *Su la congettura di Goldbach e la costante di Schrinelmann*, Seconda Memoria, Annali della R. Scuola Normale Superiore di Pisa, serie II, vol. VI, 1937 (XV), pp. 91-116.

Richstein J., *Verifying the Goldbach conjecture up to* $4\cdot 10^{14}$*, Math. Comp.*, 70 (2001), pp. 1745-1749.

Riesel H., Vaughan R. C., *On sums of primes*, Ark. Math. 21 (1983), 45-74.

Ripert L., *Nombre pair somme de deux nombres premiers*, L'intermédiaire de math. 10 (1903), 166+167.

Saouter Y., *Checking the odd Goldbach conjecture up to* 10^{20}, Math. Comp. 67 (1998), 863-866.

Shah. N. M., Wilson B. M., *Numerical data connected with Goldbach's theorem*, Proc. London Math. Soc. 18 (1920), viii.

Shah. N. M., Wilson B. M., *On an empirical formula connected with Goldbach's theorem*, Proc. London Math. Soc. 19 (1920), 238-245.

Shanks D., *Solved and Unsolved Problems in Number Theory*, New York, 1978.

Shen M. K., *On checking the Goldbach Conjecture*, BIT 4 (1964), 243-245.

Shnirelman L., *Über additive Eigenschaften von Zahlen*, Ann. Inst. Polytechn. Novocerkask, 14 (1930), 3-28, Math. Ann. 107 (1933), 649-690.

Sinisalo M., Checking the Goldbach conjecture up to *4· 10^{11}*, *Math. Comp.*, 61:204 (1993), pp. 931-934.

Smarandache Florentin, "Collected Papers", Vol. II, Kishinev University Press, Kishinev, article <Prime Conjecture>, p. 190, 1997.

Smarandache Florentin, "Conjectures on Primes' Summation", Arizona State University, Special Collections, Hayden Library, Tempe, AZ, 1979.

Stein P. R., Stein M. L., *Experimental results on additive 2 bases*, BIT 38 (1965), 427-434.

Stein P. R., Stein M. L., *New experimental results on the Goldbach conjecture*. Math. Mag. 38, 72-80, 1965.

Studnicka F. J., *Bemerkung über gerade Zahlen*, Casopis 26 (1897), 207-208.

Svoboda J., *Sur le théorème de Goldbach*, L'intermédiaire de math. 22 (1915), 132-133. (Brocard? xxx)

Sylvester J. J., *On the Goldbach-Euler theorem regarding prime numbers*. Nature 55 (1896/97), 196-197, 269.

Sylvester J. J., *On the partition on an even number into two primes* (Abstract of a talk given on November 9, 1871, London Math. Soc.). Proc. London Math. Soc. 4 (1871), 4-6. See also: Messenger of Mathematics 1 (1872), 127+128.

Vecchi M., *Un nuovo aspetto dato al teorema di Goldbach*, Rom. Acc. L. Rend. 22 No. 2 (1913), 654-659.

Vinogradov I. M., *Representation of an odd number as a sum of three primes*, Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS 15 (1937), 291-294.

VV. AA., *Mathématiques au fil des Ages*, Paris, Gauthier-Villars, 1987.

Wang Y. (editor), *Goldbach conjecture*, Series in Pure Mathematics volume 4, World Scientific, Singapore, 1984.

Waring E., *Meditationes Algebraicae*, Cantabrigiae (1770).

Weil A., *Number Theory. An approach through history. From Hammurapi to Legendre*, Birkhäuser, 1984, pp.171-172.

Young R. M., *Excursions in Calculus*, New York, MAA, 1992.

Zaccagnini A., *Goldbach Variations: Problems with prime numbers* (http://www.math.unipr.it/~zaccagni/home.html)

Zinoviev D., *On Vinogradov's constant in Goldbach's ternary problem*, J. Number Th. 65 (1997), 334-358.

Chapter Two

Foreword

The theoretical framework of this paper is the theory of didactical situations in mathematics by Guy Brousseau[32].

It is common knowledge that this theory is based on the conception of the didactic situations, where a situation is defined like the set of circumstances into which a person is (a group, a collectivity, etc.), the relations linking him to the environment and the set of data characterizing an action or an evolution, i. e. an action at a certain moment.

In particular, this work concerns an a-didactic situation, namely that part of a didactic situation which teacher's intention respect to pupils is not clear into. An a-didactic situation is really the moment of the didactic situation in which the teacher does not declare the task to be reached but he gets the pupil to think about the proposed task which is chosen in order to allow pupil to acquire a new knowledge and that it is to be looked for within the same logic of the problem.

An a-didactic situation is a such one if it allows the pupils to appropriate and to manage the staking dinamycs, to get him to be a protagonist of the process, to get him to perceive the responsibility of it as a knowledge and not as a guilt of the saught result. The pupil must accept the suggested play (a-didactic situation) but he must put into action the best strategies allowing him to win.

All that is based on solving a problem, an open problem or a conjecture. So, the aim of this research is to analyze some conceptions of pupils while they are facing a conjecture, and in particular a famous historical conjecture like Goldbach's one. Goldbach's conjecture was chosen because it has a long historical background allowing an efficient a-priori analyse, which is an important phase for the

[32] See G. Brousseau, *Théorie des situations didactiques, Didactique des mathématiques* 1970-1990, Textes rassamblés, La pensée sauvage, Grenoble 1998.

experimentation in order to foresee the possible pupils' answers and behaviours in front of the conjecture.

Moreover, it has a fascinating formulation allowing pupils to mix many numerical examples, and to discuss fruitfully about its validity and some possible attempts of a demonstration.

So, the historical context is important because it suggestes an interplay between the history of mathematics and the mathematics education.

The content of the experimentation grows around the validation or the falsification of two hyphoteses of research: the first one concerning pupils' inability to represent mentally any general method useful for a demonstration; the second one concerning their intuitive ability to recognize the validity of a conjecture. The validation or the falsification of these hypotheses are very useful in order to understand the metacognitive processes which are basic for the learning phase and the cultural growth of pupils.

Another important point for this experimentation is the fact that pupils did not know anything about the unsolvibility of Goldbach's conjecture, so that the a-didactic situation could not be disturbed by any interference due to their knowledge of the failed attempts to solve the conjecture.

Within Brousseau's theory, such an experimentation was carried out by a quantitative analysis along with a qualitative analysis.

The statistical survey for the quantitative analysis was made by two phases: in the first experiment, which was realized with a sample of pupils attending the third and fourth year of study (16-17 years) of secondary school, the method of individual and matched activity was used; the second experiment was carried out in three levels: pupils from the first school (6-10 years), pupils from primary school (11-15 years) and pupils from secondary school. The quantitative analysis of the data drawn from pupils' protocols was made by the software of inferential statistics[33] CHIC 2000 (*Classification Hiérarchique Implicative et Cohésitive*) and the factorial statistical survey S.P.S.S. (*Statistical Package for Social Sciences).*

The research pointed out some important misconception by pupils and some knots in the passage from an argumentative phase to a demonstrative one of their activity which need to be deeped.

The first experimentation

2.1 A-priori Analysis

The first statistical survey was made by using a sample of 88 pupils attending the third and fourth year of study of secondary school in Palermo (Sicily). The students worked in pairs for the part relating to interviews and individually for the production of solution protocols related to the proposed conjecture[34].

The surveyed data were analyzed by the software of inferential statistics CHIC 2000 (*Classification Hiérarchique Implicative et Cohésitive*) and the factorial statistical survey S.P.S.S. (*Statistical Package for Social Sciences*). The variables used were 15 and they were the basis for the a-priori analysis of possible answers by pupils.

Such an a-priori analysis of the problem was explained by the following steps:

1) He/she verifies the conjecture by natural number taken at random.(N-random)

2) He/she sums two prime numbers at random and checks if the result is an even number. (Pr-random)

3) He/she factorizes the even number and sums its factors, trying to obtain two primes. (Factor)

4) *Golbach's method 1*

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[33] See R. Gras, *Les fondements de l'analyse statistique implicative*, Quaderni di Ricerca in Didattica del G.R.I.M., Dipartimento di Matematica dell'Università di Palermo, n. 9, 2000, pp. 187-208.

[34] The author is grateful to Proff. Marilina Ajello, Carmelo Arena, Egle Cerrone and Emanuele Perez for their helpfulness in carrying out the experimentations into their classrooms.

He/she considers odd prime numbers lesser than an even number, summing each of them with successive primes. (Gold1)

5) *Golbach's method* 2 (*letter to Euler*)

He/she writes an even number as a sum of more units, combining these in order to get two primes. (Gold2)

6) *Cantor's method*

Given the even number 2*n*, by subtracting from it the prime numbers $x \leq 2n$ one by one, by a table of primes one tempts if the obtained difference $2n - x$ is a prime. If it is, then 2*n* is a sum of two primes. (Cant)

7) *The strategy for Cantor's method*

He/she considers the primes lower then the given number and calculates the difference between the given number and each of primes. (S-Cant)

8) *Euler*

He/she is uneasy to prove the conjecture because one has to consider the additive properties of numbers. (Euler)

9) *Chen Jing-run's method (1966)*

He/she expresses an even number as a sum of a prime and of a number which is the product of two primes. (Chen)

l0) He/she subtracts a prime number from an any even number (lower then the given even number) and he/she ascertains if he/she obtains a prime, so the condition is verified. (Spa-pr)

11) He/she looks for a counter-example which invalidates the statement. (C-exam)

12) He/she considers the final digits of a prime to ascertain the truth of the statement. (Cifre)

13) He/she thinks that a verification of the statement by some numerical examples needs to prove the statement. (V-prova)

14) He/she does not argue anything for the second question. (Nulla)

15) He/she thinks the conjecture is a postulate. (Post)

2.2 Hypotheses of search

The two experimentation were based essentially on the following hypotheses of search, which could be either validated or falsificated:

I. Pupils are not able to go beyond the empirical evidence of the conjecture because they do not know how to represent mentally any general method useful for a demonstration.

II. Pupils can reach only intuitive conclusions about the validity of Goldbach's conjecture.

2.3 Falsification of the previous hypotheses

This part of the experimentation will be treated by in the final conclusions of the work (Chapter Fourth).

2.4 The text for the individual work

The pupils working individually were expected within two hours to answer the following two questions:

Answer the following questions arguing about or motivating every answer:

***a*) Is it possible to express the given even numbers as a sum of two prime numbers (by one or more ways)?**
248; 356; 1278; 3896

***b*) If you have answered the previous question, can you show that it is valid for each even number?**

2.5 The text for the interview in pairs

The interviewes in pairs were made to two pairs of pupils, respectively 16 and 17 aged. We shall name the pupils of the first pair by the letters L, G and the others by the letters R, S. In both cases the interview lasted 30 minutes, and it was acoustically recorded.

Here is the text of the interview:

Answer the following question writing only what you have agreed on:

- Is it always correct that every even natural number greater than 2 is a sum of two prime numbers?

Let argue about the demonstrative processes motivating them.

2.6 The first interview

The transcriptions of the audio recordings and the written productions of the students were:

L. Hm ... if a natural number is the sum of two primes we can write:

2n = m + p, so $n = \frac{m + p}{2}$

G. Hm ... wait for a moment ... the last digit of a prime number is always 1-3-7-9 except for 2 and 5 ...

L. And so? Let try:

2-3-5-7-11-13-17-19-23- ...

G. Wait! If we add these final digits we always get a number divisible by 2:

1 + 3 = 4 and 2 | 4

1 + 7 = 8 and 2 | 8

1 + 9 = 10 and 2 | 10

3 + 7 = 10 and 2 | 10

3 + 9 = 12 and 2 | 12

7 + 9 = 16 and 2 | 16

....

L. We can say ... wait for ...

_2n > 2 _m, p [N: m + p = 2n

G. Let us try some example:

68 = 31 + 37

64 = 23 + 41

62 = 43 + 19

L. So, we can say that
_2n > 2 _m, p [N: ...

G. How can we prove this?

L. Wait ... Let us see if there exist an even number not representable by a sum of two prime numbers:
240 = 233 + 7
378 = 5 + 373
....

G. I think that each even number is a sum of two primes! Why we need to prove it? It's evident!

L. Wait ... perhaps ... this question ...

G. Do you want to say that it is a postulate?

L.Yes! Perhaps, it is not possible to prove it, for the simple reason that it is a postulate, here is the solution!

G. While we have tried to prove it ...

L. Instead, it is undemonstrable, because it's a postulate ...

G. Yes, but now we have some numbers ...

L. And so? Numbers or figures, we have to decide if this statement is demonstrable or not.

G. Wait a moment ... so ... we can agree on this conclusion: we are in front of a postulate ... we have not a theorem to be proved ...

L. Write down our conclusion:

After various calculations and absurde elucubrations we have come to our conclusion: the relation 2n = m + p (m, p primes) is always true for each 2n, where n N, and n is greater than 2.
In our opinion this is a postulate, so it is only verificable but not demonstrable.

2.7 Commentary

The way followed by L. and G. in order to come to their conclusion is very direct. In fact, they begin by verifying the content of the statement by various numerical examples. After some attempts in order to formalize the question, and after having tried to find a counter-example, they reach a naive conclusion:

if we are not able to prove this statement, even if it has a great numerical evidence, it means tha it will be a postulate or an undemonstrable truth!

Their conclusion is very interesting from an educational point of view, because it highlights their conception about meaning of the term “postulate”.
In their opinion the statement is a postulate because it is undemonstrable, so it seems they think that each undemostrable mathematical statement is always a postulate. Perhaps, this misleading concept derives from a wrong attribution given by them to the term “undemonstrable”. In fact, in a specific mathematical system, as we know, postulates are the principles that you request people to adher to in that system; so, they are starting points for the deductive system that follows. In this contest, the term “undemonstrable” has the meaning of “basic bricks” for the successive theoretical construction. Instead, the two pupils intend the word “undemonstrable” as a sinonyme of “to be not able to prove”, but, as we know, one thing is not to be able to prove a statement, and another one is to prove that it is not demonstrable!

2.8 The second interview

R. Hm … one asks to prove if every natural number is a sum of two primes.

S. Certainly, the experimental verification will not yelds to anything, because there can exist always a number umproving it.

R. The even number must be greater then 2 … and … if it was a postulate?

S. In fact, 2 = 1+1 and 1 is not a prime!

R. Well, wait a moment … we can state the theorem: hypothesis: if we have two primes, x, y and an even number …

S. Wait a moment … we can’t say: if we have two primes … we must say: if one has an even natural number …

R. We have to find a manner to represent a prime number …

S. A prime? But, are you sure? I think there is not a way of representing a prime …No, no … we can get an even number:

2n = n+n or n-1 + n +1 or n-2 + n + 2 …

R. In my opinion we have to find a way to represent a prime …

S. Again? No, no, I'm not sure one is able to represent a prime in general …

R. Well, if we write the sum of n-k and n+k …

S. But it's trivial! We only know that if the number is odd, then it is not a sum of two primes …
R. Hm … I'm not sure about it … look: 15=2+13; 25= … no …. Let me think … 25=2+23 …

S. Well … I was wrong … in my opinion … perhaps … why don't we attempt to find a counter-example?

R. Yes, it's a good idea … a counter-example … 2n = a + b … a, b not primes …

S. 46 = 3 + 43 … no … 52 = 3 + 5 + 7 + 37 … no, they have to be two primes … in my opinion we have to attempt to prove it in general ..,.

R. Yes, let us attempt; well, if n=2k is an even number …

S. No, we have to set the hypothesis well … let n=2k an even number …

At this point the dialogue ends and the pupils sketch the following proof:

Hp. x, y primes
a=2n
n>2, x, y, n natural numbers.
Th. a = x + y.

Proof Since *a* is even it has to be equal to the sum of two even numbers or two odd numbers.
First case: n=4 impossible for the hypothesis;
Second case: x =2k + q
y=2h + t (k, h, q, t natural numbers)
q, t are odd because they are the difference between an odd number and an even one:
x – 2k = q
y – 2h = t
We assume 2k and q coprimes, that is 2k+q is not factorizable; analogously for 2h and t; so, *a* is the sum of two prime numbers. c.v.d.

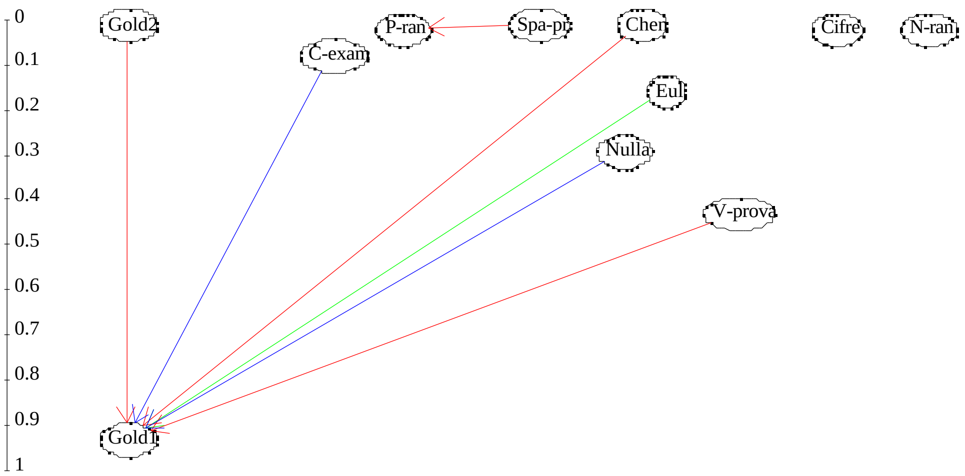
Authors's note: an even number can be intended as a sum of two primes one of which is fixed while the other is chosen among the other primes by a table of primes.
Example:
20=3+7
40=3+37
…….
Final note. The two proofs, the first one more general, the second one more specific, set some obligatory conditions, which can contradict the text, so the statement could be a postulate.

2.9 Commentary

While the first pair of pupils tries to verify the statement of the question, arriving at once to their conclusion about its characteristic essence as a postulate, the second pair is much more convinced of the possibility of proving the question. Their demonstrative attempt is praiseworthy, even if in the final part they advertise the reader that the statement can also be a postulate.
So, the final part of the two interviews in pairs is the same, even if all the pupils have not spoken each other about their experience.
This fact is very significative from an educational viewpoint .

2.10 Quantitative analysis of the statistical survey of data obtained by pupils' individual work. (Analysis by CHIC)

a) The implicative Graph



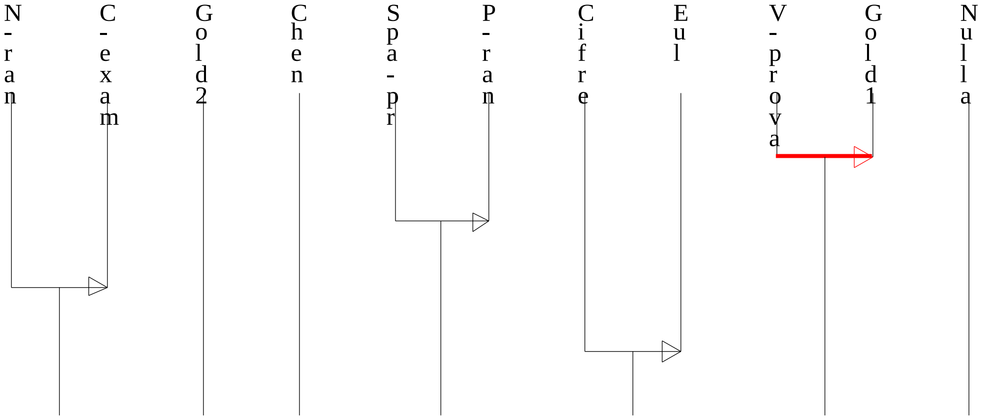
Graphe implicatif : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\goldbach1.csv 99 95 90 85

The analysis of the implicative graph shows, with percentages of 90%, 95% and 99%, that pupils' choice of following some of the strategies is strictly linked to a relevant strategy, namely Gold 1, or the one according which the pupil considers odd prime numbers summing each of them with successive primes. Hence the basis of pupil behaviour is the sequential thinking.

The strategy Spa-pr by which the pupil subtracts a prime number from an any even number (lower then the given even number) ascertaining if he obtains a prime implicates the strategy Pr-random by which the pupil sums two prime numbers at random and checks if the result is an even number. In this case linking between the two strategies lies upon choice at random either of an even number (in the first case) or of prime numbers (in the second one).

The two variables Cifre and N-random appear isolated without any linking neither between them nor to the others, and it means that pupils can use independently each other.

b) The hierarchic tree



Arbre hiérarchique : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\goldbach1.csv

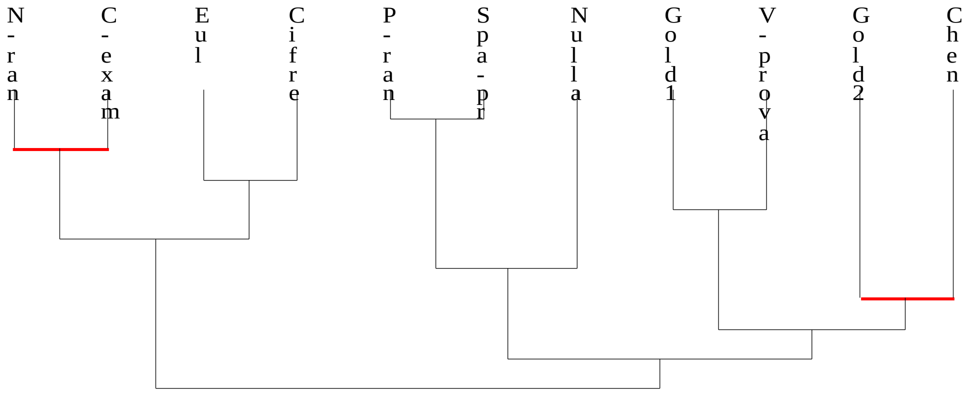
The tree shows that the most hierachic link is between the variables V-prova and Gold1, as it appears also in the implicative graph. But other implications are emphasized between:

a) N-ran and C-exam therefore pupils verifying the conjecture by natural number taken at random consider the primes lesser then the given number and calculate the difference between the given number and each of primes.

b) Spar-pr and p-rand which are therefore pupils subtracting a prime number from an any even number and ascertaining if they obtain a prime in order to verify the condition sum two prime numbers at random and check if the result is an even number.

c) Cifre and Euler therefore pupils considering the final digits of a prime to ascertain the truth of the statement are uneasy to prove the conjecture because one has to consider the additive properties of numbers.

d) Similarity tree



Arbre de similarité : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\goldbach1.csv

We observe a similarity of the first order between five pairs of variables, precisely:

a) N-random and C-exam because they are linked by the strategy of counter-example;

b) Euler and Cifre because they are linked by the ineffectiveness of method;

c) P-random and Spa- pr because they are linked by the method of sequential thinking;

d) Gold1 and V-prova because they are linked by the strong faith in a massive verification of the conjecture by many examples;

e) Gold2 and Chen because they are linked by the presence of three terms into the decomposition of an even number;

Moreover we observe a similarity of the second order between the following pairs of variables:

a) [N-random, C-exam] and [Euler, Cifre] because searching for a counter-example can be also ineffective; we name this set of pairs by **A**;

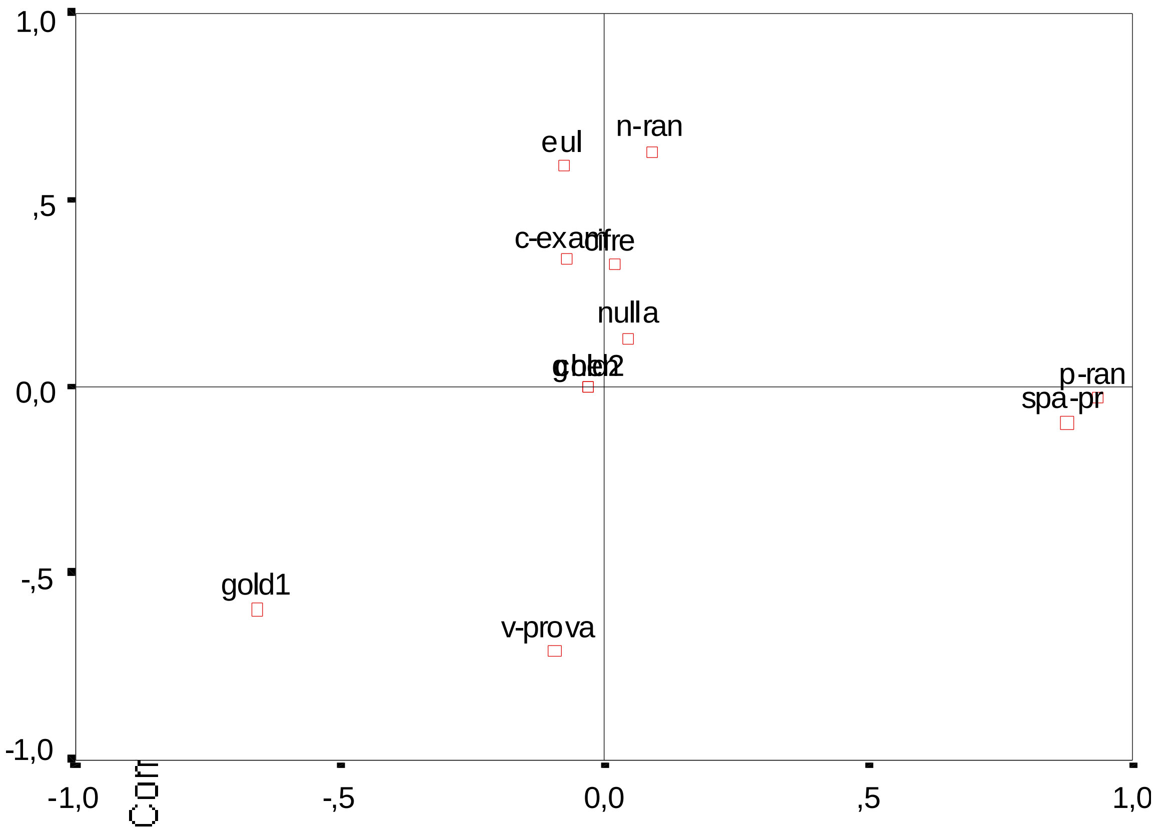
b) [P-random, Spa- pr] and [Nulla] because the method of sequential thinking can also be ineffective in order to reach a proof of conjecture; we name this set of pairs by **B**;

c) [Gold1, V-prova] and [Gold2, Chen] because they are characterized by faith in empirical evidence; we name this set of pairs by **C**.

There is a similarity of the third order between the two set B, C because they are based essentially on a lot of verifications leading to ineffectiveness; we name this set of pairs by **D**.

Finally, there is a similarity of the fourth order between A, D because they are based on a high probability of ineffectiveness.

e) The factorial analysis by S.P.S.S.



Componente 1

The graph shows that the first component is strongly characterized by the pair [P-random, Spa-pr] and [Gold2, Chen], in substance a part of pupils is inclined either to proceed by a sequential fashion or by preferring a method based on a random choice. On the other hand, the second component is characterized by many variables more or less near to it, while Gold 1 and [P-random, Spa-pr] appear isolated. This means that the characterization given by Gold 1 and [P-random, Spa-pr] is really weak while the real strong characterization of most pupils is Gold1-Chen which is much near to the intersection of the two components. Hence, this is the winning strategy among pupils to pass from an argumentation to a possible demonstration. This is a kind of a photo of the more frequent approachs to the conjecture by students.

2.11 Pupils' profiles

So, a further step was made to sketch a possible profile of a pupil approaching the problem. We made the hyphotesis that three possible profiles of pupils would be emerged, and they have been named:

a) Abdut: this is the pupil proceeding by *abduction*. Peirce introduced the term *abduction* to indicate the first moment of an inductive process, the one of choosing a hypothesis by which one may explain determined empirical facts.

On the base of such a definition, the pupil named Abdut is who observes how Goldbach's conjecture to be verified in a large number of cases, therefore he supposes it is also valid for any very large even number, and that leads him to the final thesis, that is the conjecture to be valid for evry even natural number.

In fact, in this case the pupil proceedes in the following supposed way:

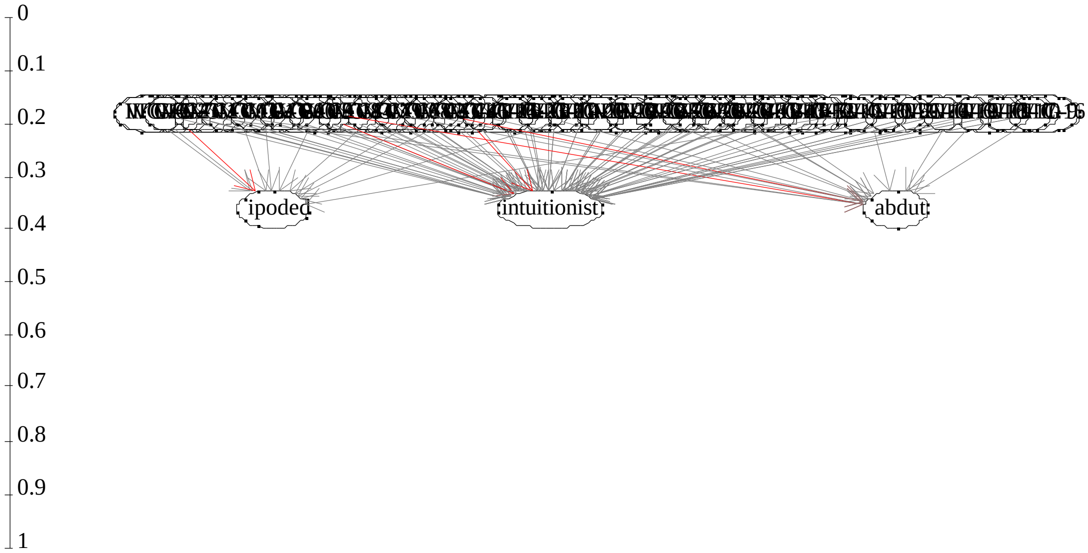
He chooses the strategy of N-random, so he approaches the problem trying to verify it by pairs of natural numbers, choosen at random; he can choose also the strategy of P-random, by choosing a prime number and looking for two natural numbers whose sum is the given prime number. This way of approaching the problem can develop toward the operation of subtracting an even number from a prime in order to obtain an even number, as the Spa-pr strategy, but all these methods can finally persuade him that a proof of the problem is impossible, and so he can fall into the eulerian case.

b) Intuitionist: (at the present who proceeds by an inductive argumentation) is instead the pupil having the N-random and Euler strategies in common with Abdut, but thinking that the demonstration of the conjecture can be deduced by a simple numerical evidence, because he is convinced that what happens for the elements of a small finite set of values can be generalized to the infinite set which the small set belongs to; so he uses the V-prova strategy. In short, in an inductive argumentation used by the intuitionist the statemet is deduced as a generic case after research from specific cases.

c) Ipoded: is just the pupil using a deductive argumentation which can be directly transposed into a deductive demonstration. It is true that he makes some trials and errors, adopting the N-random and P-random strategy as well as Abdut, but soon he follows Chen strategy or he looks for a counter-example whether to demonstrate or to disprove the conjecture.

With these new additional variables a transposed matrix has been made and the more interesting results have been the following:

a) The implicative graph



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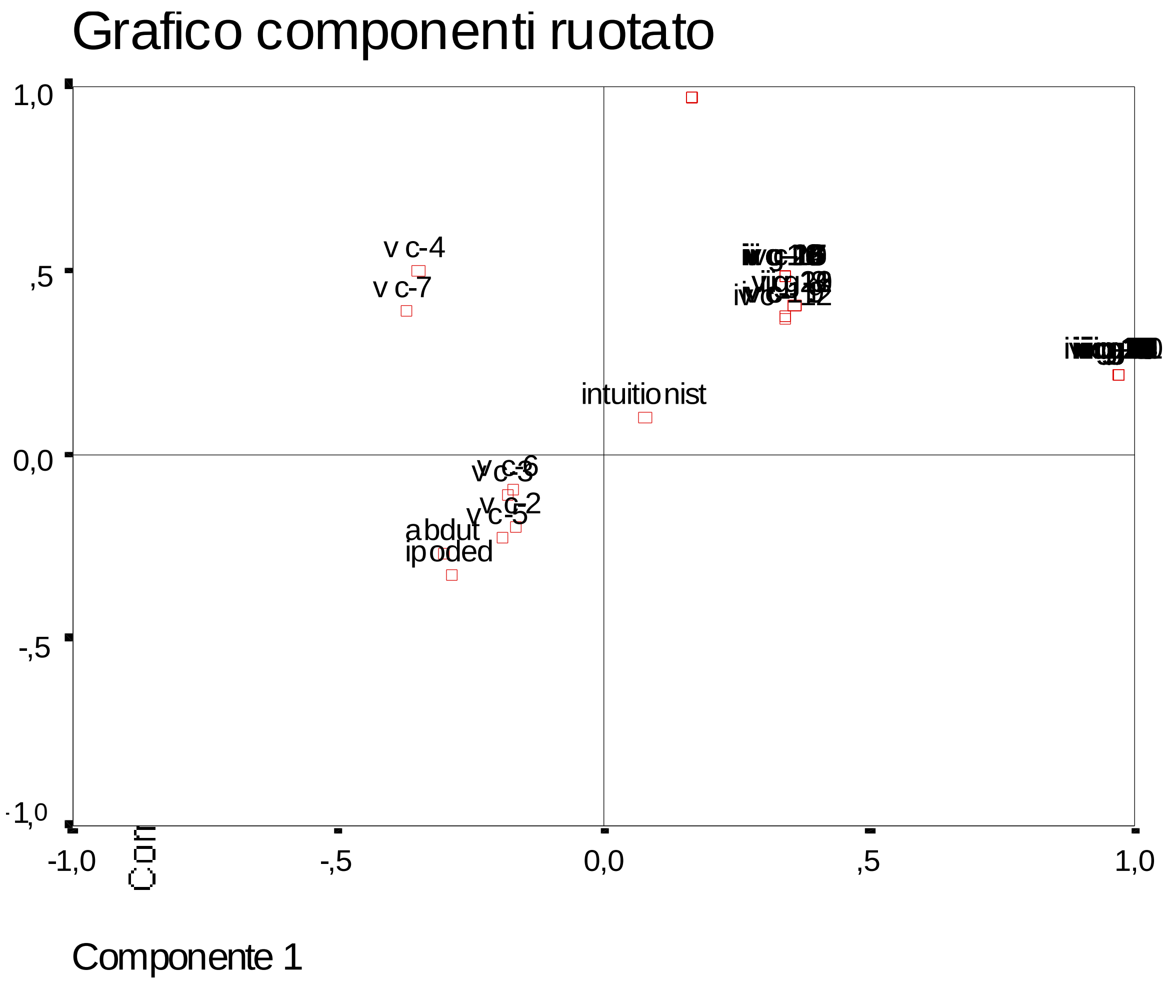
The three profiles corresponding to the additional variables are significative as much as they catalyze the outlines of reasoning of the pupils. They are real attractors for pupils' behaviours.

b) Factorial Analysis



From the viewpoint of the horizontal component the variable Intuitionist characterizes it weakly, while the variables Abdut and Ipoded with a lot of other variables characterize it much more. On the other hand, this is a paradigmatic situation which has its historical counterpart in the attemps made along centuries by different mathematicians facing the conjecture. So, Abdut and Ipoded profiles are winners, while it is less productive the intuitive method of approach. This characteristic situation is stationary also when one observes the graph from the viewpoint of the second component. This means that in any way Abdut and Ipoded methods are more interesting for pupils.

2.12 Some observations

This first experimentation about Goldbach's conjecture has pointed up that in general most pupils, while facing an unsolved historical conjecture (without knowing it is yet unsolved), start at once with an empirical verification of it which can support their intuition, but after they distinguish theirselves along three different solving tipologies:

- a congruos part of pupils bites off more than one can chew with the following conclusion: since the conjecture is true for all of these particular cases, then it has to be true anyway.
These are pupils who have a strong faith in their convinctions, but who do not know clearly enough how to pass from an argumentation to a demonstration, by using the achieved data.

- a part of pupils proceeds at the same time by an empirical verification and by an attempt of argumentation and demonstration ending in a mental stalemate. They try to clear a following hurdle: how can I deduce anything general from the empirical evidence?
These are pupils who before making any generalization want to be sure of the made steps, therefore they tread carefully.

- few pupils, after a short empirical verification, look at once for a formalization of their argumentations, but if they are not able to do that, they are not diffident about claiming they are in front of something which is undemonstrable. These pupils have a high consideration for their mental processes therefore they think that if they are not able to demonstrate anything, then it has to be undemonstrable anyway.

By this experimentation we argue that the argumentation favoured by pupils facing a historical conjecture like Goldbach's is the abductive one. Some questions arise from the results which would be advanced by other experimentations:

- Is this result generalizable?
- To what extent it is generalizable?

But the fundamental kernel of this experimentation about the interplay between history of mathematics and mathematics education is that such results could not be pointed out if the a-priori analysis had not been made by the historical-epistemogical remarks which have inspired it.

Chapter Three

THE SECOND EXPERIMENTATION

The following experimentation about Goldbach's Conjecture was made thanks to a group of teachers, coordinated by the author, in some of their classrooms of primary, middle and high school in Piazza Armerina, a provincial town of Enna[35].
It was carried out in three levels: pupils from the primary school (6-10 years), pupils from middle school (11-15 years) and pupils fron secondary school. The general subject of the experimentation was about *arguing, conjecturing and proving*.

3.1 Primary School

The experimentation in the primary school was made in two two different phases: in the first phase the pupils could answer this question:

1th Phase: The following question has been proposed to each pupil by the so-called "Playing evens" (time: 1 hour):

How can you obtain the first 30 even numbers using the prime numbers from the table which has just been made?

2th Phase: The pupils created small groups and tried to answer the following question:

Can you set even numbers obtained just by summing always and only two primes? If yes, can you say that it is always true for an even number?

3.1.1 The a-priori analysis for the first phase

[35] The author is grateful to Proff. Gabriella Termini, Salvatore Marotta, Salvatrice Sorte, Angela Milazzo, Lina Carini, Carmela Buscemi and Fabio Lo Iacona for their helpfulness in carrying out the experimentations into their classrooms.

a) The pupil sums random more prime numbers and begins to form the table of obtained even and odd numbers. [A1]

b) The pupil does sum prime numbers two by two obtaining even numbers.[B1]

c) The pupil does not make his task because he sums either prime or composite numbers. [C1]

d) The pupil does not make his task because he does not use only addition but also mutiplication between prime numbers.[D1]

3.1.2 The expected behaviours of pupils in the second phase

1. The pupils of each group confront among them by socializing the their discoveries made during the individual phase.
2. The pupils of each group begin to verify the conjecture by using a table of primes and they socialize.
3. The various groups confront and socilize the obtained results.
4. Each group tries to make own strategy acceptable by other groups.

3.1.3 The qualitative analysis of pupils' results of the first primary school

The experimentation has been made into a third classroom of a primary school, with 20 pupils, and it has been preceded by the acquisition of basic prerequisites in order to face the experimentation. The task for the pupils has not been pushed forward neither it has been explained before its realization.
The behaviours of the pupils have been pointed out by some notes and using a videocamera.

1th phase: time 1 hour: individual work

The first phase of the work, constituted by the so-called "Playing with even numbers", lies in an individual work such that pupils have to form the first 30even numbers using variously the table of primes previously formed by them. Each of them has been undertaken to form this sum and only few pupils have had a moment of confusion, that has been soon exceeded because the concept of even number is well known. Only some of them, using the table of primes, have tried to sum three prime numbers, but since the result was an odd number, they have summed after only two numbers among those of the table, in order not to disregard their task. Only a pupil (A6) has fully disregarded her task by obtaining wrong sums:

- she uses as the second term of the sum an even number not present in the table and so she obtains an odd number;

- she uses as the second term of the sum an even number but her sum is fully wrong;
- she uses two primes, but summing she makes a mistake with digits.

2nd Phase: time: 1 hour, work in group.

The second phase goes on by group. Only two groups are formed and after the teacher has read their task, each group begins to socialize. Soon after leaders come out and they supervise the dialogue helping their schoolfriends to express their ideas.
Each group talks softly so that the other group cannot listen to what it is making.
After various considerations, the following hypothesis come out, written by each group over a sheet:

- by subtracting a prime number from a greater one one obtains an even number:
Example: 11 - 5 = 6

- by summing four times a same prime number, one obtains an even number:
Example: 7 + 7 + 7 + 7 = 28

- by summing four different prime number, one obtains an even number:
Example: 5 + 3 + 7 + 9 = 24

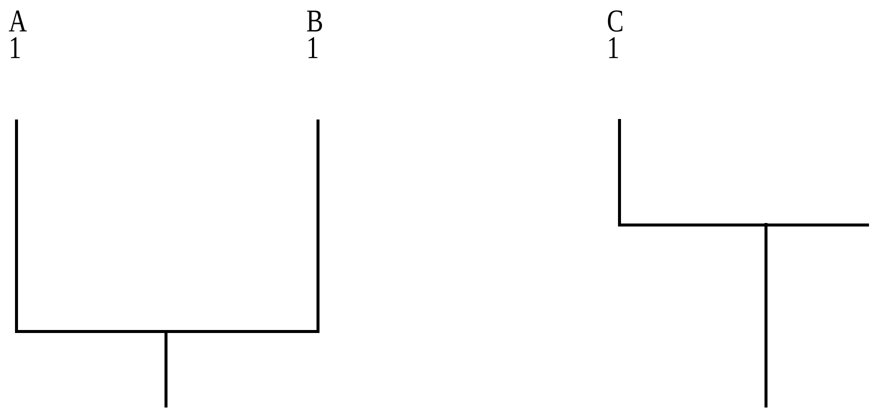
- by multiplying a prime number by four, one obtains an even number:
Example: 7 · 4 =28

- by subtracting one from a prime number, one obtains an even number
Example: 11 - 1=10

- by multiplying a prime number by any even number one obtains an even number:
Example.: 7 x 6 = 42.

3.1.4 The quantitative analysis of pupils' results of the first Primary School obtained by using the software CHIC.

1) Tree of similarity



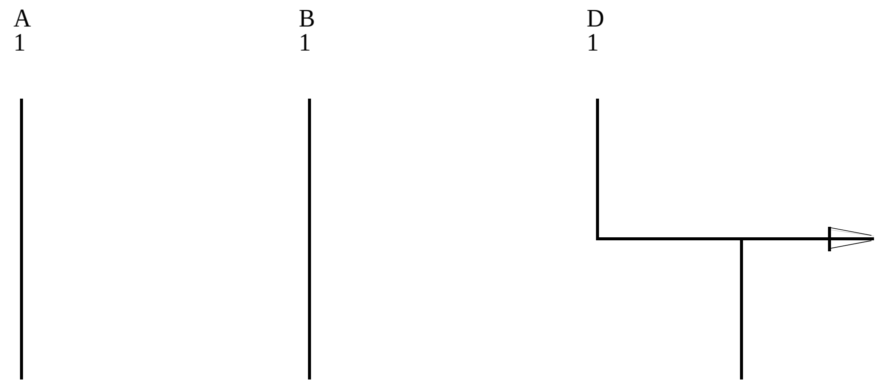
By analyzing the graph, one notes there is a similarity of the first order between the strategies A1 and B1, and between the strategies C1 and D1. The pupil who chooses the strategy A1 is following a sequential manner of thinking as soon as in the case B1.
The similarity between C1 and D1 depends on the fact that the mistake they have made is of semantic type, because the pupil sums either prime numbers or composite, and of operative type. There is also a similarity of the second order between the groups A1-B1 and C1-D1, because the two sets of strategies are based upon the sequential thinking.

2) The implicative Graph

(See Appendix B)

By the analysis of the implicative graph it emerges that there are not inplications among the variables; it means that the choosen variables for foreseeing the results are independent, therefore they allow the pupil to work independently of other distinctive characters.

3) The hierarchic tree



By the analysis of the hierarchic tree it emerges that there is not any hierarchy between the variables A1, B1 and between their type of answer. Instead, there is a hierarchy between the variables D1 and C1, namely, the pupil choosing the strategy D1 could choose the strategy C1 too.

Factorial Amalysis

(See Appendix B)

As for the first factor, namely the horizontal axis, the variable C1 characterizes strongly the first factor, with the variable D1, even though the latter influences in a lower manner the horizontal factor. As for the latter one, the variables A1 and D1 appear isolated, and it means that they do not influence the characterization of the first factor.

As for the second factor, the variable A1 gives the greatest charatcterization, while the variables D1 and C1 appear as an isolated structure. All this has a clear corrispondence with the similarity tree

3.2 The qualitative analysis of pupils' results of the second Primary School

1th phase: time 1 hour: individual work.

Eighteen pupils have a hand in the proposed activity. They are first euforic and very interested for the presence of a videocamera. After receiving a sheet with their task, they read the text and they are very surprised in seeing the table of prime numbers previously obtained by themselves.They are silent and someone is looking for some explanation looking at the teacher. Some of them are sure they have taken the right street to solve the question, and so they begin to write.Then they stop; they are the most clever. The teacher with the video camera goes closer to see why they have stopped. The pupils are rereading the text because some doubts have come out; they are whispering among them that by summing random numbers of the table they obtain great numbers, and they are adding that one cannot obtain a number greater than 60. The teacher in order to encourage them rereads the text aloud pausing over the significative sentences of the text. After almost half an hour most pupils sum the prime numbers two by two obtaining the even numbers. Those who find difficulty in calculations make the sum wrong.

2th phase: time: 1 hour, work in group.

The pupils are divided into four groups. Each group, after having read their task, begins to chat vivaciously, but as soon as the teacher goes nearer with the videocamera to make a shot several of them feel awkward and so they do not speak. Being pressed, the pupils more clever begin to discuss among them, they confront but they try to to make their statements emerged and to make them validated by the group. The more unsafe pupils erase, rewrite, ask their schoolfriends the right strategy. When the time is out the teacher picks up the sheets, but the pupils continue to talk on the made work.

By their papers and by the videocassettes the teacher notes that the pupils have reached the conclusion that an even number cannot be obtained al ways and only by summing two primes, and they prove such a statement by executing the three operations: addition, subtraction and multiplication. A very little group does not know how to argue neither to demonstrate.

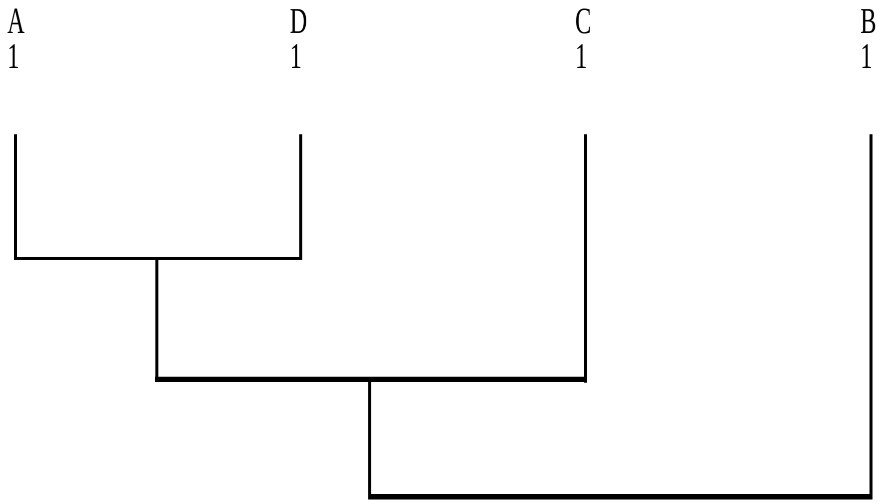
3.2.1 The quantitative analysis of pupils' results of the Second Primary School obtained by using the software CHIC.

1) The implicative graph

(See Appendix B)

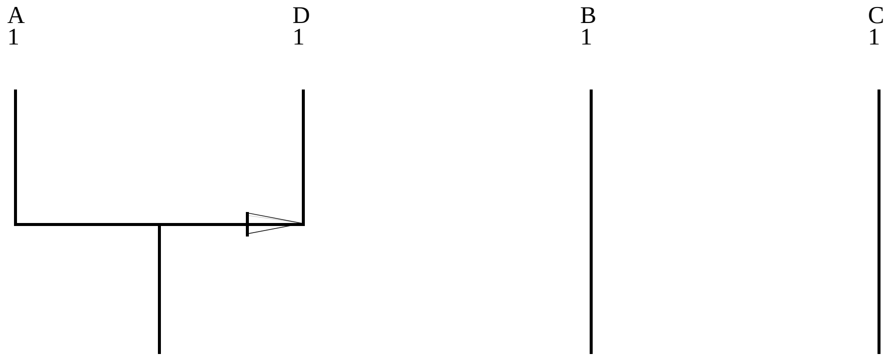
By the analysis of the implicative graph it emerges that there are not inplications among the variables; it means that the choosen variables for foreseeing the results are independent, therefore they allow the pupil to work independently of other distinctive characters.

2) The similarity tree



By analyzing the graph, one notes there is a similarity of the first order between the strategies A1 and D1, because both are based upon a random choice either of the prime numbers or of the operations to be made. A second order similarity existes between the group (A1-D1) and C1, because C1 is based on the random choice of summing either primes or composite numbers. There is also a third order similarity between the group ((A1-D1), C1) and B1 because the choice of the prime numbers to be summed is random.

3) The hierarchic tree



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By the analysis of the hierarchic tree it emerges that there is not any hierarchy between the variables B1 and C1 and between their types of answer. Instead, there is a hierarchy between the variables A1 and D1, namely, the pupil choosing the strategy A1 could choose also the strategy D1.

4) Factorial analysis

(See Appendix B)

By the factorial analysis one deduces that as regards the first factor the variable C1 assumes a determinate role and that it correlates itself to the variable B1, even though this latter does not influence the factor; the other two variables seem very distant.

Instead, as regards the second factor the variable A1 characterizes it fully, while variables D1 and C1 form a group, and in this second case too the variable B1 remains distant, and so it is irrelevant in order to characterize either the first factor or the second one.

3.3 Middle School

Experimentation's text:

Is the following assertion always true?

Is it always possible to resolve an even number into the sum of two primes? Let you argue your assertions.

Lead time: 100 minutes.

Organization of the work:

3.3.1 The Phases of the activity:

a) discussion about the task in couples (10 min.);
b) individual written description of a chosen solving strategy (30 min.);
c) dividing of the class into two groups discussing the task (30 min.);
d) proof of a strategic processing given by the competitive groups (30 min.).

3.3.2 The qualitative analysis of the results of the experimentation

Ith Phase – The pupils, after having received their task, theGli alunni, ricevuta la consegna, confront t wo by two in order to realize if they have understood the question; they argue about even numbers, prime numbers and about the factorization of numbers. It is during this phase they begin to understand the problem.

IInd Phase -The phase of the personal individuation of the resolutive strategies begins. Some of the pupils detect soon the strategies, others need to think; all of them become responsible and are seeking solutions by using also a table of prime numbers.

IIIrd Phase – The classroom is divided into two teams, the pupils elect two foremen and they line up in little groups (two groups of four pupils for each team). The pupils begin to discuss among them, they confront and try to make their strategy emerged making them validated by the other groups. Some of the pupils does not speak, while others emerge as foremen trying to in volve all of them in, because they have understood the success is not personal but of the team.

The pupils choose the more valid strategies writing them by as more as an appropriate and scientific language. The two teams do not communicate between them not to get out any strategy to the other team.

IVth Phase – The competition between the two teams begins: the foremen and the leaders explain the strategies alternatively at the blackboard, whiele the rival team is looking for some counterexamples to confute them. All the pupils are interested because they know that the victory will be for the team making more points. One point is assigned for each valid strategy, while three points are assigned to a team if it prooves that a strategy of the rival team is not valid. The two teams end in a tie, achieving 6 points for each of them.

Group A	Group B
6	6

The not valid strategies are bloated out, and on the blackboard remain the ones that "proove" the conjecture.

The strategies detected by pupils are enclosed into the ones detected by the following a-priori analysis:

3.3.3 A-priori analysis

1. He/she verifies the conjecture by summing progressive prime numbers and verifying if their sum is an even number or not. [A]

2. He/she chooses an even number and considers prime numbers lesser then it; then he/she verifies the conjecture by choosing one of these prime numbers and noting if its complementary (the difference between the even number and the prime number considered) is also a prime (using tables of primes). [B]

3. He/she resolves the even number into a sum of units; then, he/she applies the associative property until he/she obtains two prime numbers suche that their sum is the given number.[C]

4. He/she resolves the even number into prime factors and sums the factors trying to obtain two primes.[D]

5. He/she verifies the conjecture by considering prime numbers choosen at random. [E]

6. He/she restes on the final figures of a prime number to ascertain the truth of the statement. [F]

7. He/she verifies if the even number is factorable by two primes added to another prime number. [G]

3.3.4 The qualitative analysis by indicators

By the analysis of the works and records either in a seat of little group or of a team, one deduced that the most part of the classroom group argued about the conjecture, produced definitions and generalized. All were looking for strategies, only some of them justifing the same strategies, the most part using linguistic indicators of conditionality and generality.

Particularly, by the video, the following typologies of argumentation were evident:

b) He/she definies and produces argumentations of local type ("by summing two prime numbers at random one obtains an even number"), by using linguistic indicators of an ostensive and general type;

c) He/she definies and classifies, making reference to the theory;

d) He/she generalizes and uses linguistic indicators of generality ("by factorizing multiples of 10 I verify that one always obtains two prime numbers");

e) He/she works out and verifies hypothesies making reference to a mathematical knowledge;

f) He/she verifies hypothesis and gets down to make them valid by examples, presented by linguistic indicators of condizionality and returns on a strategy, proving it;

g) He/she verifies the conjecture and gets down to make it valid in a experimental manner;

h) He/she produces conjectures and verifies them by exemplifications making reference to the theory;

i) He/she enunciates, hierarchizes, detects regularities and comprises the role of defining in mathematics;

j) He/she verifies the conjecture by resolving successive even numbers, by using linguistic indicators of condizionality.

Two counterexamples are given:

a) An ostensive counterexample to confute the conjecture, focusing attention on number 2;

b) A counterexample on an argumentative base to confute an hypothesis.

Only a few of the pupils do not produce argumentations or argue in a tautological manner.

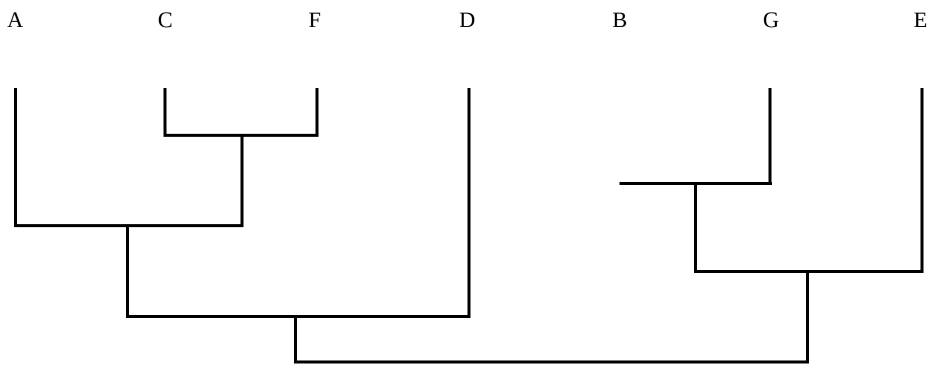
On the whole the experience has been useful in order to get the following targets:

- A development of logical abilities;
- A development of the abilities for arguing on a problem;
- The socialization.

The teacher's role was that of a guide.

3.3.5 The Quantitative Analysis by CHIC

1) The similarity tree



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By observing the graph of similarity, a major affinity is evidenced between the strategies C and F and those B and G. In fact, by analizing the strategies B and F one notes that who has adopted the strategy C (resolving an even number into a sum of unities), and the strategy F (considering the last digits of a prime number) has a predisposition for thinking sequentially. It seems also there is an affinity

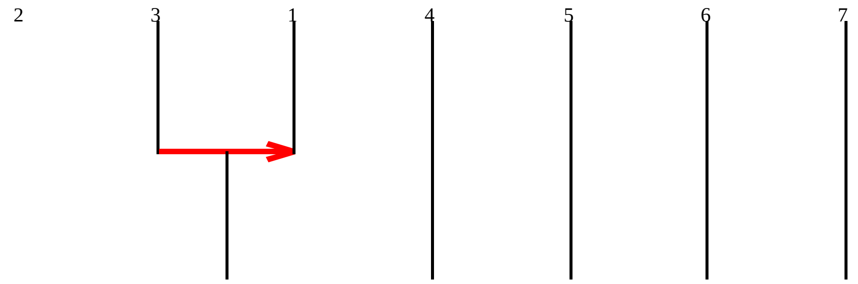
between who chooses the strategy B and who chooses the G one (see the a-priori analysis), because who has choosen the two strategies has really based always on the operation of subtracting and of choosing at random prime numbers.
Beside, there exists some similarities of second order, precisely between strategy A and the similarity group (C-F), because A predisposes also for a sequential reasoning; of the same order is the similarity between groups B-G and E, because they are based always on numbers choosen at random.
There exists a similarity of the third order between assembling (A-(C-F)) and the tipology D, because the latter is of sequential type. Finally, ,there is a similarity of the fourth order between the assemblings ((A-(C-F))-D) and ((B-G)-E), because in the latter group, near the more evident character of causality, one picks out also some parts of sequentiality.

2) The implicative graph

(See Appendix B)

The implicative graph, at present, informs us on the fact that the chance of strategies of an approach by pupils to the conjecture has been made so that each strategy is autonomous enough, therefore each of them has some characters of univocity, which allow pupils to do to the bitter end the way started independently of other considerations.

3) The hierarchic tree



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By the graph is evident a marked hierarchy between strrategies B and A, because the pupil choosing to resolve an even number into sums of unities, in order to obtain two primes by bringing together unities, will choose surely to verify the conjecture by summing consecutive prime numbers.

4) Factorial Analysis

(See Appendix B)

The graph of the factorial analysis of data shows a marked characterization of the horizontal factor by the strategies A, C and F; as regards this axis the group G-B-E and the strategy D are clearly isolated, in accord with the similarity tree.

As regards the vertical axis representing the second factor of tha analysis, are pi cked out the groups of strategies G-B-E and A-C-F, which are very far from the strategy D.

3.4 The First SecondarySchool

3.4.1 The text for the experimentation

Is the following statement always true?

"Ii is always possible to represent an even number as a sum of two prime numbers."

Let argue your statements.

Lead time: 2 hours.

Organization of work:

a) discussion about the task in couples (10 min.);
b) individual written description of a chosen solving strategy (30 min.);
c) dividing of the class into two groups discussing the task (30 min.);
d) proof of a strategic processing given by the competitive groups (30 min.).

3.4.2 A-priori analysis

A1: He/she verifies the conjecture by summing progressive prime numbers and verifying if their sum is an even number or not.

A2: He/she chooses an even number and considers prime numbers lesser then it; then he/she verifies the conjecture by choosing one of these prime numbers and noting if its complementary (the difference between the even number and the prime number considered) is also a prime (using tables of primes).

A3: He/she resolves the even number into a sum of units; then, he/she applies the associative property until he/she obtains two prime numbers suche that their sum is the given number.

A4: He/she resolves the even number into prime factors and sums the factors trying to obtain two primes.

A5: He/she verifies the conjecture by considering prime numbers choosen at random.

A6: He/she restes on the final figures of a prime number to ascertain the truth of the statement.

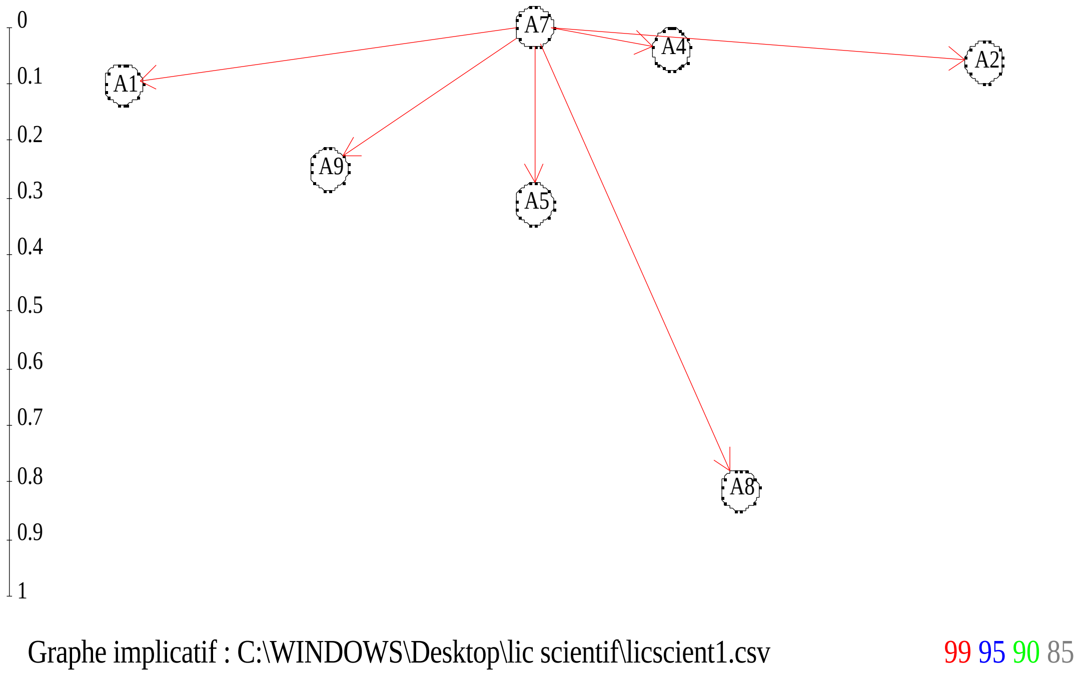
A7: He/she verifies if the even number is factorable by two primes plus another prime number.

A8: He/she verifies the conjecture by taking even numbers at random or progressively.

A9: He/she verifies the conjecture by basing upon the fact that the sum of two odd numbers is always an even number and observing the particularity of number 2, he/she concludes the conjecture is true for even numbers greater than 2.

3.4.3 Quantitative analysis

1) The implicative tree

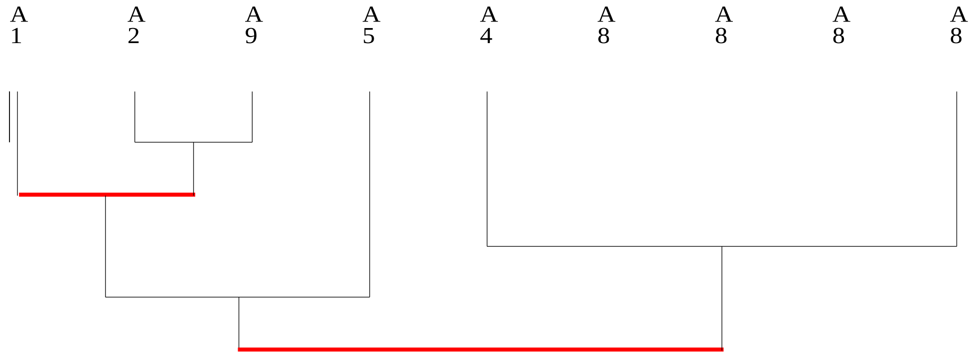


The implicative tree shows that the conceptions A7, A4 are prerequisites for the other strategies used by puypils. As regard the strategy A7, if the pupil is able to verify if an even number is factorizable by product of two primes plus another prime, then he is able to use other strategies, in particular A4 by which he factorizes the even number and sums its factors trying to obtain two primes.

One observes that the strategy A7 implies all the others because it is really Chen Jing-Run's theorem (1966) by which an even number is the sum of a prime plus the product of at the most two prime number.

This is the winning strategy for proving Goldbach's conjecture and it is incredible that a pupil has arrived to it. A4 is a strategy similar to A7 because both base upon factorization of an even number.

2) Tree of similarity



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The tree shows a similarity of the first order between the strategies A2, A9. This similarity is justfied by the pupils' misconcept concerning the similarity between prime and odd numbers. In fact, some pupils have written: "You know the sum of two primes is always even. Therefore since, apart 2, all of prime numbers are odd, then Goldbach's conjecture is always true if the two prime numbers are either different each other or equal 2." The strategy A1, according to which the pupil verifies the conjecture by summing progressive prime numbers is linked to the strategies A2, A9.

The strategies A4, A8 are similar too, and so there is similarity between the strategies (A9, A5) and A8.

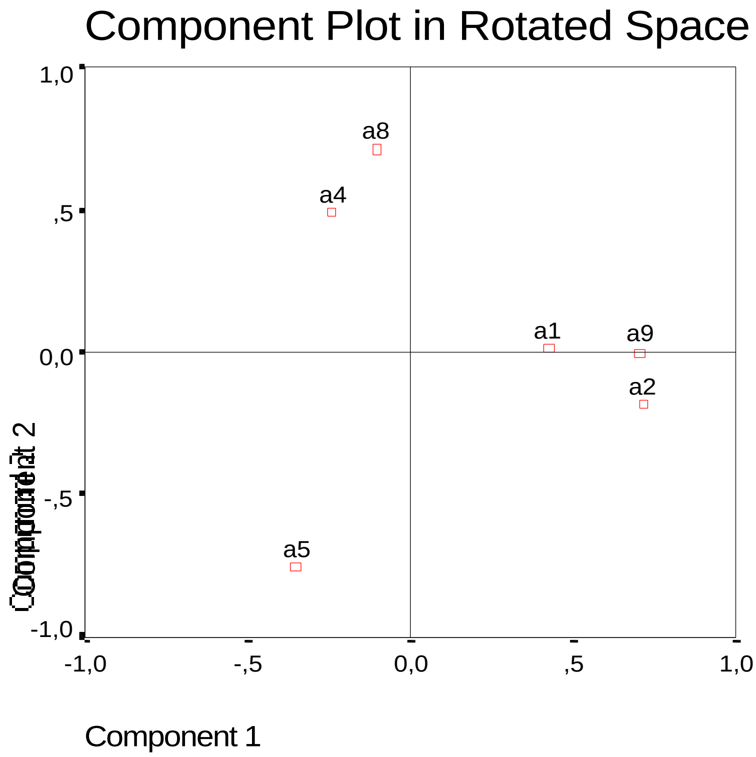
The strategies A1, A2, A5, A8, and A9 are indepent among them, because there is not any mutual implication.

3) Hierarchic Tree:

(See Appendix B)

The graph shows a marked hierarchy between strategies A3-A1 and A6-A2, besides the startegy A7 implies (as we have observed by the implicative graph) the strategies A3 and A1. The strategies A6-A2 imply strategy A9. The strategies A4, A5, A9 appear alone as regards other strategies.

4) Factorial Analysis



The strategies A1, A9 and A2 identify the factors along the horizonthal axis. The strategies A4, A8 are opposite to the strategy A5 as regards the above-mentioned strategies. We observe that perfectly agrees with the similarity tree. The strategies A4 and A8 are both sequential, and are opposite to the strategy A5 because it is not sequential but random.

3.5 The Second Secondary School

3.5.1 The text for the experimentation

Goldbach Conjecture:

Is the following statement always true?

"Can an even number be represented as a sum of prime numbers?"

Argue your claims.

Lead time: 2 hours
Organisation of the work:

Ith phase: Individual thinking about the given question (1 hour);

IInd phase: Arguing in an arragement of small group and acoustically recording of individual strategies (1 hour).

3.5.2 A-priori analysis of the first phase

A1: He/she verifies the conjecture by summing consecutive primes numbers and verifying if the sum is even or not.

A2: He/she chooses an even number and considers prime numbers less than it ; then, he/she verifies the conjecture by choosing one of these prime numbers and noting if its complementary (the difference between the given even number and the considered prime) is prime too.(use of tables).

A3: He/she resolves the even number into a sum of unities; then he/she applies the associative property until he/she obtains two prime numbers such that their sum is the given number.

A4: He/she factorizes the even number by its prime factors and sums the factors, trying to obtain two primes

A5: He/she verifies the conjecture by considering prime numbers choosen at random.

A6: He/she bases on the last digits of a primew number to ascertain the truth of the statement.

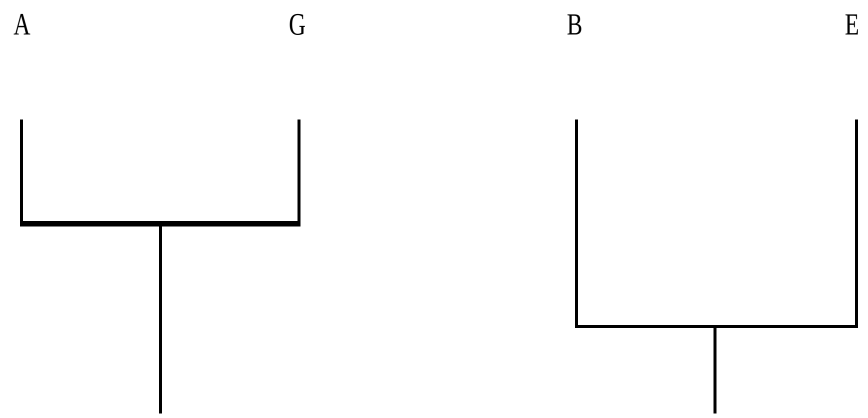
A7: He/she verifies if the even number is factorizable into the product of two primes and another prime.

A8: He/she verifies the conjecture by considering natural even numbers at random or consecutive.

A9: He/she verifies the conjecture by basing on the knowledge thatthe sum of two odd numbers is always an even number and after having observed the particuylarity of number 2, he/she concludes that the conjecture is true for even numbers greater than 2.

3.5.3 The quantitative analysis of data

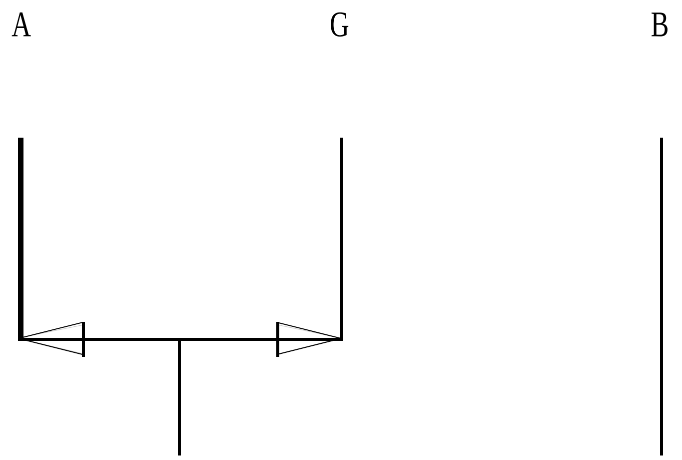
1) The tree of similarity



By observing the similarity graph one deduces that there is a greater affinity between strategies A,G and B,E. In fact, by analyzing strategies A and G, one notes who has adopted the strategy A, by summing successive prime numbers, seems to find as an easy use of the strategy G (multiplying and then summing them); while, who has adopted the strategy B (choosing a prime number and verifying if its complementar as regards the first prime number is also a prime) finds convenient to adopt the strategy E (summing two primes choosen at random), namely, he first subtracts them and then sums the ones.

Finally, one points out a similarity of the second order between the groups (A,G) and (B,E), because of their sequential character.

2) The hierarchic tree



The graph points out a marked hierarchy between the strategies G and A, because the pupil choosing to represent the given even number as a sum of two prime numbers probably uses the same factors multiplying and after summing them. Nothing is pointed out between strategies B and E.

3) The implicative graph

(See Appendix B)
By the analysis of the implicative graph it emerges that there are not inplications among the variables; it means that the choosen variables for foreseeing the results are independent, therefore they allow the pupil to work independently of other distinctive characters.

The factorial analysis
(See Appendix B)
The graph of the factorial analysis of data does not show a marked characterization of the horizontal factor by the strategies A, B, E and G; as regards this axis the strategies B, E appear isolated, while A and G are overlapped because they have been used by same pupils..

As regards the vertical axis representing the second factor of the analysis, there is not any characterization by the strategies used.

Chapter Fourth

The falsification of the initial hypotheses and final conclusions about the two experimentations

Writing this thesis I asked myself some questions answering some of them, not others.
My first task was to point up the following points:

- pupils' conceptions in relation to a conjecture faced during the historical development of mathematics;
- pupils' attempts proving a conjecture reclaimed from history and compared with their argumentative processes;
- to what extent the history of mathematics can favour the study of pupils' conceptions about arguing, conjecturing and proving;
- their reaction to a conjecture's terms seemingly simple to solve;
- their approach in the solving of a conjecture;
- their abilities in carrying out non-standard solving strategies (lateral thinking).

Now I am analyzing each of these points trying to deduce some endeavours which will help the educational activity.
In order to answer the first two points the first experimentation was very useful, because it pointed out to me a seemingly unexpected conception of pupils about arguing, conjecturing and proving.
In fact, the two initial interviews closed with a strong claim by both of the pairs, namely that Goldbach's conjecture was really a postulate, so an undemonstrable assertion; this is a strong conclusion because it implies that there is a misconception by pupils about the meaning of “postulate”, and it should be advanced by further experimentations.
As for the third point it is clear the role played by history, because without an a-priori analysis based upon historical attempts by mathematicians throughout centuries it should not be possible to analyze profitably pupils' attempts.

Pupils' reaction to terms of Goldbach's conjecture seemingly simple to solve was without any misunderstanding because they knew the meaning of all of the terms involved by the conjecture.

As for pupils' approach in the solving of Goldbach's conjecture, both experimentations showed that essentially most of them based on numerical evidence, and only some of them extrapolated their results from a finite set of values to the infinite set of positive integers, but without showing how they passed from trial and errors to the conviction of the general validity of Goldbach's conjecture. This is a delicate point which should be advanced by further investigations:

- how do pupils pass from an argumentative phase to the demonstrative one?
- which is the borderline between argumentation and demonstration?
- which is the event that push them from the supposition into the conviction?

The last point, namely lateral thinking, was really what pupils did not use for facing the conjecture, but this is not surprising because they are not generally accustomed to think in a not sequential manner.

The two hypothesis (see 2.2) which our work based on allowed me their verification either their falsification or not. They were:

I. Pupils are not able to go beyond the empirical evidence of the conjecture because they do not know how to represent mentally any general method useful for a demonstration.

II. Pupils can reach only intuitive conclusions about the validity of Goldbach's conjecture.

In order to reach a conclusion I shall use the most significative implicative graphs of the two experimentations, namely the graph relating to the first experimentation (2.8), and the graph relating to the second experimentation (3.4.3).

The following table where I pose:

B1 = N-random, B2 = Pr-random, B3 = Nulla, B4 = Gold1, B5 = Gold2, B6 = Cifre, B7 = V-pr, B8 = Euler, B9 = Chen, B10 = Spa-p, B11 = C-ex; summarizes the variables of the a-priori analysis involved by the implicative graph relating to the first experimentation (2.8):

	A-priori Analysis
B1	He/she verifies the conjecture by natural number taken at random. (N-random)
B2	He/she sums two prime numbers at random and checks if the result is an even number. (Pr-random)
B3	He/she does not argue anything for the second question. (Nulla)
B4	He/she considers odd prime numbers lesser than an even number summing each of them with successive primes. (Gold1)
B5	He/she writes an even number as a sum of more units, combining these in order to get two primes. (Gold2)
B6	He/she considers the final digits of a prime to ascertain the truth of the statement. (Cifre)
B7	He/she thinks that a verification of the statement by some numerical examples needs to prove the statement. (V-prova)
B8	He/she is uneasy to prove the conjecture because one has to consider the additive properties of numbers. (Euler)
B9	He/she expresses an even number as a sum of a prime and of a number which is the product of two primes. (Chen)
B10	He/she subtracts a prime number from an any even number (lower then the given even number) and he/she ascertains if he/she obtains a prime, so the condition is verified. (Spa-pr)
B11	He/she looks for a counter-example which invalidates the statement. (C-exam)

I representthis implicative graph by an informal fashion using a square table formed by the implicated variables. A coloured cell in red means that a variable on the *x*-axis implies the correspondent variable on the *y*-axis with 99% of statistical percentage, in bleu means that the percentage is 95%, in green that the percentual is 92%.

	Transposition of the implicative graph										
B11											
B10											
B9											
B8											
B7											
B6											
B5											
B4			■		■		■	■	■		■
B3											
B2										■	
B1											
	B1	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**	**B11**
			bleu		red		red	green	red	red	bleu

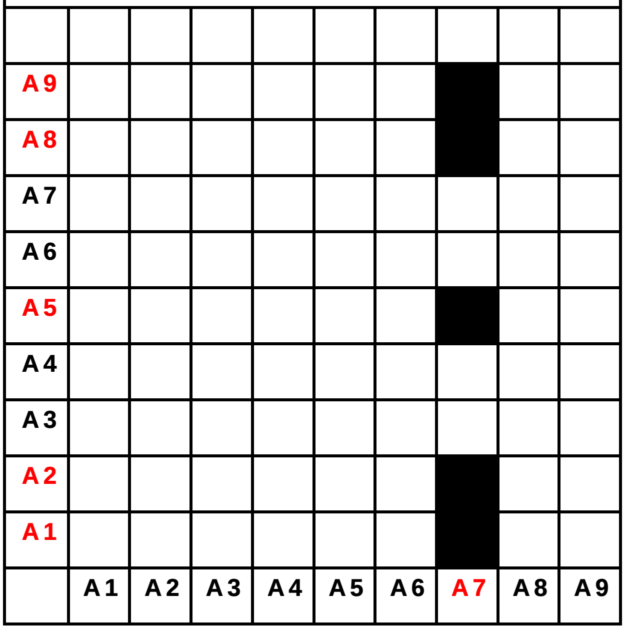
By the graph one notes that the variable B4 is implicated by the others with an exception: B10 implies B2. So, in the final analysis, all of the trials of pupils come down to the historical attempt of Golbach. So, this validates our two hypothesis.

The following table summarizes the variables of the a-priori analysis involved in the implicative graph relating to the second experimentation (3.4.3):

	A-priori Analysis
A1	He/she verifies the conjecture by summing progressive prime numbers and verifying if their sum is an even number or not.
A2	He/she chooses an even number and considers prime numbers lesser then it; then he/she verifies the conjecture by choosing one of these prime numbers and noting if its complementary (the difference between the even number and the prime number considered) is also a prime (using tables of primes).
A3	He/she resolves the even number into a sum of units; then, he/she applies the associative property until he/she obtains two prime numbers such that their sum is the given number.
A4	He/she resolves the even number into prime factors and sums the factors trying to obtain two primes.
A5	He/she verifies the conjecture by considering prime numbers choosen at random.
A6	He/she restes on the final figures of a prime number to ascertain the truth of the statement.
A7	He/she verifies if the even number is factorable by two primes plus another prime number.
A8	He/she verifies the conjecture by taking even numbers at random or progressively.
A9	He/she verifies the conjecture by basing upon the fact that the sum of two odd numbers is always an even number and observing the particularity of number 2, he/she concludes the conjecture is true for even numbers greater than 2.

Also in this case we are representing the implicative graph by an informal fashion using a square table formed by the implicated variables. A coloured cell in red means that a variable on the *x*-axis implies the correspondent variable on the *y*-axis with 99% of statistical percentage, in bleu means that the percentage is 95%, in green that the percentual is 92%.

Transposition of the implicative graph



red

By the graph one notes that the variable A7, which is equivalent to B4, implicates the others with three exceptions: A3, A4 and A6. So, also in this case, all of the trials of pupils are implicated by the historical attempt of Goldbach. So, this validates our two hypothesis.

Moreover, we want to point up that the second experimentation pointed out a characteristic behaviour of pupils from primary to high school while facing Goldbach's conjecture.

First of all it is clear that pupils of primary school could not proceed if not by a sequential fashion, because they did not yet reach the phase of the demonstration; they were still within a phase of naive argumentation.

Most pupils of middle and high school faced the conjecture by a solving methodology based on random choice and on sequential thinking.

There was a difference between methods of facing the conjecture by pupils of middle and secondary school. Pupils of middle school in general faced the conjecture basing on an empirical approach, also arguing their choices; but their task went on until a certain point of verification and not beyond.

On the contrary, there was the presence either of argumentation and attempt of proving in the approach of pupils of secondary school. Really,

many of them tried to infer a demonstration by their argumentation, and some of them reached also to Chen strategy, wondering me. I came down to the same conclusions by the results of the two interviews made during the first experimentation. Well, in these cases my initial hypotheses were falsificated, and this was a fine surprise.

I noted a close analogy between strategies of most pupils and Abdut profile releaved by the first experimentation.

Some of them, finally, were wrong because they exchanged the statement of the conjecture by the converse, which is trivial.

The results of the experimentation realized some questions which would be deeped:

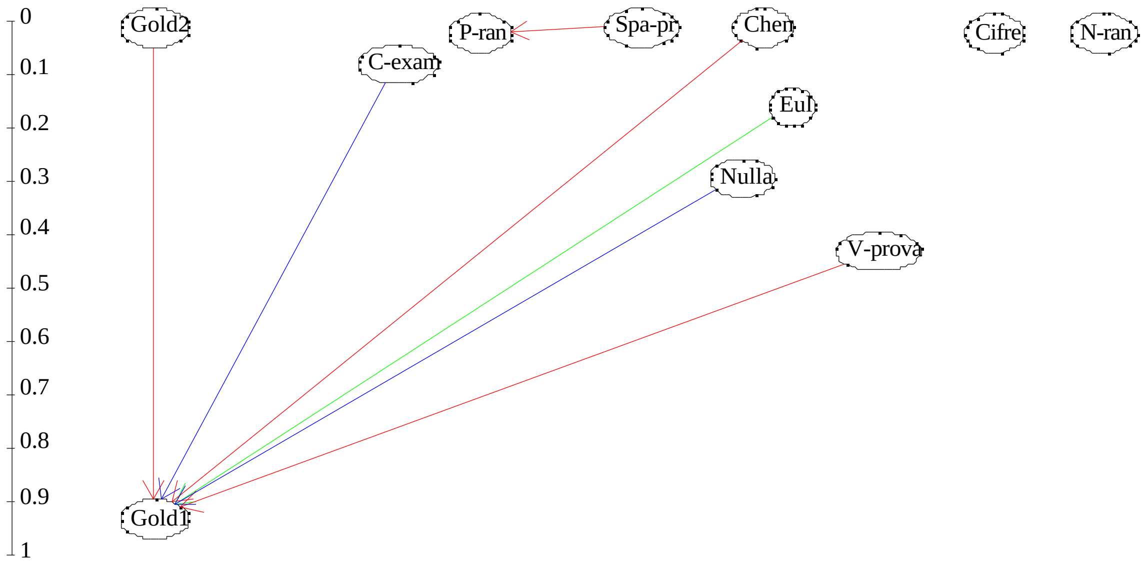
- how do pupils get consciousness of a demonstrative process?
- how do pupils get consciousness of the necessity of a demonstrative process?
- how pupils are able to pass from an argumentation to a demonstration?
- are pupils fully conscious of the difference between a verification and a proof?

These and similar questions can give rise to significant experimentations in order to comprise even better metacognitive processes which are basic for the learning phase of pupils and their cultural growth.

Appendix A

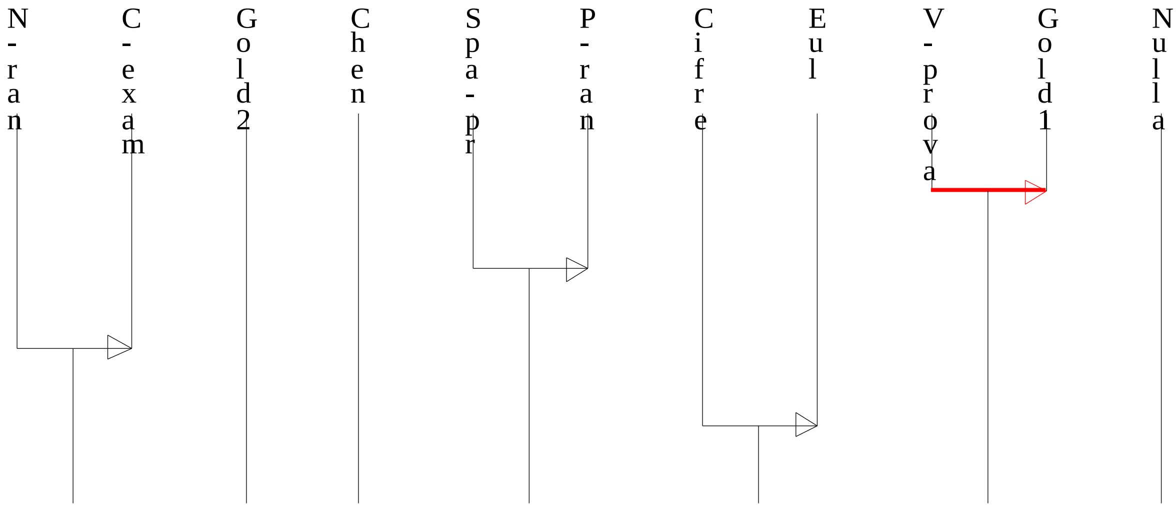
The First Experimentation

b) The implicative Graph

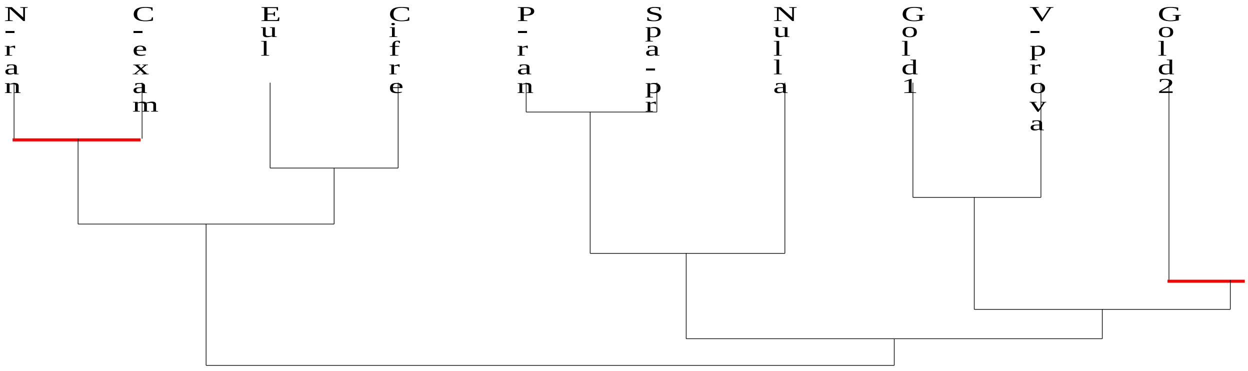


Graphe implicatif : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\goldbach1.csv 99 95 90 85

c) The hierarchic tree

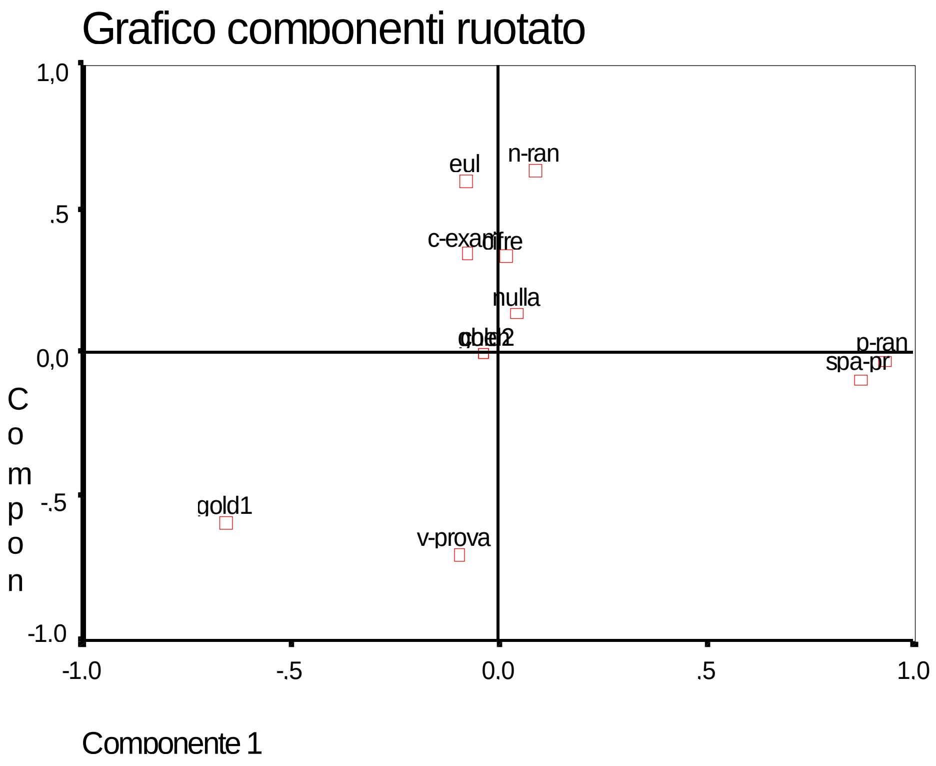


Arbre hiérarchique : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\goldbach1.csv *d) The similarity tree*



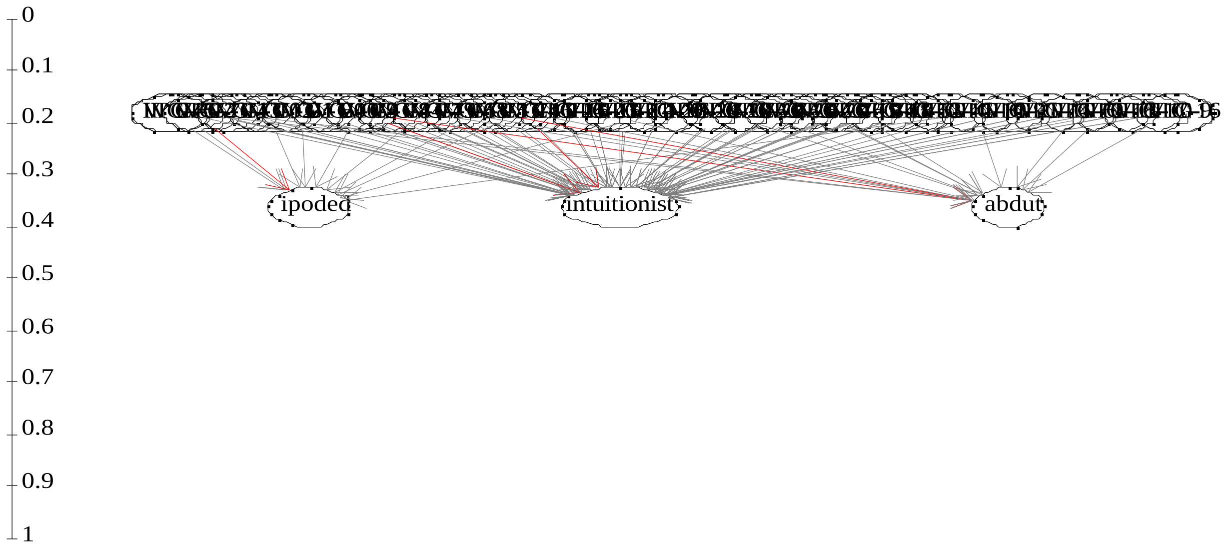
Arbre de similarité : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\goldbach1.csv

d) Factorial Analysis



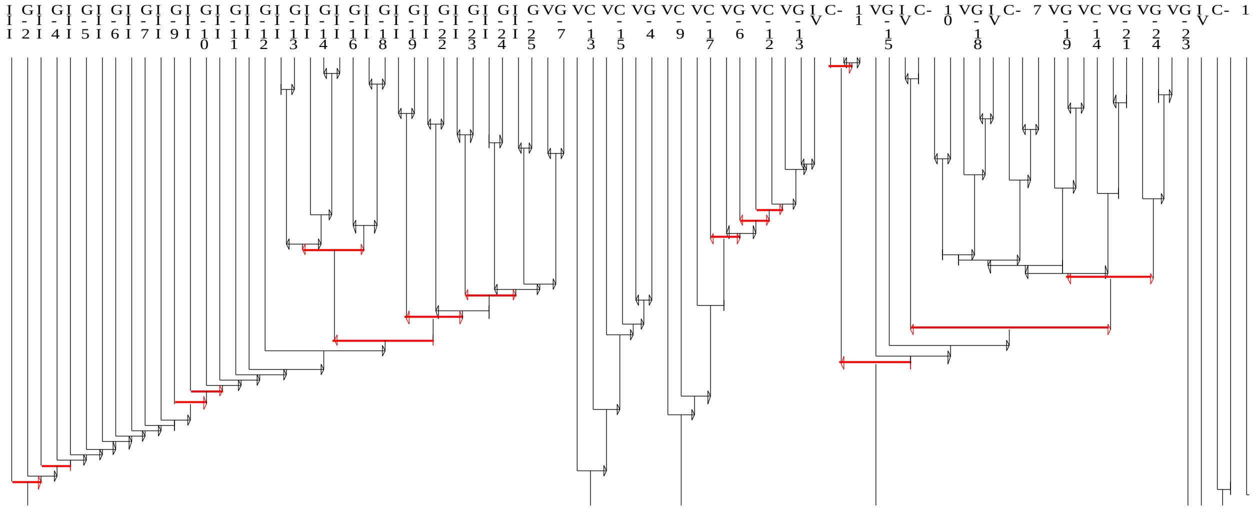
Graphs with added variables

a) The implicative Graph

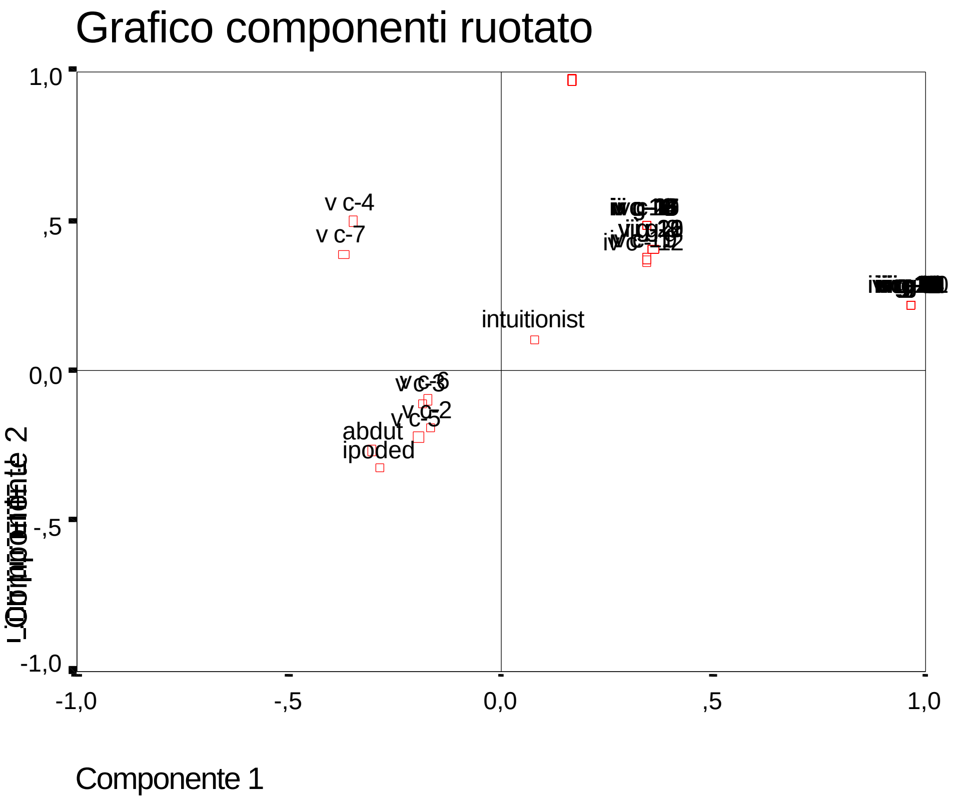


Graphe implicatif : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\Goldtrasp2.csv 99 9

b) The hierarchictree



Arbre hiérarchique : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\Goldtrasp2.csv *c) Factorial Analysis*

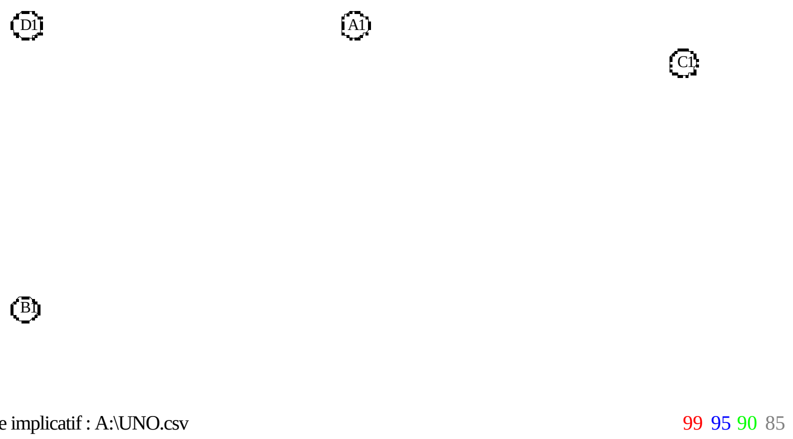


Appendix B

The Second Experimentation

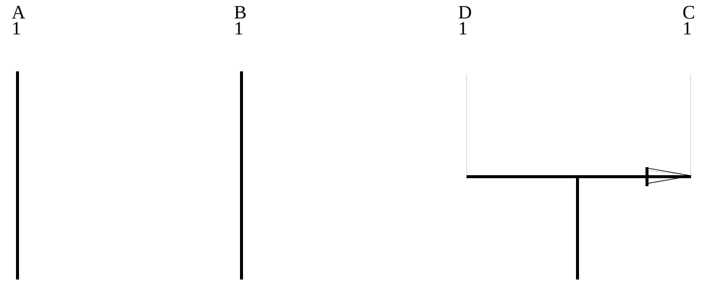
The First Primay School

a) The implicative Graph



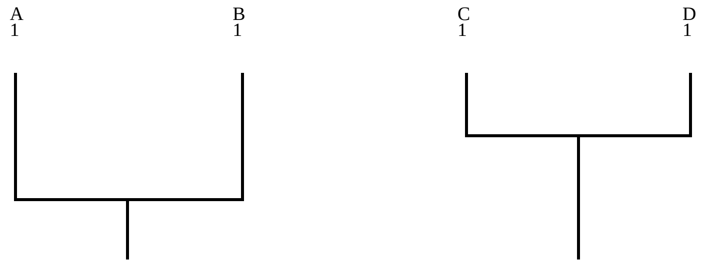
Graphe implicatif : A:\UNO.csv

b) The hierarchic tree



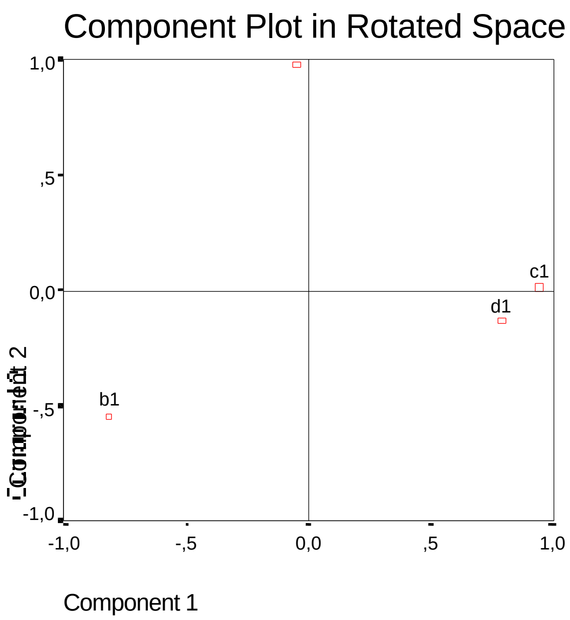
Arbre hi?rarchique : A:\UNO.csv

c) The similarity tree



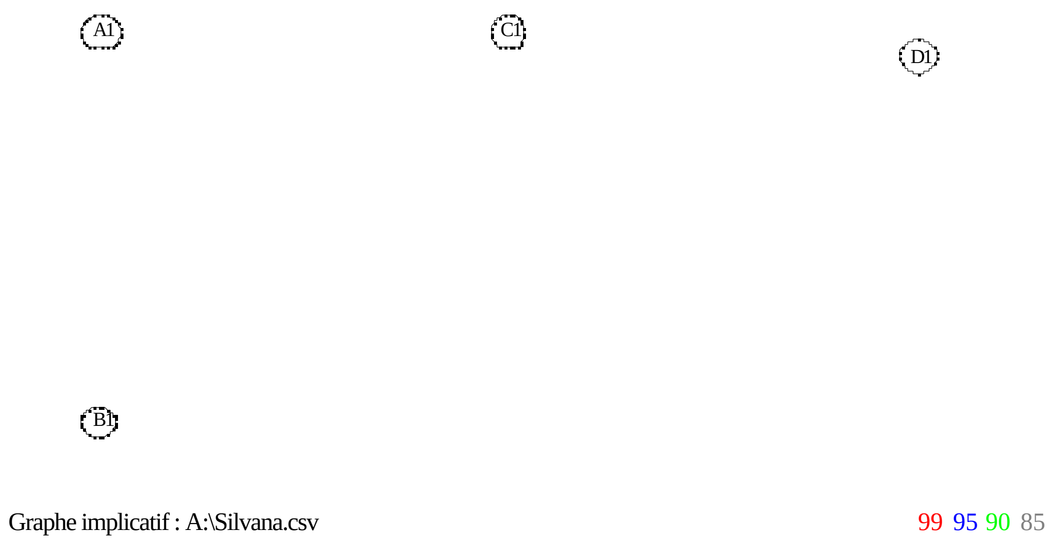
Arbre de similarit? : A:\UNO.csv

d) Factorial Analysis

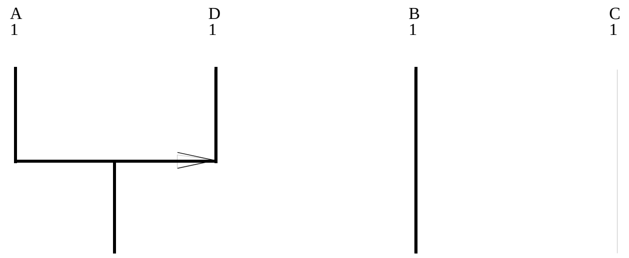


The Second Primay School

a) The implicative Graph

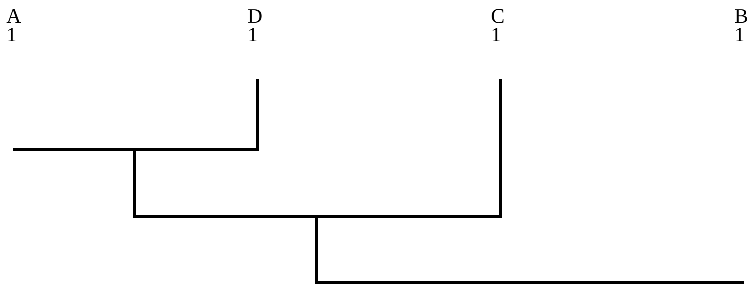


b) The hierarchic tree



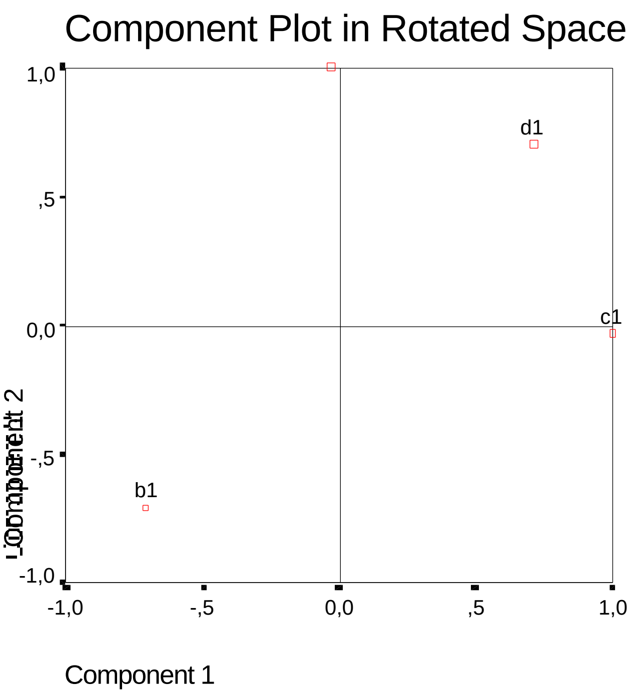
Arbre hi?rarchique : A:\Silvana.csv

b) The similarity tree



Arbre de similarit? : A:\Silvana.csv

c) Factorial Analysis



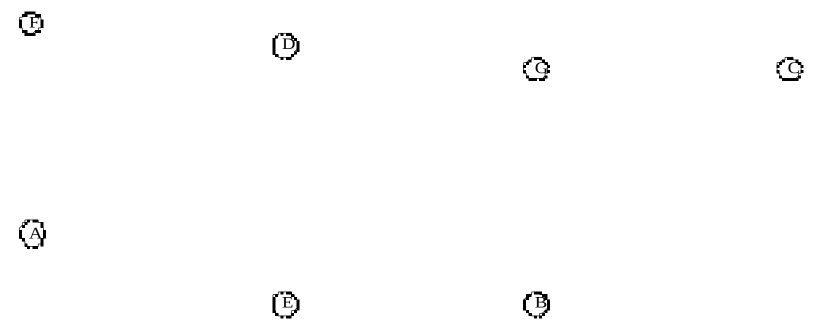
Middle School

Table of the quantitative analysis
Legenda:
A1 ÷ A16 pupils

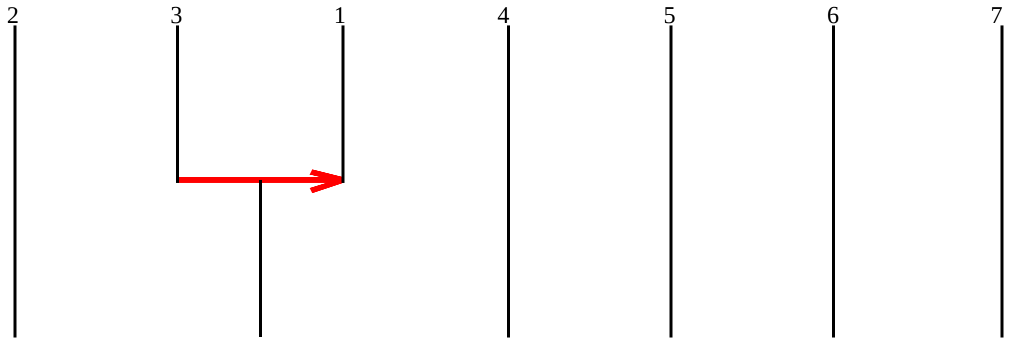
A ÷ G: strategies

	A	**B**	**C**	**D**	**E**	**F**	**G**
A1	1	1	0	0	0	0	0
A2	0	1	0	0	1	0	0
A3	0	1	0	0	1	0	0
A4	1	1	0	1	1	0	0
A5	0	1	0	0	1	0	0
A6	1	0	1	0	1	0	0
A7	1	1	0	0	1	0	0
A8	1	1	0	0	1	0	1
A9	0	1	0	0	1	0	0
A10	1	1	1	0	1	0	0
A11	1	1	0	0	1	0	1
A12	1	1	0	0	1	0	1
A13	1	1	0	0	0	0	0
A14	1	0	0	0	1	0	0
A15	1	0	0	0	1	0	0
A16	1	1	1	0	1	1	0

a) The implicative graph

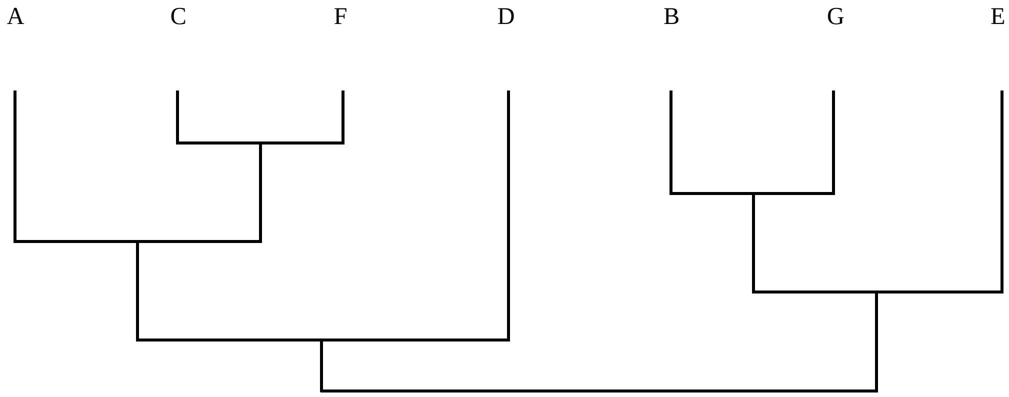


Graphe implicatif : C:\WINDOWS\Desktop\Grp2scuolamedia.csv 99 95 90 85 *b) The hierarchic tree*



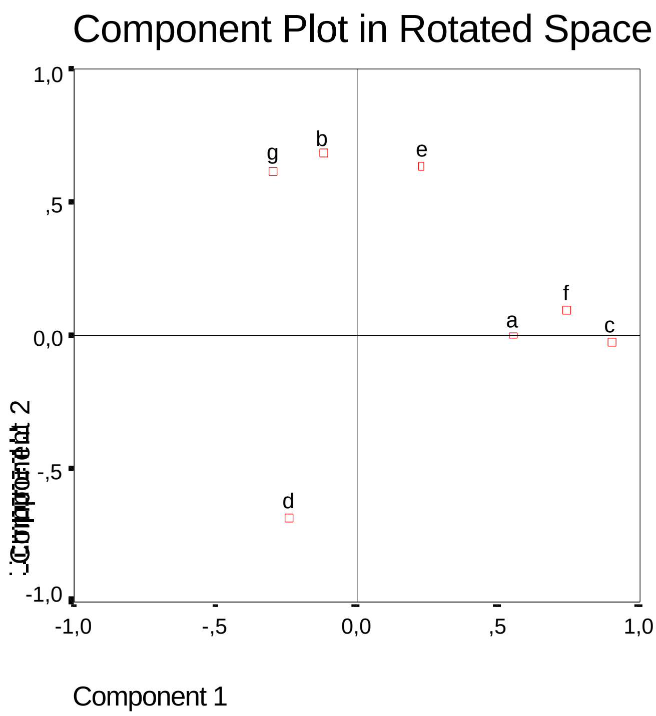
Arbre hi?rarchique : A:\Angelxcel1.csv

c) The similarity tree



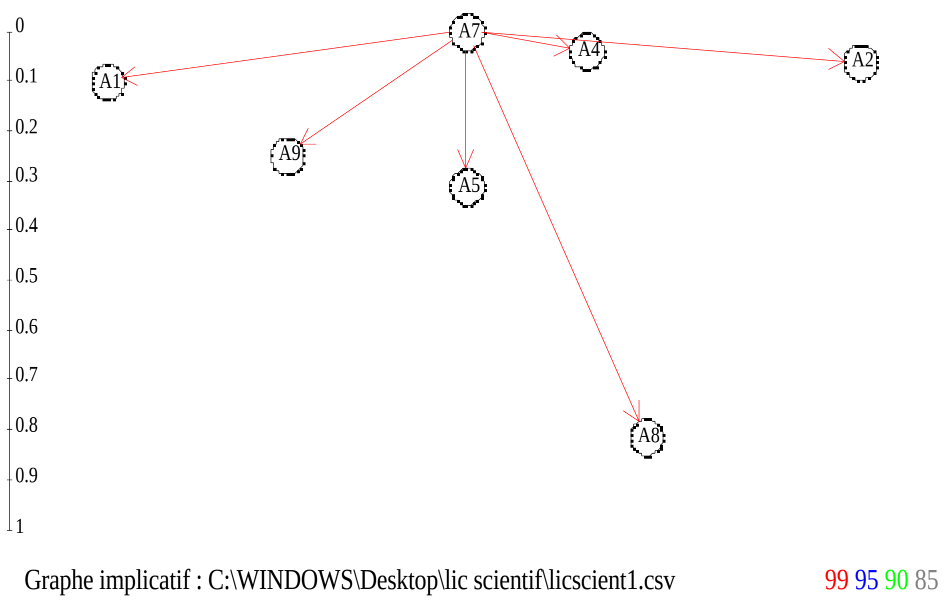
Arbre de similarit? : C:\WINDOWS\Desktop\Grp2scuolamedia.csv

d) Factorial analysis

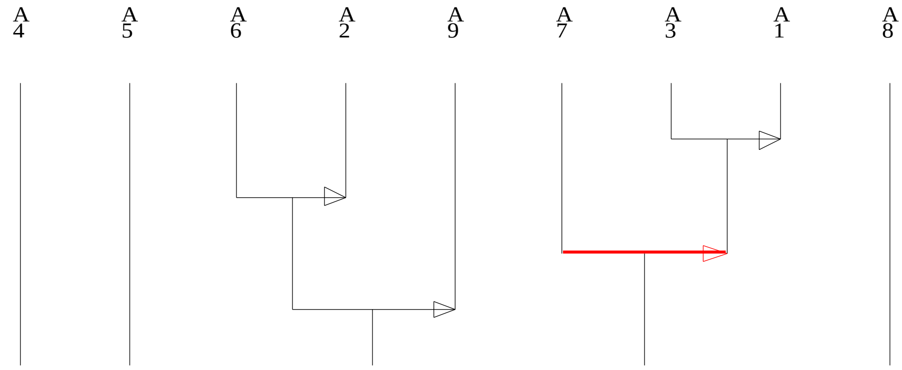


The First Secondary School

a) The implicative graph

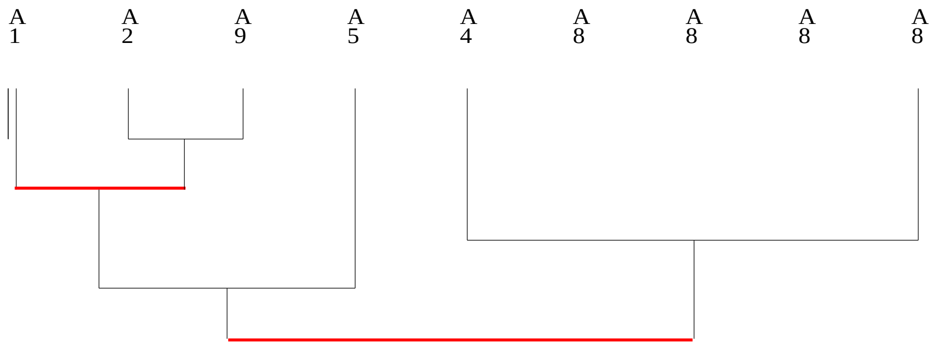


b) The hierarchic tree

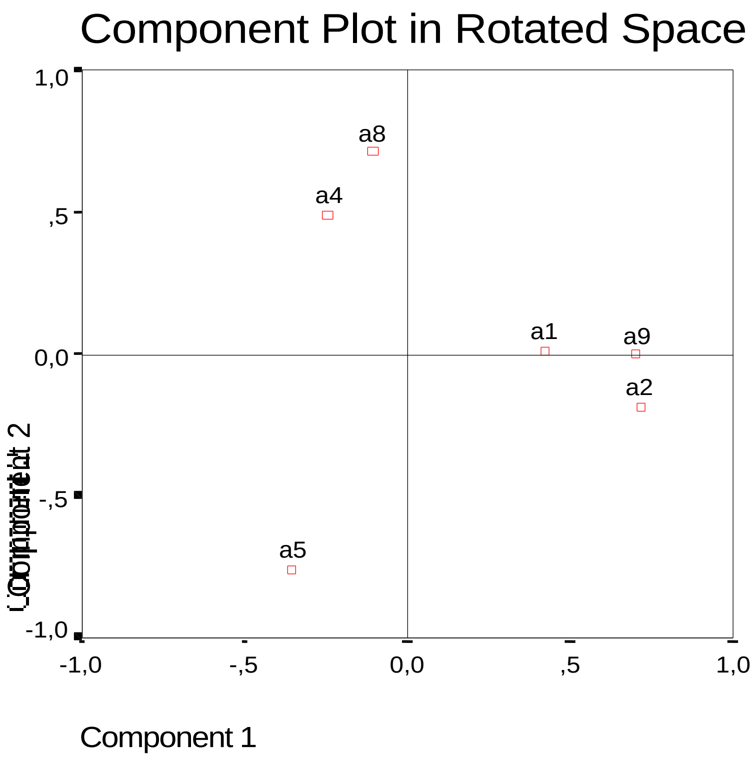


Arbre hiérarchique : C:\WINDOWS\Desktop\lic scientif\licscient1.csv

c) The similarity tree

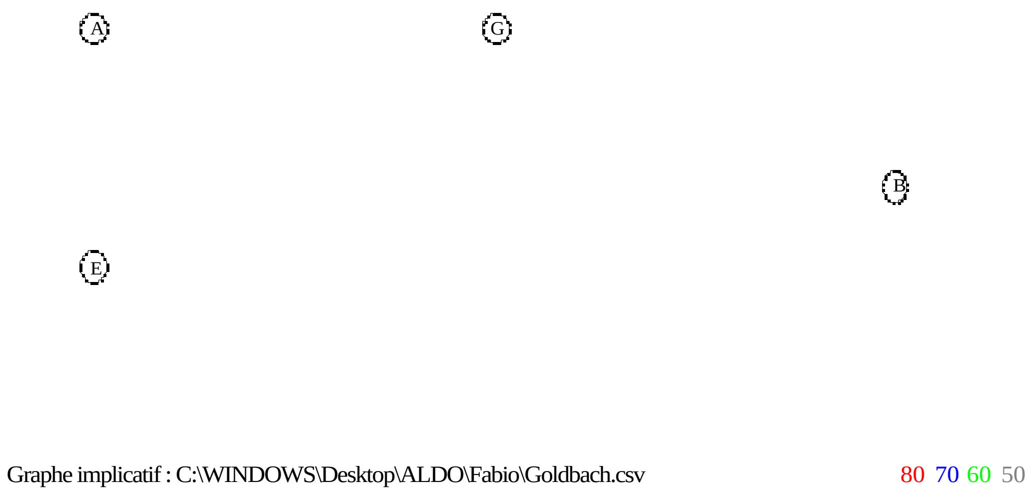


Arbre de similarité : C:\WINDOWS\Desktop\lic scientif\licscient1.csv *d) Factorial Analysis*

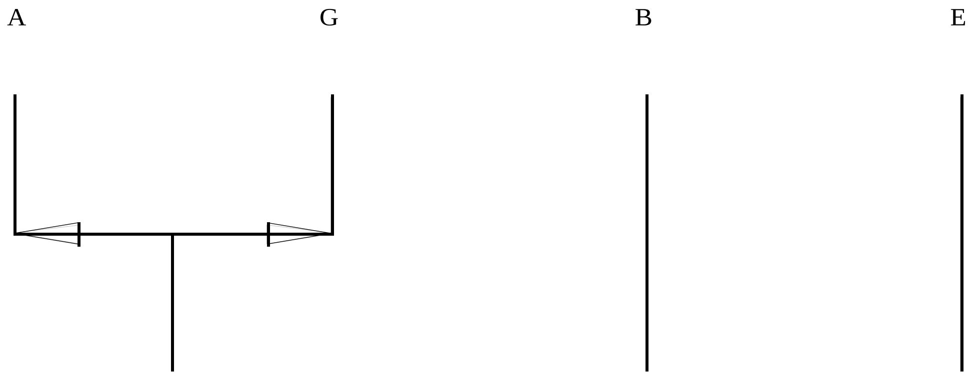


The Second Secondary School

e) The implicative graph

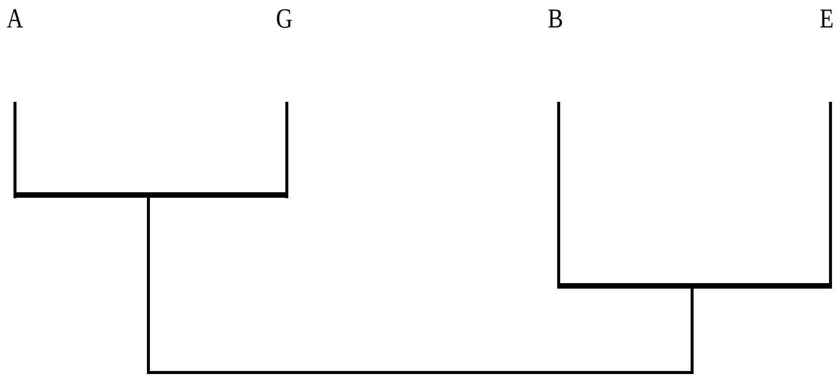


f) The hierachic tree

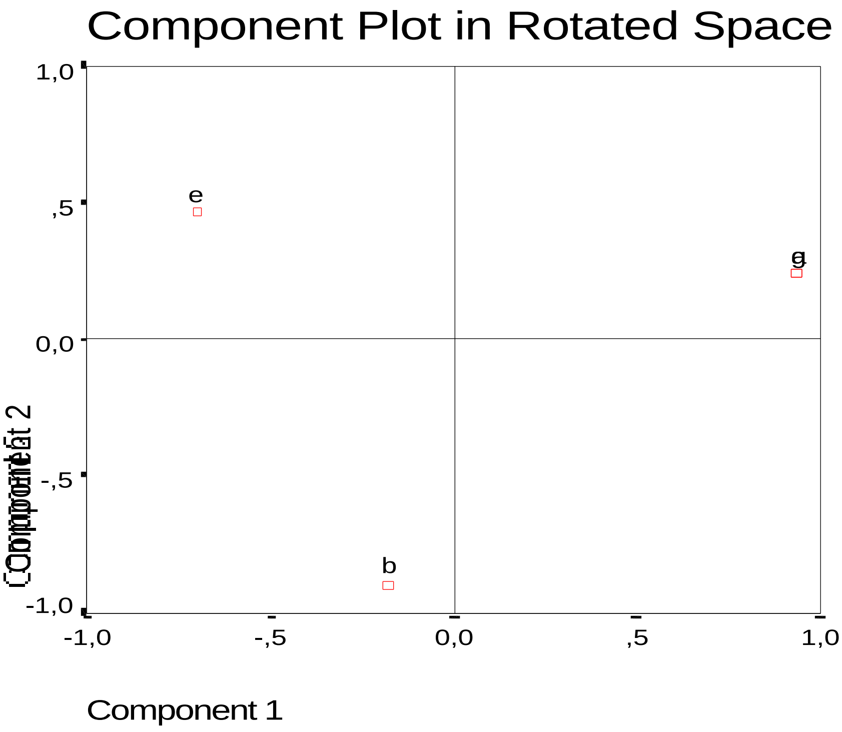


Arbre hi?rarchique : C:\WINDOWS\Desktop\ALDO\Fabio\Goldbach.csv

g) The similarity tree



Arbre de similarit? : C:\WINDOWS\Desktop\ALDO\Fabio\Goldbach.csv *h) Factorial Analysis*



Bibliografy

Accardi S., Calisti S., Spagnolo F., *"Una esperienza di apprendimento della matematica nell'ambito dell'attività di sostegno di un alunno portatore di handicap nella scuola media"*, Quaderni CIDI, n.16, 1983, Franco Angeli editore.

Ajello M., *L'analisi a-priori come strumento per la strutturazione dell'insegnamento per moduli*, GRIM di Palermo, Quaderni di ricerca in Didattica, n. 9, 2000.

Ajello M., Trizzino G., *Due esperienze di autoanalisi in ambiente scolastico attraverso l'analisi implicativa delle variabili*, GRIM di Palermo, Quaderni di ricerca in Didattica, n. 9, 2000.

Ajello M., Grillo B., *Il pensiero proporzionale: analisi di un lavoro sperimentale*, Quaderni di ricerca in Didattica, n. 7, 1998.

ArzarelloF., Micheletti C., Olivero F., Paola D., & Robutti O., *A model for analysing the transition to formal proof in geometry*, Proc. PME XXII, Stellenbosch, vol. 2, 24-31. 1998.

Balacheff N., *Conception, connaissance et concept*, Didactique et tecnologies cognitives en mathématiques, séminaires, Université Joseph Fourier (Grenoble 1), 1994-1995, pp. 219-244.

Ball D. L., *Teaching mathematics for understanding: What do teachers need to know about subject matter?*, Teaching academic subjects to diverse learners (M. M. Kennedy, ed.), Teachers College Press, NY, 1995, pp. 63-83.

Berger P., *The hidden dimension in maths teaching and learning processes: Exploring entrenched beliefs, tacit knowledge, and ritual behavior via metaphors*, Proceedings of the Eighth European Workshop on Research on Mathematical Beliefs - MAVI-8 (University of Cyprus, Nicosia, Cyprus) (G. Philippou, ed.), 1999, pp. 8-17.

Boaler J., *Participation, knowledge and beliefs: A community perspective on mathematics learning*, Educational Studies in Mathematics 40 (1999), 259-281.

Bologna V., Di Leonardo M.V., Malisani E., Marino T., Profumo M., Salvo C., Scimone A., Spagnolo F., *Considerazioni su alcuni articoli di didattica della matematica della rivista "Il Pitagora"*, "La matematica e la sua didattica", 4(1994), 383-389, Editrice Pitagora, Bologna.

Bologna V., *Esperienze francesi sulla dimostrazione*, Quaderni di Ricerca in Didattica del GRIM, Palermo, 2 (1992), 5-23.

Brandau L., *The power of mathematical autobiography,* Proceedings of the 1988 Annual Meeting of the Canadian Mathematics Education Study Group (Winnipeg, Manitoba, Canada) (L. Pereira-Mendoza, ed.), 1988, pp. 143-159.

Brousseau G. , *Theory of didactical situations in mathematics*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997, Edited and translated by N. Balacheff, M. Cooper, R. Sutherland, and V. Warfield.

Brousseau G., *Fondaments et méthodes de la didactique des mathématiques*, Recherchesen didactique des mathématiques, Paris, 1986.

Coen S., *Ascoltando Giovanni Prodi*, Bollettino U.M.I., La Matematica nella Società e nella Cultura, Serie VIII, Vol III-A, Agosto 2000, 147-173.

Cutrera M., Lo Verde D., Aritmetica (epistemologia del numero naturale, Trasposizione didattica di approcci al numero, Analisi a-priori di una situazione didattica), Sigma Edizioni (E-Mail: Promopress@imd.it), Palermo, 1999.

Cutrera M., Spagnolo F., *Le difficoltà di apprendimento in matematica nelle immagini dei docenti,* L'Educatore, XXXVIII, 4(1990), 27-30, Fabbri Editore.

D’Amore B., Oliva P., Numeri. Teoria, storia, curiosità, giochi e didattica nel mondo dei numeri, Milano, Angeli, 1994.

D'Amore B., Problemi. Pedagogia e psicologia della matematica nell'attività del problem solving, Milano, Angeli, II ed., 1996.

D'Amore B., Elementi di Didattica della Matematica, Pitagora Editrice, 1999.

Dantzig T., Number. The language of Science, Macmillan, New York, 1954.

Davis B., *Listening for differences: An evolving conception of mathematics teaching*, Journal for Research in Mathematics Education 28 (1997), 355-376.

De Bellis V. A. and Goldin G. A., *Aspects of affect: Mathematical intimacy, mathematical integrity*, Proceedings of the 23rd International Conference of the International Group for the Psychology of Mathematics Education (Haifa, Israel) (O. Zaslovsky, ed.), vol. 2, 1999, pp. 249-256.

De Bellis V. A., *Mathematical intimacy: Local affect in powerful problem solvers*, Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Raleigh, North Carolina) (S. Berenson et al., ed.), vol. 2, 1998.

Di Leonardo M. V., Marino T., Spagnolo F., *The "0" is it an obstacle?*, Amman (Giordania), 18-23 Novembre 2000, International Conference on Mathematics Education iinto the 21st Century.

Di Leonardo M.V., Marino T., Spagnolo F., *Lo "0 é un ostacolo?"*, Quaderni di Ricerca in Didattica, n.8, 1999.

Ellis D. B., *Becoming a master student*, Houghton Mifflin, New York, NY, 1985, Updated and reprinted in January 2000.

Ernest P., *Social constructivism as a philosophy of mathematics*, SUNY Press, Albany, NY, 1998.

Even R., *Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept*, Journal for Research in Mathematics Education 24 (1993), 94-116.

Gras R., *Les fondaments de l'analyse statistique implicative*, Quaderni di Ricerca in Didattica, n. 9, Palermo, 1999.

Hannula M., *Cognitive emotions in learning and doing mathematics*, Proceedings of the Eighth European Workshop on Research on Mathematical Beliefs - MAVI-8 (University of Cyprus, Nicosia, Cyprus) (G. Philippou, ed.), 1999, pp. 57-66.

Lakatos I., Prove e confutazioni, Torino, Boringhieri, 1980.

Malisani E., *Incidenza di diversi tipi di struttura logica di un problema sulla condotta di risoluzione*, Quaderni di Ricerca in Didattica del GRIM, Palermo, 3 (1992), 65-86.

Marchetta U., Spagnolo F., *"Analisi delle capacità di valutazione della misura di superfici piane: Indagine sperimentale su due gruppi di studenti di ambiente urbano e rurale"*, Stampatori Tip., Palermo, Luglio 1977.

Margolinas C., *Le milieu et le contrat, concepts pour la construction et l'analyse de situations d'enseignement*, in Actes de l'Université d'été, IREM Clermont-Ferrand, 1998.

Marino T., Spagnolo F., *Gli ostacoli epistemologici: Come si individuano e come si utilizzano nella ricerca in Didattica della Matematica,* L'insegnamento della Matematica e delle Scienze Integrate, Centro U.Morin Paderno del Grappa, vol.19B, n.2(1996), 129-152.

Mariotti M. A., *La démonstration en mathématiques*, Actes de la conférence "Enseigner les Mathématiques", Enseigner les mathématiques, Juillet 2000, Grenoble.

Mason J., Spence M., *Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment,* Educational Studies in Mathematics 28 (1999), 135-161.

Mostacci C., Spagnolo F., *Svantaggio e apprendimento della matematica,* L'Educatore, XXXVII, 20(1990), 32-36, Fabbri Editore.

Nastasi P., Scimone A., *Una polemica catanese degli anni 30 sulla trattazione dei numeri decimali,* La Matematica e la sua didattica, VI, 1(1992), 29-35.

Niceta F., Profumo M., *L'informatica a scuola: ruolo, aspettative,esperienze,*Quaderni di Ricerca in Didattica del GRIM, Palermo, 2 (1991), 55-75.

Pedemonte B., *Some cognitive aspects of the relationship between argumentation and proof in mathematics*, Proceedings of the 25th conference of the international group for the Psychology of Mathematics Education Utrecht, Editor: van den Heuvel-Panhuizen M., 1999.

Santos-Trigo M., *On the implementation of mathematical problem solving instruction: Qualities of some learning activities*, CBMS Issues in Mathematics Education: Research in Collegiate Mathematics Education III 7 (1998), 71-80.

Savoja M.G., Bufalino M., Picardo L., Matematica, un approccio tematico, Salvatore Sciascia Editore, Caltanissetta, 2000.

Schoenfeld A. H., *Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics*, Handbook of Research on Mathematics Teaching and Learning (D. A. Grouws, ed.), Macmillan, New York, NY, 1992, pp. 334-370.

Scimone A. et alii, *Il metodo dimostrativo in Euclide e in Cartesio,* Insegnare, 1 (1992), pp. 18-21.

Scimone A., Spagnolo F., (a cura di) *Atti dei Convegni dell'A.I.C.M. n.1, Michele Cipolla, la figura e l'opera: Le conferenze*, Palermo, 1998.

Selden A., Selden J., Hauk S., and Mason A., *Why can't calculus students access their knowledge to solve nonroutine problems?*, CBMS Issues in Mathematics Education: Research in Collegiate Mathematics Education IV (2001).

Sfard A., *On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin*, Educational Studies in Mathematics 22 (1991), 1-36.

Spagnolo F., *"I nuovi programmi per la scuola elementare: Alcune considerazioni sull'insegnamento della matematica"*, "Funzione Docente" Organo dell'IRRSAE-Sicilia, Dic. 1989, Anno VIII.

Spagnolo F., *"La comunicazione matematica assistita dal computer: Il CABRI-Geometria"*, Convegno nazionale Inforscuola, Palermo Dicembre 1996.

Spagnolo F., *"L'intervento della nozione di Operatore negli ampliamenti numerici"*, L'Educazione Matematica, Cagliari, Anno VII, serie II, I, 2(1986), 169-185.

Spagnolo F., *"Sull'Impostazione di certi metodi risolutivi dei problemi nella tradizione cinese"*, L'Inseganmento della Matematica, IX, 8(1986.), 81-91.

Spagnolo F., *"Un modèle théorique-expérimental pour l'individuation des obstacles épistemologiques"*, (Presentato alla Scuola estiva per ricercatori in didattica delle matematiche, Houlgate, Francia), Pubblicato su: A multidimensional approach to learning in mathematics and sciences, University of Cyprus, Cyprus, Nicosia, 1999.

Spagnolo F., *A theoretical-experimental model for research of epistemological obstacles*, Cairo (Egitto) 14-18 Novembre 1999, International Conference on Mathematics Education into the 21st Century.

Spagnolo F., *Gli IREM in Francia*, Lettera PRISTEM, Marzo 1991, Milano.

Spagnolo F., *I lavori di M. Cipolla sull'assioma di Zermelo,* Atti del convegno dell'AICM n.1, Palermo 1998.

Spagnolo F., *I linguaggi della matematica, Atti* del *Convegno* "Le *tre* culture: *Umanistica*, Scientifica, *Multimediale*", Editrice Scolastica Italiana, Marsala (1996), 123-130.

Spagnolo F., Insegnare le matematiche nella scuola secondaria, La Nuova Italia Editrice, Ith Ed. 1998, 1999 (the first reprint).

Spagnolo F., *L'analisi a-priori e l'indice di implicazione statistica di Gras*, Quaderni di ricerca in didattica GRIM, n. 7, Palermo, 1997.

Spagnolo F., *La recherche en didactique des mathématiques: un paradigme de référence*, Zbornik, Bratislavskeho seminara z teorie vyucovania matematiky, Bratislava, 1999.

Spagnolo F., La ricerca in didattica nel lavoro dell'aula, Convegno nazionale organizzato dall'IRRSAE-Sicilia sul tema Ricerca e d Autonomia, Palermo, Dicembre 1999.

Spagnolo F., La ricerca in didattica, Sito internet della Treccani, ITER, Dialogare, 1999. (http://www.treccani.it/site/iniziative/scuola/documenti/Spagnolo.pdf)

Spagnolo F., *Le competenze di lunga durata e i saperi irrinunciabili nel primo anno della scuola secondaria superiore*, Convegno Regionale CIDI, Palermo Febbraio 2000.

Spagnolo F., *Lo sviluppo del pensiero algebrico nel periodo Federiciano: Una disputa tra Abacisti e Algoritmisti,* Annuario della Scuola Altavilla, Palermo, (1995), 71-75.

Spagnolo F., Margolinas C., *Un ostacolo epistemologico rilevante per il concetto di limite: Il postulato di Archimede,* La Matematica e la sua Didattica, 4(1993), 408-427, ed. Pitagora, Bologna.

Spagnolo F., *Michele Cipolla e la didattica delle Matematiche,* Atti del convegno dell'AICM n.1, Palermo 1998.

Spagnolo F., *Obstacles Epistèmologiques: Le Postulat d'Eudoxe-Archimede, Tesi di Dottorato di Ricerca,* Università di Bordeaux I (Francia), Luglio 1995, Pubblicata dall'Atelier National de Réproduction des thèses Microphiches (BP - 38040 Grenoble Cedex 9, Francia) e dai Quaderni GRIM di Palermo per la diffusione in Italia (Supplemento al n.5, 1995).

Spagnolo F., *Scholastic leakage in professional schools at Bagheria (Palermo).* Pupils aged 13-15, Cairo (Egitto) 14-18 Novembre 1999, International Conference on Mathematics Education into the 21st Century.

Spagnolo F., *Semiotic and hermeneutic can help us to interpret teaching/learning?*, Palm Cove (Cairns, Australia), Agosto 2001, International Conference on Mathematics Education into the 21st Century.

Spagnolo F., *The role of history of Mathematics in Research in Mathematics Education,* Amman (Giordania), 18-23 Novembre 2000, International Conference on Mathematics Education into the 21st Century.

Spagnolo F., *Una prospettiva sulle ricerche riguardanti la didattica della statistica,* Convegno nazionale CIRDIS, Palermo, Settembre 1997, Induzioni, n.2, 1998.

Spagnolo F., Valenti S., *"Errori matematici: Un'occasione didattica",* L'insegnamento della Matematica (Paderno del Grappa), VII, 1(1984), 18-36.

Spagnolo F., Valenti S., *"Le riflessioni sulle proprietà dei Naturali, come spunto e metodologia didattica",* L'Ins. della Matematica, X, 6 (1987), 556-574.

Szydlik J. E., *Mathematical beliefs and conceptual understanding of the limit of a function*, Journal for Research in Mathematics Education 31 (2000), 258-276.

Trognon A., *How does the process of interaction work when two interlocutors try to resolve a logical problem?*, Cognition and Instruction 11 (1993), 325-345.

Zahorik J., *How Children Learn: Encouraging-and Challenging-Students' Understandings*, Education Leadership, (1997) 30-32.

Zaskis R. and Gunn C., *Sets, subsets, and the empty set: Students' constructions and mathematical conventions,* Journal of Computers in Mathematics and Science Teaching 16 (1997), 133-169.