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THE CONCEPT OF VARIABLE IN THE PASSAGE FROM THE ARITHMETICAL LANGUAGE TO THE ALGEBRAIC LANGUAGE IN DIFFERENT SEMIOTIC CONTEXTS

Doctoral Thesis by

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To Carolina, Andrea and Andrés

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INTRODUCTION

STORY OF THE PRECEDING WORKS

The aim of the experimental research effected in Malisani (1990, 1992) was to study the cognitive performance of the students between the ages of 14-15 in the assignment of resolution of algebraic and geometric problems. We wanted to know more specifically how the different kind of logical structure of a problem affect the resolving performance (types of solutions, steps of the resolving algorithm and errors); and if it is verified that the isomorphism of logical structures in the algebraic and geometric contexts does not implicate isomorphism in the performance of the students.

The problems belonging to the algebraic context are concerned with the resolution of equations of first degree with one unknown, of the type: $y = k_1 \cdot (x - k_2)$ [1] for a determined value of *x* or *y*, being k_1 and k_2 positive constants and such that $k_2 < x$. In this type of equation the variables *x* and *y* and the constants k_1 and k_2 can represent any elements, therefore, they have only a *formal significance*.

The geometric problems consider, instead, the application of the "theorem of the sum of the interior angles of a convex polygon" that has equation: $s = 180°. (n - 2)$ [2], in which every variable and every constant represent determined geometric objects or relations among these objects. For example: *s* is the sum of the interior angles of a convex polygon, *180°* is the sum of the interior angles of a triangle, *n* is the number of sides of a polygon, $n - 2$ is the number of triangles that are determined in the polygon tracing the diagonals from a vertex to the others. In this case the variables and the constants have a *geometric significance*.

We observe that the equations [1] and [2] are isomorphic with regard to their logical structure, because if we fix a variable (for example: *y* and *s*) they require the same steps for their resolution. These equations are of *arithmetical* kind, using the terminology of Gallardo and Rojano (1988), because to resolve them it is necessary to manipulate only the numerical values of the equation (actions in the arithmetic context) and not the quantities to find or unknowns.

From the results obtained we deduce that the *geometric significance* of a problem:

- Ø would affect partially the achievement of correct answers, only in those problems that introduce greater logical difficulty (7 or more different steps).
- Ø influences positively the economy of steps of the resolution.
- Ø affects the number and the kind of errors made by the pupils.

Therefore, the intuitive support that the geometric problems offer and a good comprehension by the pupils of the conceptual relations between the elements that compose the equation of the theorem favour the saving of steps in the resolution and decrease the number of errors, above all the errors of calculation. That is, the saving of steps in the resolution does not always implicate a greater quantity of errors, contrary to what is usually supposed.

At that time the formulated conclusions affirmed that the resolution of problems that involves equations requires something more than the domain of certain operations (arithmetical and algebraic); the subjects must have the necessary conceptual knowledge to understand and to represent conveniently the information of the problem.

The individualization and the diagnosis of the errors effected in Malisani (1990) and Malisani (1992) led us to deepen the principal works of research carried out in these last decades on the cognitive processes associated with the learning of algebra (Matz, 1982; Kieran & Filloy, 1989; Kieran, 1991; Gallardo & Rojano, 1988; Lee & Wheeler, 1989; Chiappini and Lemut, 1991; Herscovics & Linchevski, 1991). These studies deal with matters concerning the difficulties and obstacles that the beginner students of algebra meet, regarding the conceptual changes necessary in the transition from the arithmetic thought to the algebraic thought. These changes refer especially to the concept of equality, the conventions of notation and the interpretation of the concept of variable. We also examined the results of some researches on the interpretation and simplification of algebraic expressions and the resolution of equations and algebraic problems (Malisani, 1993).

Successively we carried out a research on the individualization, diagnosis and classification of errors in the resolution of algebraic and geometric problems that involve arithmetical equations of first degree (Malisani, 1993). Even if the resolving procedure of the algebraic and geometric problems is isomorphic two by two, the results we obtained point out that the students do not make the same types of errors. For example, the percentages of errors related to the use of the equal sign and to the transport of terms from a member to the other of the equation are lower in the geometric context. On the other hand, the percentage of errors concerning the formulation of an answer consistent with the meaning of the variables that represent the results is lower in the algebraic context.

Several experimental studies (Harper, 1987; Sfard 1992) seem to confirm that some difficulties of the students can be grouped around some obstacles met in history (Cfr. Arzarello, pp. 7-8). The elements that allow to identify these obstacles have to be searched in the analysis of the resistances emerged in the historical development and in the debates that have overcome them. But history alone is not sufficient; the historical epistemological analysis must be completed by a study of the grounding of mathematics (Spagnolo, 1995, pp. 18-19). If we consider this point of view, it could be useful to take into consideration the history of the algebraic thought that leads us to go over the steps of the construction of the algebraic language.

The historical analysis effected in Malisani (1996, 1999) shows that for many centuries algebra stayed behind in comparison with geometry and that the construction of the symbolic language was too slow and difficult. The lack of an adequate algebraic language conditioned the evolution of the resolutive procedures. The ancient mathematicians often explained these procedures through their application to some examples. They used other languages: natural, arithmetical and geometric.

PURPOSE OF THE RESEARCH

To deepen the conclusions previously expressed a new research is proposed. It is founded on the necessity of studying and analyzing the obstacles that the students meet in building up and assimilating certain concepts, in the passage from the arithmetical thought to the algebraic thought.

From some effected studies (Matz, 1982; Wagner, 1981, 1983) it emerges that the point of critical transition between the two kinds of thought is the introduction of the concept of variable. This notion could take on a plurality of conceptions: ***generalized number*** ($2+4 = 4+2$ is generalized with $a+b = b+a$); ***unknown*** (resolution of equations); ***"something that varies"*** (relation among quantities, functional aspect); ***entirely arbitrary sign*** (study of the structures); ***register of memory*** (in computer science) (Usiskin, 1988).

The study of the various aspects that this concept can take constitutes a very wide field of research and requires different confirmations, provided by historical-epistemological

and experimental investigation and by setting up the didactical situations built *ad-hoc*. Therefore it is necessary to circumscribe the dominion of survey.
The aim of this research is to study some characteristics of the period of transition from the arithmetical language to the algebraic language. We want to analyze if the different conceptions of variable are evoked by the students in the resolution of problems and if the notion of variable in its double aspect – unknown and relational-functional – represents an obstacle for the pupil.

APPLICATIONS

This research is set as a contribution to Mathematics Education, particularly, to the studies that are being carried out within the GRIM, on the epistemological and didactical obstacles concerning the passage from the arithmetic language to the algebraic language.
This experimental study will supply us some necessary tools to analyze in details whether the concept of variable, in its different aspects, represents an epistemological obstacle or an obstacle of didactical origin.
We could also determine how the semiotic context influences the conceptions of variable from the pupil's point of view. We could study more specifically the interaction of other contexts –natural language, geometric language, perceptive schemes, etc.– with the operating of the pupils in a strictly algebraic context.
Moreover, it will be possible to draw some tools to set up appropriate a-didactical situations and to get at a more deep comprehension of the communicative processes.
From a general point of view, this research can help to clarify matters concerning the representations of the arithmetical and algebraic knowledge and the operating in the resolution of problems from the pupil's point of view.

STRUCTURE OF THE THESIS

The thesis is composed by five chapters. The first one is about history and introduces the construction of the algebraic language and the evolution of the methods and of the strategies of resolution of equations in the periods that preceded the formalization.
The second chapter has the purpose to study some aspects of the period of transition from the arithmetical language to the algebraic language. We want to analyze if the pupils evoke the different conceptions of variable in the resolution of problems and if

the natural language and/or the arithmetic language prevail as symbolic systems in absence of an adequate mastery of the algebraic language.

The third chapter intends to study the relational-functional aspect of the variable in problem-solving, considering the semiotic contexts of algebra and analytical geometry. The aim is to investigate whether the notion of unknown interferes with the interpretation of the functional aspect, and whether the procedures in natural language and/or the arithmetical language prevail as resolutive strategies in lack of an adequate knowledge of the algebraic language.

The aim of the fourth chapter is to analyze how the conceptions of unknown and of functional relation are activated and used in the process of resolution of a problematic situation. We want also to study the process of translation from the algebraic language into the natural one and the representation of the syntax-semantics relation within the algebraic code.

In the fifth chapter the final conclusions of the thesis are presented.

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CHAPTER ONE:

HISTORICAL EVOLUTION OF THE ALGEBRAIC LANGUAGE

1.1 INTRODUCTION

The preliminary study of the epistemological and historical-epistemological representations is fundamental to be able to deal with the experimental contingency. In fact, the more deepened this analysis will be, the greater the possibility will be of inferring the phenomenon of teaching/learning and of being able to reproduce it under other analogous conditions (Spagnolo, 1998). If we consider this point of view, it could be useful to take into consideration the history of the algebraic thought that brings us to go over the steps of the construction of the algebraic language again.

The historical analysis shows that for many centuries the algebra stayed behind in comparison with the geometry and that the construction of the symbolic language was very slow and difficult. And then, in absence of an adequate language and an appropriate knowledge on the numerical sets, how were the different types of equations represented? Which algorithms of resolution were used? How did the arithmetical and geometric knowledge influence the development of the algebraic language and the resolutive techniques? How has the historical conception of equation developed? Was it possible to classify the problems according to the algorithms of resolution, in absence of symbolism or with a very rudimentary one? What is the origin of the notions of unknown and of variable? How are these concepts developed? In this chapter we will try to give answer to some of these questions.

The algebraic thought is favourite by the use of a suitable symbolism and therefore, not only the history of the concepts is very important in the history of algebra, but also that of the systems of symbols used for expressing them (Arzarello *et al.*, pp. 10-11). According to Nesselman three distinct periods can be individualized:

1 - RHETORICAL PHASE: anterior to Diophantus in Alexandria (250 AD), in which the natural language is used exclusively, without resorting to any sign.

2 - SYNCOPATED PHASE: from Diophantus up to the end of the XVI century, in which some abbreviations for the unknown and the relations of more frequent use have been introduced, but the calculations are performed in natural language.

3 - SYMBOLIC PHASE: introduced by Viète (1540-1603), in which the letters are used for all the quantities and the signs to represent the operations, the symbolic language is utilised not only to resolve equations but also to demonstrate general rules.

Some most recent studies point out that, in the historical route, it is not possible to individualize in a precise way certain, distinct and separate phases that mark the development of the algebraic thought. Every phase certainly has not supplanted suddenly the preceding one, the passage was slow and gradual (Cfr. Malisani, 1996).

According to Ferreri and Spagnolo (pp. 90): "The study of the historical conceptions is the study of the meanings connected to a certain language in a determined historical period. A language arises with semantic ambiguities as well as wealth of meanings within the grammar. When the language is formalized, a meaning is assigned to every formula and the preceding meanings are lost".

The goal of this work is to study the construction of the algebraic language with its semantic ambiguities and its wealth of meanings, in relation to the evolution of the methods and strategies of resolution of equations, in the two historical periods that precede the formalization: rhetorical and syncopated. Indeed, the passage between a meaningful semiotic field *"the arithmetic"* and the attempt to set a new language *"the algebra"*, relative to a certain class of problems *"the resolution of equations"*, is found precisely in the phase of transition between the arithmetical thought and the algebraic thought. The epistemological obstacles are actually tied to this passage (Spagnolo, 1995, pp. 81; Marino e Spagnolo, pp. 131).

This chapter is divided into five parts. In the first one, we introduce the historical construction of the symbolic language of algebra; in the second one, we describe the principal methods of resolution of equations used up to 1500; in the third one, we analyze the incidence of certain aspects of the arithmetical language in the development of the algebraic language; in the fourth one, we illustrate the different levels of generality of the methods of resolution; and in the fifth one, we show the historical evolution of the concept of variable.

1.2. THE SYMBOLISM

The analysis of the historical development of algebra demonstrates that the construction of the symbolic language is very slow and difficult; some periods show progressive improvement while others, instead, regression and paralysis. For example, the Babylonians (2000 B.C.), the Egyptians (1700 B.C.), the Greek (600-200 B.C.) and the

Chinese (300 B.C.-300 A.C.) used exclusively the natural language without resorting to any sign. Historians recorded isolated attempts of introducing some name or abbreviation to represent the unknown, but these proofs have not been effected in systematic manner(1).

Diophantus (250 A.D.) introduced, for the first time in the History of Mathematics, of the abbreviations (Greek letters) to represent the unknown of an equation and its powers (Cfr. Kline., pp. 162-163):

$x \rightarrow \zeta$		called	*"the number of the problem or arithme"*
$x^2 \rightarrow \Delta^{\Upsilon}$			*"square" o "power"*
$x^3 \rightarrow K^{\Upsilon}$			*"cube"*
$x^4 \rightarrow \Delta^{\Upsilon}\Delta$			*"square - square"*
$x^5 \rightarrow \Delta K^{\Upsilon}$			*"square - cube"*
$x^6 \rightarrow K^{\Upsilon}K$			*"cube - cube"*
$1/x \rightarrow \zeta^{\chi}$			

Diophantus marked the addition by writing the terms one after the other, for the subtraction he used the symbol /|\ and for the equality ι^{σ}. There were no symbols to represent the multiplication, the division and the generic coefficients. He effected the calculations in *natural language* and he wrote the solution in a continuous text. It is interesting to observe that Diophantus introduced an important concept in Algebra: the *"arithme"* or *the number of the problem* that represents "an undetermined quantity of units", that is the *unknown* of the problem (Ver Eecke, pp. 2; Radford, pp. 43).

Beginning from the 7th century the Indians created a quite efficient algebraic symbolism that allowed them to develop new procedures of resolutions of equations. Brahmagupta (born in 598) in his work *Brahmasputasiddhanta*, uses some abbreviations to represent the unknown and its powers (Cfr. Bortolotti, 1950, pp.. 637):

$x \rightarrow$ *ya* [first syllable of the word *yavattavat (so much-as)*]
$x^2 \rightarrow$ *va*
$x^3 \rightarrow$ *gha*
$x^4 \rightarrow$ *vava*
$x^9 \rightarrow$ *ghagha*
$x^{1/2} \rightarrow$ *ka* [first syllable of the word *karana (square root)*]

The Indians did not use any symbol to denote the addition and the product (that was represented writing the two factors one after the other); for the subtraction, instead, a point was written above the subtracting whereas for the equality of two quantities they just wrote the two members in two consecutive lines. When several unknowns were present in a problem, one of them was represented with the syllable *ya* and the others with objects of different colours: in practice they used the first syllables of the words related to the each colour. This symbolism, however rudimentary, is sufficient to classify the Indian algebra as *almost-symbolic*; in this sense, it is surely superior to the syncopated algebra of Diophantus. The problems and the solutions were written in this syncopated style, but the different passages were not accompanied with motivations or demonstrations.

The Arabs (≈800-1300 A.D.), heirs of the Greek's and Indian's works, did not use symbols. Some authors like al-Khowârismî (≈780-≈850) used some particular names to represent the unknown and its powers, but in general they developed an algebra entirely rhetoric and this represents a step back in comparison with the algebra of Diophantus and the Indian one.

Leonardo Pisano(2) (≈1170 - 1250), called Fibonacci, introduced in Europe the Indian-Arabic numeration system and the arithmetical procedures used by the Arabs and Indians. Thus the characteristics of the Arabic algebra spread in Europe, exerting a strong influence for more than three centuries. We observe that in the work of Leonardo and in the essays of abacus of the Middle Ages, for example in the *Trattato d'Algibra*(3) (Anonymous of the 14th century) the algebraic developments use fundamentally the natural language. It is important to underline that a certain tendency toward the symbolism appears in the *Trattato d'Algibra*, in that the unknown and its powers are called with some particular names:

x	*cosa (o chosa)*
x^2	*censo*
x^3	*chubo*
x^4	*censo di censo*
x^5	*chubo di censi*
x^6	*censo di chubo.*

The abbreviations used in the 16th century are derived precisely from these words. In the work of Pacioli (1445-1514?) we observe a meaningful progress as for the use of the syncopated language. This author performs the calculations in natural language, but he represents the unknown and its powers (up to the twentieth) through names and particular abbreviations, for example (Loria, pp. 476):

x	*cosa*	*co*	
x^2	*censo*	*ce* o *Z*	
x^3	*chubo*	*cu* o *C*	
x^4	*censo di censo*	*ce ce*	
x^5	*primo relato*	*p° r°*	etc.

Pacioli also used other *abbreviations* as *p* (for the sum), *m* (for the subtraction or to mark a negative number) and *ae* (for equal: *aequalis*), R^2 and R^3 (crossed by an oblique bar) to indicate the quadratic and cubic roots.

Bombelli (≈1526-≈1572) is responsible for an authentic transformation of the algebraic language with the introduction of a special symbol to represent the unknown and its powers: a semi-circumference on which a number was written, this denotes the exponent of the power (in this article, to simplify the notation, the semi-circumference will be denoted with a circumference):

x	*tanto*	•	
x^2	*potenza*	‚	
x^3	*cubo*	ƒ	
x^4	*potenza di potenza*	„	
x^5	*primo relato*	…	and so on.

This represents an important evolution of the symbolic language, because most of the changes of notation effected until that moment were essentially abbreviations of the natural language. Bombelli uses this "S*yncopated-Advanced"* symbolism, a combination between *natural language* and *algebraic symbolism*, to formulate the rules of the numerical operations and with the polynomials and the procedures of resolution of equations. This symbolism shares precisely the characteristic of *"auto-explanation"* with the symbolic algebra of Viète (1540-1603), although Bombelli always needs to accompany the developments carried out by its rhetorical version and he shows the validity of the express equalities, in the different types of equations, through the geometric constructions. This demonstrates that the syncopated-advanced language used by Bombelli is not self-sufficient, because it is necessary to apply other languages, natural and geometric, that are richer semantically, to complete the communication (Colin and Rojano, pp. 141 - 142).

It is important to observe that many changes of notation effected until the 16^{th} century were accidental and it is clear that the researchers of this epoch were not able to appreciate the enormous importance that would have meant the symbolism for algebra.

Almost all the symbols currently known have been introduced between 16th and 17th century, but the process was very slow, the symbolic algebra did not supplant suddenly the syncopated one.

Some authors (Kline, pp. 303; Loria, pp. 468) think that the Germans introduced the signs + and - to denote the weights in excess or in defect of the cassettes; these signs were adopted then by the mathematicians Widman (15th century) and Stifel (1486?-1567). Rapisardi (pp. 169), instead, attributes the invention of these signs to Leonardo da Vinci, (1452-1519). The sign = had introduced in 1557 from Recorde (1510-1558) that wrote the first English essay of algebra. Viète (1540-1603), who at first used the word *aequalis*, then adopted the symbol ~ to indicate equality; Descartes (1596-1650), instead, used α. Oughtred (1574-1660) invented the sign $\times$ of the product and Harriot (1560-1621) used the signs > and < to denote the inequalities. The round brackets appear in 1544, the square and brace brackets, used by Viète, dates back to about 1593.

The square root $\sqrt{\ }$ and cubic root $\sqrt[3]{c}$ appear in the 17th century with Descartes (Cfr. Kline, pp. 304).

The exponents have been introduced gradually. Chuquet (1445?-1500?) wrote 8^3, 10^5, 12^0 e 7^{1m} to indicate $8x^3$, $10x^5$, 12 e $7x^{-1}$ in his work *Triparty*. Bombelli used a semi-circumference on which he wrote the exponent of the power and Stevin (1548-1620) also used the fractional exponents: *1/2* for the square root and *1/3* for the cubic root.

The most meaningful change in the construction of the algebraic language was produced by the symbolism of Viète. This author was the first to adopt deliberately and systematically the letters to represent all the quantities (the unknown, its powers and the generic coefficients). He usually used the consonants for the known terms and the vowels for the unknown; he employed the symbolic language to solve equations, but also to demonstrate general rules. Viète called his symbolic algebra "*specious logistic*", in contrast with the *"numerical logistic"*: he considered that algebra is a method to operate on the kinds or the forms of things whereas arithmetic, *the numerical* one, is interested in numbers. In this way algebra became the study of the general types of forms and of equations, because what is applied to the general case is valid in all the infinite particular cases (Kline, pp. 305).

1.3. METHODS OF RESOLUTION OF EQUATIONS

The aim of this section is to introduce a wide variety of methods of resolution of equations and to show the influence of the arithmetical and geometric knowledge on the evolution of the resolutive techniques. The procedures have been grouped according to the type of equations: first, second and third degree and indeterminate equations. At the end of the section we describe synthetically the methods used in Europe by Fibonacci, by a book of the abacus (representative of the Mediaeval and Renaissance algebra) called *Trattato d'Algibra,* and by the algebraists of the 16^{th} century.

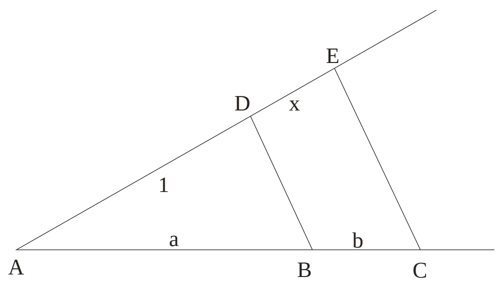
1.3.1 METHODS OF RESOLUTION OF THE EQUATIONS OF FIRST DEGREE

Here follows a description of the geometric procedure of Euclid, of the methods of the false position and of the *"regula infusa"*.

1.3.1.1 THE GEOMETRIC PROCEDURE OF EUCLID

The Euclid's "*Elements"* contains some important results of modern algebra, but treated geometrically, for example: the resolution of equations of first degree.

The proposition 12 of the 6^{th} Book of the Elements (1930, pp. 107) asks: "Find the fourth proportional from three given segments".



$A\,B : BC = AD : DE$

Figure 1

The application of this proposition allows to solve "geometrically" equations of first degree of the type $ax = b$ with positive coefficients, considering as segments: $AB = a$, $BC = b$, $AD = 1$ e $DE = x$.

1.3.1.2 THE METHODS OF THE FALSE POSITION

During the Middle Ages these procedures were called with the name of *regula al-chataim* (word of oriental origin) or *regula falsorum*. Their origin is very ancient and is found precisely in the Egyptian and Chinese mathematicians. These techniques were often used by the Indians and by the Arabs in the resolution of problems and they appear in most of the texts of arithmetic from the Middle Ages until the beginning of our age (Cfr. Guillemot, pp. 1).

The methods of the false position were applied to solve equations of first degree with one unknown, and, in some cases, systems of linear equations and equations of second degree. There are two kinds of methods: simple false position and double false position.

1.3.1.2 .1 THE METHOD OF THE SIMPLE FALSE POSITION

This procedure consists in assigning a *particular value to the unknown* and in effecting the necessary calculations to obtain the exact result: from here the name of simple *false position*. This rule was applied to solve linear problems, therefore the concept of direct proportionality is used basically in the calculations.

The origin of this method is found in the papyrus Rhind (ca. 1700 B.C.). His author, Ahmes, applies it in the problem-solving of the type: $x + (1/n)\, x = b$, with n and b positive integers and $x \in E$, being E the numerical set used by the Egyptians and composed of the positive integers, of the fraction $2/3$ and of the fractions of the type $1/n$ with n positive integer[(4)].

For example, the problem 24 of the papyrus asks "find a quantity that if increased by its seventh part is equal to 19". The problem translated to the symbolic language of modern algebra corresponds to the equation: $x + (1/7)\, x = 19$. Ahmes resolves it in this way:

1 - He adopts *the false position 7*, that is $x = 7$, and then he obtains $7 + (1/7)\, 7 = 8$ rather than 19.

2 - He divides 19 into 8 and multiplies the result by 7, namely he applies the direct proportionality: $19 : 8 = x : 7$ and he obtains as result $x = 16 + 1/2 + 1/8$ (Cfr. Guillemot, pp. 3).

The manipulation of the fractions of the set E resulted enough complex for the Egyptians, therefore they tried to avoid them effecting the lesser number possible of calculations. The method of the simple false position was applied to the preceding problem; this allows precisely replacing the elementary division of 19 by 8 with that of 19 by $(1 + 1/7)$, very difficult when using the Egyptian rules. Besides in every equation of the type: $x + (1/n)\, x = b$, Ahmes chooses the false position $x_0 = n$, so he obtains in the first member an integer value: $n + 1 = b_0$, after he divides b by b_0 and he multiplies the result by x_0, that is: $x = \frac{b}{b_0} \cdot x_0$. Thus the author chooses to work with integers. This demonstrates that the difficulties found in effecting the calculations with the fractions led the ancient mathematician to search for alternative methods, to solve the proposed problems more easily.

1.3.1.2.2 THE METHOD OF THE DOUBLE FALSE POSITION

This procedure consists in assigning *two particular values to the unknown* (from this the name of *double false position*), in effecting the necessary calculations to find the committed errors (in replacing these values) and, then, in applying the formula of linear interpolation.

The mediaeval authors do not succeed in establishing with exactness the field of application of every method of the false position. According to Pellos (1492): "The subtlest and more difficult problems can be resolved with the method of the double false position; their resolution without this rule would represent a big effort...". Often the proposed examples can be resolved also, through the application of the method of the simple false position. From an accurate analysis of the texts it is possible to determine that *the subtlest and most complex problems* frequently correspond to the resolution of: equations of first degree in which the unknown is found in both members, systems of linear equations and equations of second degree (approximately) (Guillemot, pp. 12 - 13).

The Arabic Al-Qalasadi (1423-1494/5) and Beda Eddin (1547-1622) propose simple problems that could be resolved applying this rule. For example: "Find a number that increased by *2/3* of itself and by *1* is equal to *10"*. Algebraically it corresponds to the equation: $x + (2/3)\,x + 1 = 10$ with $x \in Q$, that the author resolves this way:

1 - He adopts the false position: $x_1 = 9$, therefore the first member is equal to *16* and the difference with the second member is $d_1 = 6$.

2 - He considers the false position: $x_2 = 6$, then the first member is equal to *11* and the difference is $d_2 = 1$.

3 - He applies the formula of linear interpolation:

$$x = (x_2 d_1 - x_1 d_2) / (d_1 - d_2) = (6 \,.\, 6 - 9 \,.\, 1) / (6 - 1) = 5 + 2/5.$$

This procedure allows to resolve equations of the type $ax = b$ with $x \in Q$, and it can be translated to the modern algebraic language this way:

1. We adopt the false position x_1 and we get to $a\,x_1 = b + d_1$ [1]
2. We suppose the false position x_2 and we find to $a\,x_2 = b + d_2$ [2]

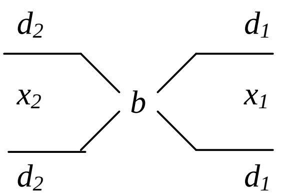
d_1 and d_2 are called *differences or errors*, obtained in considering x_1 and x_2 like values of the unknown.

3. We resolve the system composed by the equations [1] and [2] in function of a and b and we get:

$$a = (d_1 - d_2) / (x_1 - x_2) \quad \text{e} \quad b = (x_2 d_1 - x_1 d_2) / (x_1 - x_2). \qquad [3]$$

4 - Since $x = b/a$ we find: $$x = (x_2 d_1 - x_1 d_2) / (d_1 - d_2). \qquad [4]$$

Because the Arabs did not dispose of the formula, Al-Qalasadi used the idea of the plates of a scale to introduce in a clearer and more precise way the performed algorithm. Other authors used a graphic scheme, in which they represented in a different way the positive and negative differences (Loria, pp. 345-346):

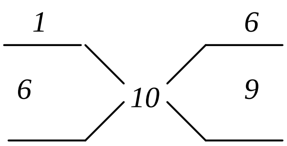


position of the differences with positive sign

position of the differences with negative sign

Figure 2

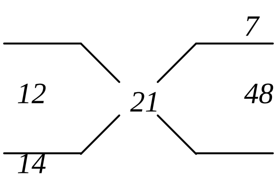
The preceding example corresponds to the following scheme:



$x = (6 . 6 - 9 . 1)/(6 - 1) = 5 + 2/5.$

Figure 3

Al-Qalasadi proposes the problem: "The sum of the third part and the fourth part of a number is equal to 21. Which is the number?". The equation to be solved is: $x/3 + x/4 = 21$; considering $x_1 = 48$ and $x_2 = 12$ he gets respectively the differences $d_1 = 7$ and $d_2 = -14$, therefore the corresponding scheme is the following:



$x = (12 . 7 + 48 . 14)/(7 + 14) = 36 .$

Figure 4

The author of the *Trattato d'Algibra* (opera of the 14^{th} Century) resolves some systems of linear equations through the application of this algorithm. For example, the problem 38 can be translated, according the modern symbolic language, in a system of four equations with four unknowns. The author transforms it through successive substitutions in a system of two equations with two unknowns of the type (Cfr. Franci and Pancanti, pp. 145-150):

$$\begin{cases} 7y = 13x + 4 & [5] \\ 4y = 2x + 176 & [6] \end{cases}$$

that he resolves in this way:

1. He adopts the false position $y_1 = 40$ and in the equation [5] he calculates $x_1 = 21 + 3/13$.
2. He replaces these two values in the equation [6] and he find *160* in the first member and $218 + 6/13$ in the second member. Since the two members would be equal, the difference is $d_1 = 58 + 6/13$.
3. Likewise he adopts the false position $y_2 = 80$, calculate $x_2 = 42 + 10/13$ and $d_2 = -(58 + 6/13)$.
4. He applies the formula [4] and he obtains:

$$y = [80.(58 + 6/13) + 40.(58 + 6/13)] / (58 + 6/13 + 58 + 6/13) = 60.$$

5. He replaces $y = 60$ in the equation [5] and he finds $x = 32$.

1.3.1.3 THE "REGULA INFUSA"

The *"regula infusa"* is a technique used by the Indians and by the Arabs to solve equations of first degree. It appears in a text of arithmetic, the author of which seems to have been Ajjub Basri, the first Arab who mastered different Indian methods of solving equations (Cfr. Charbonneau & Radford, pp. 2). The Latin version of this text is called *Liber augmentis et diminutionis* and it was translated by Abraham ben Ezra (in the 11th Century). This is the version that was in circulation in Europe. It also contains numerous problems resolved with the rule of the false position.

The author does not give a precise definition of the regula infusa, but he explains it through its application to some practical situations that are translated in equations of the general form: $x + x/n = k$. Accordingly, this technique allows to solve linear equations that present difficulty in manipulating fractional terms.

For example, one of the problems is the following (Libri 1838-1841, pp. 321): "A treasure is increased by a third [of it]. Then a fourth of this sum is added to the first sum. The new sum is 30. How much was the treasure originally?".

The problem expressed in the present symbolic language becomes:

$$x + \frac{1}{3}\cdot x + \frac{1}{4}\cdot\left(x + \frac{1}{3}\cdot x\right) = 30$$

We remember that in the mediaeval algebra the unknown is pointed out with the word *thing* or *res* that, in this case, represents the treasure. We have symbolized it with *x*.

The author divided the problem into two simpler sub-problems. In the first one he considered $x + \frac{1}{3}\cdot x$ like a *res*, namely, in modern notation $y = x + \frac{1}{3}\cdot x$. Then the first sub-problem was to resolve the equation: $y + \frac{1}{4}\cdot y = 30$. He calculated the value of y, therefore the second sub-problem was to find the solution of $y = x + \frac{1}{3}\cdot x$.

Here following we give a table with the solution in natural language, as it appears in the text and the relative translation to the algebraic language (Cfr. Charbonneau & Radford, pp. 3):

Solution proposed in the Liber augmentis et diminutionis	**Translation to the symbolic language of algebra**
Assume one res and add its fourth to it and you have a res and a fourth of res.	$y + \frac{1}{4}\cdot y$
How much must you take away from one *res* plus a fourth of *res* to obtain a *res*? You will find that it is one fifth of it.	$y + \frac{1}{4}\cdot y = 30$ therefore $\frac{5}{4}\cdot y = 30$. In order to reduce this one "y", $\frac{1}{5}$ of $\frac{5}{4}\cdot y$ has to be subtracted from each side: $\frac{5}{4}\cdot y - \frac{1}{5}\cdot\frac{5}{4}\cdot y = 30 - \frac{1}{5}\cdot 30$
Subtract therefore from 30 its fifth and 24 will remain.	That is y = 30 – 6 = 24
Then take the second *res* and add its third to it and you will have a *res* and its third.	$x + \frac{1}{3}\cdot x$
How much must you take away from one *res* and a third of a *res* to get one *res*? You will find that that it is one fourth of it.	$x + \frac{1}{3}\cdot x = 24$ therefore $\frac{4}{3}\cdot x = 24$. In order to reduce this one "x", $\frac{1}{4}$ of $\frac{4}{3}\cdot x$ ad has to be subtracted from each side: $\frac{4}{3}\cdot x - \frac{1}{4}\cdot\frac{4}{3}\cdot x = 24 - \frac{1}{4}\cdot 24$
Therefore subtract from 24 its fourth and 18 will remain	x = 24 – 6 = 18

1.3.2 METHODS OF RESOLUTION OF THE EQUATIONS OF SECOND DEGREE

Following we introduce the geometric procedure of Euclid, the method of al-Khowârizmî and the *cut-and-paste geometry*.

1.3.2.1 THE GEOMETRIC PROCEDURE OF EUCLID

In the Euclid's *"Elements"* we also find the resolution of equations of second degree from a geometric point of view.

Beginning from the propositions 28 and 29 of the 6th Book (1930, pp. 146-150), the equations of second degree that admit at least a positive root can be resolved geometrically(5). For example, the equation $ax - x^2 = b^2$ correspond to the geometric problem: *On a given segment (a), taken as the base, build a rectangle (altitude = x) that exceeds the square of the altitude (x^2) of an area equivalent to a given square (b^2)* (Cfr. Zapelloni, pp. 150). To solve it Euclid proceeds as follows:

Let a be the given segment and let C be the square of area b^2:

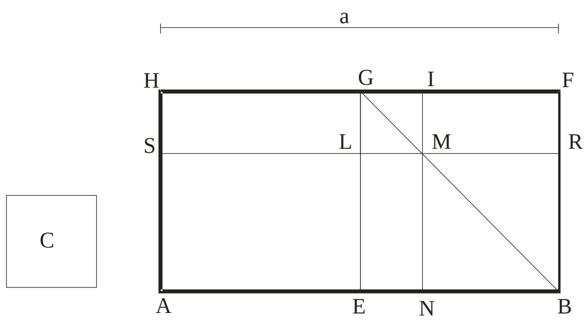


Figure 5

1. Divide the segment $a = AB$ into two equal parts, in the point E; build the square *EBFG* on *EB* and complete the square *AEGH*. The area of the square *AEGH* has to be greater than or equal to b^2, otherwise the problem has no solution.
2. If the area of the square *AEGH* is b^2, then $x = AH$ and the problem is resolved.
3. If the area of the square *AEGH* is greater than b^2, build the square *LMIG* of an area equal to the differences of the areas. Then the squares *LMIG* and *NBRM* are disposed around the same diagonal (prop. 26, 6th Book). Trace the diagonal GB and complete the figure.
4. The area of the figure *LEBFIM* is equal to b^2 in the construction. It can be easily demonstrated that the area of the rectangle *ANMS* is equal to the area of LEBFIM and therefore is equal to b^2. Then $x = SA$.

1.3.2.2 THE PROCEDURE OF AL-KHOWÂRIZMÎ

The Arabs resolved the equations of second degree considering five different cases separately:

$a\,x^2 = bx$, $a\,x^2 = c$, $a\,x^2 + bx = c$, $a\,x^2 + c = bx$, $a\,x^2 = bx + c$

so that the coefficients *a, b* and *c* are always positive. This way of proceeding without negative numbers is similar to that proposed by Diophantus; but it represents a step back in comparison with the Indian algebra, that considered the "general form" of the equation of second degree in that the negative coefficients were allowed.

One of the equations which Al-Khowârizmî (6) studied is the following: "A square and ten of its roots are equal to nine and thirty (for thirty-nine), namely you add ten roots to a square and the sum is equal to nine and thirty" (Kline, pp. 226). This statement, translated into the symbolic language of algebra, corresponds to the equation: $x^2 + 10\,x = 39$. The author uses the method of the completion of the square or calculating the positive solution:

Solution proposed by al-Khowârismî:	Modern algebraic notation:
1. *"Consider half of the number of the roots, in this case five, then multiply by itself, the result is five and twenty" (for twenty-five)".*	$x^2 + 10\,x = 39$
2. *"Sum this number to nine and thirty (for thirty and nine), that gives sixty-four".*	$(x + 5)^2 = 39 + 25 = 64$
3. *"Take the square root, that is eight".*	$x + 5 = 8$
4. *"Subtract from it half of the number of the roots, that is five, and it remains three".* 5. *"This is the root"*	$x = 3$

Some variations of this rule are found in the Babylonian and the Indian mathematics that very probably were already known by the Arabs. But al-Khowârizmî first finds the numerical solutions of the five types of equations, then demonstrates geometrically the truth of the same problems. For example, his geometric approach to the equation $x^2 + 10\,x = 39$ is the following (Cfr. Gheverghese Joseph, pp. 320-321):

1. He considers a square *ABCD* of side *x*.
2. He prolongs *AD* and *AB* until *E* and *F*, in way that $DE = BF = 5$.
3. He completes the square *AFKE*, he prolongs *DC* up to *G* and *BC* until *H*.
4. From the diagram it results that the area of $AFKE = x^2 + 10\,x + 25 = (x + 5)^2$.
5. He adds *25* to both the members of the equation $x^2 + 10\,x = 39$, therefore he obtains $x^2 + 10\,x + 25 = 39 + 25 = 64$

6. From the equality he derives that a side of the square *AFKE*, let say *EK* is $x + 5 = 8$ and then $EH = x = 3$.

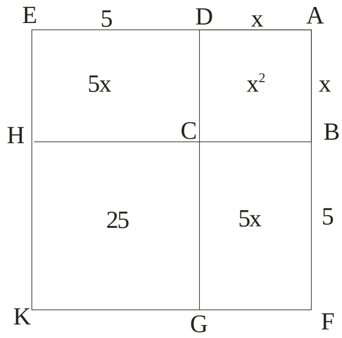


Figure 6

1.3.2.3 THE CUT-AND-PASTE GEOMETRY

The *Liber Mensuratonium* by Abû Bekr (ca. 9th Century) is a text that contains numerous problems solved with two different methods. One of these methods uses the syncopated algebra whereas the other one does not have a specific name and Høyrup, (1990) has called it *"cut-and-paste-geometry"*.

For example, the statement of the problem 25 says that: "The area is 48 and the sum of the two sides is 14, how much does every side measure?".

When expressed in algebraic language this problem results: $x \,.\, y = 48$ and $x + y = 14$, that corresponds to the equation: $x^2 - 14\,x + 48 = 0$. The author applies the method of the "cut-and-paste geometry" and explains the resolution in this way (Cfr. Charbonneau & Radford, pp. 5):

1. Divide in half *14*, the result will be *7*.
2. Multiply *7* by itself and you will get *49*.
3. Subtract from it *48* and 1 will remain, from which is obtained the root, which is *1*.
4. If you add half of *14* to it, the result will be the longer side.
5. If you subtract it from the half of *14*, the result will be the shorter side.

Although the author does not declare it explicitly, the problem is to find out the length of the sides of a rectangle that satisfy determined conditions. Charbonneau & Radford (pp. 5) think that probably the solution was accompanied by some drawings that were not inserted in the text. The text would have played only the role of supporting the memory. These authors propose the following sequence of drawings (pp. 6):

1.

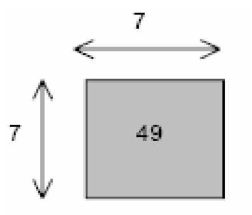


fig. 1

Build a square whose side is equal to half of 14: steps 1 and 2 of the procedure previously explained.

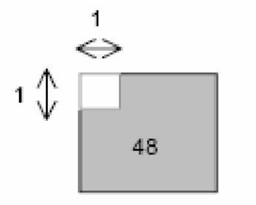
2. 

fig. 2

Step 3

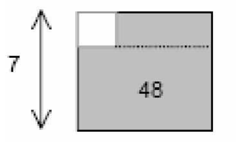
3. 

fig. 3

Apply the method of the "cut-and-paste geometry"

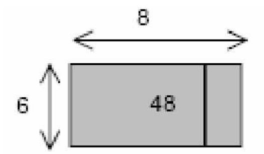
4. 

fig. 4

Steps 4 and 5

1.3.3 METHODS OF RESOLUTION OF THE EQUATIONS OF THIRD DEGREE

1.3.3.1. THE PROCEDURE OF AL-KHAYYAM

One of the most interesting progresses of the Arabic mathematics is the resolution of cubic equations through the intersection of conic sections. After the diffusion of the *Essay of Algebra (Al-jabr w'al muqâbala)* of al-Khowârizmî, two currents of ideas developed:

- Ø certain geometric problems can go back to the resolution of an algebraic equation with an unknown;
- Ø the resolution of an equation of third degree can go back, for example, to a geometric construction.

According to Rashed, the most important contribution of the Arabic mathematics is precisely the starting of the development of this correspondence between geometry and algebra five centuries before Descartes and Fermat.

With al-Khayyam (1038/48-1123) *algebra* becomes the *general theory of the algebraic equations* of degree less than or equal to three and with positive integers coefficients. This author solves the equations of second degree with positive roots using the geometric procedure of Euclid. He also finds the general solution to the equations of third degree (with positive roots and that cannot can go back to equations of second degree) through intersections of conic curves (Cfr. Ballieu, pp. 12). For example, to solve the equation: $x^3 + ax = b$ being a and b positive numbers, al-Khayyam writes the homogeneous form $x^3 + p^2x = p^2q$ with $p^2 = a$ and $p^2q = b$.

Afterwards he builds the parable of equation $y = x^2/p$. He draws the circumference of diameter QR whose length is equal to q; this curve corresponds to the equation $x^2 + y^2 - q\,x = 0$. For the point P of intersection of the two curves (different from the origin of the coordinates) he traces the perpendicular segment PS and he demonstrates that the solution of the equation is QS. From the geometric construction he deduces that this type of equation always admits a positive root.

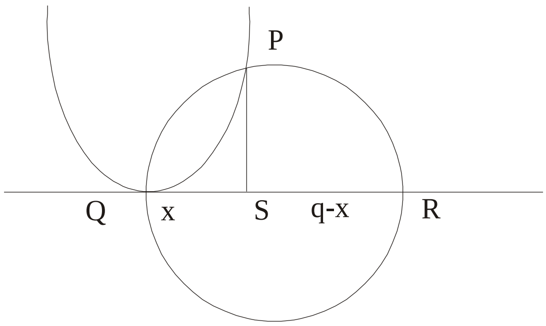


Figure 7

Al-Khayyam carries out a demonstration of synthetic kind, using the theory of the proportions. He applies the property of the parable discovered by Apollonius of Perga: $x/PS = p/x$. [1]

He considers the rectangle triangle QPR, in which the altitude PS is proportional mean between QS and RS: $x/PS = PS/(q - x)$. [2]

From [1] and [2] he deduces that: $p/x = PS/(q - x)$. [3]

Beginning from the equation [1] he obtains that $PS = x^2/p$. He replaces this value in the equality [3] and demonstrates that x satisfies the equation: $x^3 + p^2x = p^2q$ (Cfr. Kline, pp. 227-228)

Al-Khayyam also solves equations like: $x^3 + a = bx$ for a and b positive, with the aid of the parable $y = x^2/\sqrt{b}$ and of a branch of the hyperbola equilateral $x^2 - y^2 - (a/b).x = 0$. He shows that this type of equations can admit: none, one or two positive solutions (he did not take into consideration the negative solutions).

He also determines the roots of the equation: $x^3 + a\,x^2 = c^3$ through the intersection of a hyperbola and a parable and the roots of the equation: $x^3 \pm a\,x^2 + b^2\,x = b^2\,c$ by the intersection of an ellipse with a hyperbola.

1.3.3.2. THE PROCEDURE OF AL-TUSI

Al-Tusi (1130 -?) classifies the equations of degree less than or equal to three, according to the existence of positive roots. Particularly he studies five kinds of equations that admit -using his expression- "impossible cases", that is, the cases that do not admit positive solutions:

$x^3 + c = ax^2$ $\qquad$ $x^3 + bx + c = ax^2$

$x^3 + c = bx$ $\qquad$ $x^3 + c = ax^2 + bx$.

$x^3 + ax^2 + c = bx$

Every equation of this type can be written in the form $f(x) = c$ where f is a polynomial. Al-Tusi characterizes the "impossible cases" studying the intersection of the curve $y = f(x)$ with the straight line of equation $y = c$ for $x > 0$ and $f(x) > 0$. The existence of solutions depends on the position of the straight line $y = c$ in comparison with $f(x_0)$, where x_0 is the maximum of the polynomial function. If the straight line intersects the function, al-Tusi determines the roots of $f(x) = 0$. This allows him to frame the roots of $f(x) = c$, that is the roots of $f(x) = 0$ determine the interval that contains the roots de $f(x) = c$ (Cfr. Ballieu, pp. 16). Al-Tusi calculates the roots with the aid of a procedure that is analogous to the method of Ruffini-Horner. Ballieu (pp. 16) thinks that this method was used in the calculation of the square and cubic roots in the 11th Century and that al-Tusi generalized this procedure applying it to the resolution of polynomial equations.

Al-Tusi applies so the local analysis: to find the maximum of $f(x)$, he solves an equation which translated into the modern symbolic language corresponds to $f'(x) = 0$. That is, he introduces the notion of derivative, but he applies it only to some examples, without formalising the concept. This author uses a local and analytical approximation which is opposed to the global and algebraic procedure adopted by al-Khayyam. The used language is lacking in formalism, therefore it does not lend itself easily to the handling of such mathematical structures. However Ballieu (pp. 16) thinks that for the first time in the History of the Mathematics the idea of calculating the maximum of a polynomial function is found. Thus al-Tusi studies the variation of the function in the proximities of the extreme points. He handles new concepts, without the rigor of Newton obviously, but we have to remember that this occurs in the 12th Century!

1.3.4 METHODS OF RESOLUTION OF THE INDETERMINATE EQUATIONS

1.3.4.1. THE PROCEDURE OF DIOPHANTUS

Some authors (Bourbaki, pp. 122) think that Diophantus has been the first mathematician to discuss the problems of indeterminate analysis in a systematic way. Therefore, Diophantus is considered the promoter of that branch of algebra today called "Diophantine analysis".

The most extraordinary characteristic of the work of Diophantus is precisely the resolution of indeterminate equations. He solves *linear equations with two unknowns* of the type: $ax + by = c$ being a, b and c positive numbers. He gives a value to one of the unknowns, for example $x = x_o$ being x_o less than the ratio c/a. Therefore the rational positive number $y = (c - a\, x_o\,)/b$ satisfies the equation. In the case of the *quadratic equations,* Diophantus expresses some unknowns in relation to an "indeterminate", chosen so that the solutions are rational positive numbers. For instance, to solve the equation: $x^2 + y^2 = a^2 + b^2$, he considers $x = \lambda\xi - a$, $y = \mu\xi - b$ with λ, μ arbitrary constants and ξ as indeterminate quantity. So he finds: $\xi = 2\,(\lambda a + \mu\, b)/(\,\lambda^2 + \mu^2)$ and then x and y are rational (Cfr. Loria, pp. 202-203). In the case of *systems of two quadratic equations*: $y^2 = A\, x^2 + B\, x + C$ and $z^2 = D\, x^2 + E\, x + F$ he considers only particular cases in which $A, B, ..., F$ are special numbers. He assumes that y and z can be expressed in relation to x and he solves in comparison to x. But Diophantus understands that in choosing certain expressions or certain values for some unknowns, he only finds some particular solutions and that the assigned values are somehow arbitrary.

Generally Diophantus is satisfied in obtaining a positive rational solution and in exceptional cases he searches integer solutions (the modern Diophantine analysis looks only for integer solutions).

1.3.4.2 THE METHOD OF PULVERIZATION

The resolution of *indeterminate equations and problems* of the first and second degrees is a field in which the Indians reached results of notable interest. They look for all the integer solutions, while Diophantus is generally satisfied in obtaining a positive rational solution. They solve equations of first degree of the type: $ax \pm by = c$, with a, b and c positive integers and equations of second degree of the form: $x^2 - ay^2 = 1$, in which a is not necessarily a perfect square. They recognize that these equations are fundamental to solve those of type: $cy^2 = ax^2 + b$.

The method to resolve linear indeterminate equations: $ax \pm by = c$, with a, b and c positive integer, are introduced by Ãryabhata (b. 476) and improved by his successors. This procedure is called *method of pulverization (Kuttaka)* and corresponds to the strategy followed by Euler. For instance, the integer solutions of $ax + by = c$ are obtained in this way (Cfr. Kline, pp. 218 - 220):

1. If a and b have a common factor m that does not divide c, then the problem does *not admit integer solutions*, because the first member is divisible by m while the second one does not verify this property. If a, b and c have a common factor, the equation is divided by this factor and then, for the preceding observation, it is sufficient to consider the case in which a and b are relatively prime.

2. We divide a by b using the Euclid's algorithm to find the greatest common divisor of two integers. We consider $a > b$. This algorithm requires to divide indeed a by b so as to obtain $a = a_1 b + r$, where a_1 is the quotient and r the remainder. Therefore $a/b = a_1 + r/b$, that we can also express in the form $\frac{a}{b} = a_1 + \frac{1}{\left(\frac{b}{r}\right)}$ [1]

3. The second passage of the Euclid's algorithm consists in dividing b by r so as to obtain $b = a_2 r + r_1$ or $b/r = a_2 + r_1/\text{r}$. Substituting this value of b/r in the equality [1] we draws: $\frac{a}{b} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\left(\frac{r}{r_1}\right)}}$

4. We continue to apply the Euclid's algorithm and obtain the so-called *continuous fraction*: $\frac{a}{b} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}$

   This procedure is also applied when $a < b$. In this case a_1 is zero and the other passages are the same as those described previously. Since a and b are integers the continuous fraction is finite.

5. The fractions obtained by stopping at the first, second, third and, in general, n^{th} quotient are called first, second, third and n^{th} convergent respectively. Since, when a and b are integers, the continuous fraction is finite, there is a convergent that precedes of one place the exact expression of a/b. If p/q is the value of this convergent, we can demonstrate that: $aq - bp = \pm 1$.

6. We consider $aq - bp = 1$ and we return to the starting equation $ax + by = c$, then $ax + by = c (aq - bp)$ from which we obtain $(cq - x)/b = (y + cp)/a$.
   If t is the common value of these two fractions, we will have
   $x = cq - bt$ and $y = at - cp$. [2]
   If we assign to t integer values then we obtain integer values for x and y because all the other quantities are integer.

7. If the starting equation is $ax - by = c$ or the relation $aq - bp = -1$ is verified then it is necessary to effect small changes to the case previously introduced.
   It is interesting to point out that Brahamagupta (b. 598) arrives at the solutions [2] even if he does not use any generic letters a, b, p and q.

1.3.4.3 THE PROCEDURE OF ABU KAMIL

The problems of indeterminate analysis are also found in the Arabic literature. Abu Kamil dealt with a category of matters that the Chinese indicated with the name of "Problems of the one hundred birds", because the number 100 appears often in them.

These problems, translated into the modern symbolic language, are introduced by a system of equations of this kind: $x + y + z +... = m$ and $ax + by + cz +... = n$ with m and n positive integers (they often take the value 100). Abu Kamil resolves them by substituting in the second equation the value of one unknown drawn from the first one. Then he looks for all the positive integer solutions of the resultant indeterminate equation.

The ability of Abu Kamil in solving this category of problems is demonstrated by the fact that he finds 2676 possible solutions to the system: $x + y + z + u + v = 100$ and $2x + (1/2)by + (1/3)z +(1/4)u + v = 100$ (Loria, pp. 344-345).

1.3.5 EUROPEAN METHODS UP TO 1500

In his work the *Liber Abaci* (1202), Fibonacci resolves numerous problems of practical order (related to the commercial transactions). In doing so he uses the numerical succession that today has his name (every number is drawn by the sum of the two immediate precedents) or the indeterminate analysis of first and second degree. It is interesting to observe that Leonardo follows the Diophantine and Arabic style to solve equations of second degree, considering five different cases separately so that the coefficients result always positive. He finds the solutions using the geometric reasoning of Euclid in every case. He solves the innumerable matters of indeterminate analysis by applying different Diophantine artifices or the method of the false position (Cfr. Loria, pp. 386-391).

The author of the *Trattato d'Algibra* (14th Century) establishes 25 rules to resolve equations of the first four degrees. He considers different particular cases separately for the equations of the same degree, greater than the first one, so that the coefficients result always positive(7). He continues with the Arabic tradition accepting only the nonzero positive real solutions. He resolves the first 22 equations applying the transport of terms from one member to the other and the quadratic formula of the equations of second degree to calculate the positive root only. He transforms the biquadratic equations into quadratic ones and some equations of third and fourth degree into equations of second degree dividing them by the unknown or by its square. It is important to underline that these observations can seem obvious to the person that is accustomed to use the algebraic symbolism, but they are not very obvious to the author that formulated it only counting upon the natural language. The last three rules(7) correspond to equations of third degree of the type: $ax^3 + bx^2 = c$, $ax^3 = bx^2 + c$ and $ax^3 + c = bx^2$. The author carries out some adequate substitutions (for example, in the first equation he uses $x = y - b/3a$) to calculate these roots, transforming these equations into others of the type $x^3 = px + q$. After he calculates them by attempts, because he does not know its resolving formula. According to Franci and Pancanti (pp. XX) the importance of these rules is still greater if we consider that the resolution of the general equation of third degree $x^3 + a x^2 + b x + c = 0$, passes actually through the resolution of the equations of the type $y^3 + p y + q = 0$, obtained through the transformation $x = y - a/3$. This rule is precisely the one proposed in the *Trattato d'Algibra* and it is the first rule of this kind in mathematical literature.

Around 1500 Scipione Dal Ferro (1465-1526) enunciated the resolving formula of the cubic equations of the type $x^3 + px = q$ with p and q positive, using the natural language. This formula, translated into the symbolic language of algebra, corresponds to the expression:

$$x = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2} + \frac{q}{2}} - \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2} - \frac{q}{2}}$$

In 1535 Tartaglia discovered the resolving formula for the cubic equations with positive coefficients of the type: $x^3 + px = q$ and $x^3 + q = px$ in an independent manner.

These formulas were published in 1545 by Cardan in his work *Ars magna*. This author quotes the method of resolution of the cubic equations and carries out a geometric demonstration for every obtained rule, following the Arabic tradition. He also introduces the resolving procedure for some equations of fourth degree discovered by Ferrari. Cardan establishes the conditions for which the number of the roots of an equation (of second and third degree) is equal to its degree, together with the rules to lower the degree of an equation of which a root is known (Cfr. Bortolotti, 1950, pp. 656-657).

In his work *L'Algebra* (1966), Bombelli develops the theory of the equations of the first four degrees[(8)]. He considers many particular cases of equations of the same degree separately, greater than the first one, so that the coefficients are always positive. For every type of equation he enunciates (in rhetorical language) the practical rule of resolution, he carries out the geometric construction (when is possible) to justify the validity of the formulated equality in the equation and he analyzes the nature and the multiplicity of the roots. He follows the Arabic and Mediaeval tradition in accepting only the nonzero positive real solutions, because the negative or complex roots are difficult to interpret adequately, in relation to the problem-solving.

Bombelli uses the geometric construction to solve algebraic problems; but his procedure is inverse in comparison with the geometric algebra of the ancient mathematicians. This author does not resolve the geometric problem directly in order to obtain the analytical solution from the arithmetical interpretation of the accomplished construction, but he uses precisely the algebraic resolution to draw the geometric construction (Bortolotti, 1966, pp. XLIII).

1.3.6 CONCLUSIONS ON THE METHODS OF RESOLUTIONS

The historical analysis presented so far shows a wide range of procedures conceived on purpose to resolve equations. These methods point out the necessity to apply to others languages: natural, arithmetical or geometric, since there is not an adequate symbolic language. The arithmetical language was used widely by the ancient people, Diophantus and the Chinese and Indian mathematicians. This language constitutes also the base of the method of the false position, applied by the Egyptian, Chinese, Indian, and Arabic mediaeval mathematicians. The geometric language is used in the resolutive methods by the classical Greek, al-Khowârizmî and al-Khayyam and in the procedure that Høyrup calls "cut-and-paste geometry". Some protomathematic notions of analysis were used by al-Tusi [9]. It is interesting to emphasise that on the one hand, the natural language was used as a medium up to 1500 and later as a support to the reflections of the mathematicians; on the other hand, arithmetic and geometry were two languages supporting the expression and/or procedures. Particularly, geometry has contributed to the process of reasoning-demonstration notably. In all these cases the level of development of the algebraic language was very scarce. The mathematicians had to use other languages (natural, arithmetical, geometric or analytical) to obtain the solution of the problem, beginning from the interpretation of the performed procedures. Bombelli also used the geometric construction to justify the validity of the formulated equalities in the equations or to solve algebraic problems, but his procedure was different from those cited so far. In this situation other languages -natural or geometric- were utilized only to complete the communication, they were not useful to resolve the problem, because Bombelli used a different scheme of reasoning combining algebraic and Euclidean tools.

The semantics of the algebraic language is less rich than those corresponding to the natural, arithmetical or geometric languages. Therefore, in the syncopated phase to rely on other semantics is necessary to formulate the rules, to give an adequate interpretation to the problem-solving, to obtain its solution or even to justify the passages effected algebraically. The semantic ambiguity and the richness of meaning allow precisely to set little by little the symbolic language.

The "regula infusa" and the "method of the false position" use the natural language and the arithmetic language. From a comparison between the two procedures it can be deduced that the first one is more restrictive, because it is used only to resolve equations of the type $x + x/n = k$. While the second one has got a wider field of application: linear equations, systems of linear equations and approximate resolution of quadratic equations. On the other hand, the equality has a different meaning in these two procedures. In the method of the false position it points out the result of an arithmetical operation, obtained by substituting any value to the unknown. In the "regula infusa", instead, the equality represents, somehow, the equivalence between the two styles of expressing the same quantity: k is interpreted as the $(n + 1)/n$ parts of the unknown. This notion is nearer to the conception used in algebra (Cfr. Charbonneau & Radford, pp. 4).

In the procedure of al-Khowârizmî and in the cut-and-paste geometry the concept of equality represents, instead, the equivalence (equality between the areas) of plane figures. These methods are based fundamentally on the application of a series of transformation to an initial figure to arrive at a final figure of known area. The demonstration of Euclid for the resolution of equations of second degree also passes through the concept of equality of areas.

It is interesting to underline that the aim of the procedures described so far is the problem-solving, through the calculation of one or more unknowns. Even if in the indeterminate equations of Diophantus, of the Indians and of the Greek some unknowns are expressed beginning from another (the arithme or an indeterminate one), the predominant conception of variable is that of unknown.

The geometric procedure of al-Khayyam allows to resolve equations of third degree through the intersections of two conics, but the author considers them like curves, he does not apply the concept of function. Al-Tusi resolves some cubic equations by studying the intersection of a polynomial curve with a straight line, but his procedure is different from that applied by al-Khayyam. Al-Tusi uses the protomathematic notion of derivative and utilizes a local and analytical approximation that is opposed to the global and algebraic procedure adopted by al-Khayyam. Al-Tusi studies the variation of the curve in the proximities of the extreme to calculate the maximum of a polynomial. This seems to point out that the author considers in an implicit manner the dependence among variables.

1.4 THE NEGATIVE NUMBERS AS OBSTACLE. THE INCOMPLETE NUMERICAL FIELD

Although from a certain point of view the use of the arithmetic language supports the development of the algebraic language, from another one it can represent a strong limitation. The calculations with the fractions carried out by Egyptians were laborious and complicated; according to some authors this is one of the reasons that prevented the algebraic language of Egyptians to overcome the first level of development(4). Diophantus, the Arabs and the European mathematicians until 1500 did not accept the negative numbers. This is the reason why they avoided the negative coefficients in the formulation of the rules of resolution and they admitted only the positive roots (the negative roots resulted difficult to interpret adequately, in relation to the problems that they allowed to resolve).

This represents a step back in comparison with the Indian algebra that considered the general form of the equation of second degree and it also admitted the negative solutions in some cases (when it was possible to find an interpretation for them). In the same way, the lacking of acceptance of the complex numbers is the reason why Bombelli did not admit them like roots of the equations. Some authors (Bortolotti, 1966, pp. 182) think that probably the same demonstrations and the geometric constructions of the algebraic solutions of the equations averted the mathematicians' eyes (also of Bombelli) from this kind of roots. However, in the 4th Book of *L'Algebra*, Bombelli introduced the negative segments and the negative or nonzero areas to be able to operate with them. We think that the true difficulty to accept the negative roots is found precisely in the same negative numbers as an epistemological obstacle at arithmetical level (Cfr. Glaeser).

Leonardo Pisano had already made some observations and then the mathematicians of 1500 recognized that the impossibility to resolve certain equations of third degree depended on the incompleteness of the numerical field that did not contain the suitable elements to express the solution. Thus Bombelli carried out the successive extensions of the Euclidean field of rationality with the introduction of the cubic radicals first and then of the complex numbers.

It is important to underline that the necessity to widen the numerical field with the complex numbers did not appear with the resolutions of the quadratic equations, but of the cubic ones. That is, the obstacle of the complex number did not depend on the type of equation or problem, but on the procedure followed in the resolution. In fact, up to that moment the presence of the square root of a negative number meant the absence of solutions; while this did not happen in the equations of third degree: at times it was possible to find an imaginary expression in the procedure of resolution, even though the three roots were real(10). Therefore the procedure of resolution was incomplete for lack of adequate algebraic transformations that allowed to conclude it. Accordingly the resolutive formula of Dal Ferro-Tartaglia did not offer the possibility to calculate the positive root, whose existence could often be verified through a simple substitution. That is, the impossibility to perform a computational process aroused the necessity of introducing new algebraic objects of more abstract nature: the complex numbers. Bombelli had defined the rules of calculation with the cubic irrationalities and with the complex numbers, but the mathematicians of the time did not accept them as "true" numbers, that is as abstract objects. It is important to underline that an operational conception of the irrational and complexes numbers is still found in *L'Algebra*, the structural conception of these numbers (as true objects) will arrive in the following centuries (Arzarello *et al.*, pp. 9).

In the historical development of the algebraic language we frequently find that the mathematicians manifest some ambiguities to operate in certain situations with new abstract objects, for example: on the one hand, we observe the lack of acceptance of the negative numbers as coefficients or roots of the equations; on the other hand, if they are necessary to complete the process of resolution of a particular problem, then they are used in these functions. Numerous examples of this kind are found in the *Trattato d'Algibra* and in *L'Algebra* of Bombelli. But the problem persists whether these ambiguities are or not the consequence of the epistemological obstacle that the negative numbers represent (Malisani, 1996, pp.68).

1.5 GENERALIZATION OF THE PROBLEMS

A very important aspect of the construction of the algebraic language is the possibility to hypothesize the generalization of problems. The ancient and oriental mathematicians did not have general methods; they resolved every problem in a different way, that is they did not look for analogies to classify them in groups of similar problems. In the *Liber Quadratorum* (1225) of Leonardo Pisano, instead, a certain tendency is already manifested to resolve problems trying to insert them in families or classes of problems (Cfr. Leonard de Pisa, pp. 43). In the *Trattato d'Algibra*, the author classifies the problems according to the rules of resolution and he transforms the biquadratic equations in quadratic ones and some equations of third and fourth degree in equations of second degree, dividing them by the unknown or by its square. In the *L'Algebra* by Bombelli we observe a qualitative leap with the use of an adequate symbolic language. The aim of the author is to arrive at a generalization of the problems that he tackles: he resolves the arithmetical problem proposed in an analytical form, then he formulates a general rule of solution setting aside the numerical values and finally he applies this rule to the resolution of an analogous equation (with the coefficients expressed in connection with an indeterminate quantity). This shows precisely the importance that assumes the symbolic language in the processes of generalization.

From the historical analysis effected it results that also Fibonacci and the author of the *Trattato d'Algibra*, using only the rhetorical language, arrive to formulate certain generalizations, naturally of inferior level than those of Bombelli. Consequently, in the process of construction of the algebraic language it is possible to distinguish two levels of conceiving the generality of a method: one related to the possibility of applying it to a variety of specific cases and the other one regarding the possibility of expressing it through the language of symbolic algebra (Cfr. Colin & Rojano, pp. 158).

Then it would be interesting to consider a wider syncopated phase that includes not only the introduction of abbreviations for the unknowns, its powers and certain relations of frequent use, but also the first level of generalization of a method. The second level of generalization could belong either to the syncopated phase or to the symbolic one depending on the degree of development of the symbolic language. Even if Leonardo and the author of the *Trattato d'Algibra* use the natural language, they insert the problems in classes of problems (that is they apply algorithms), therefore we can affirm that they use the syncopated algebra. According to this vision the algebraic thought starts before the symbolism (Cfr. Arzarello *et al*., pp. 10).

1.6 THE VARIABLE AS "THING THAT VARIES"

The notion of variable as "thing that varies" is very ancient; but it results difficult to establish in an exact way the origin of this concept in the history of algebra. We find some traces of this notion in the ancient Babylonian tablets, precisely in the astronomic tables and in those of the reciprocal numbers. The first trigonometric table of the history of mathematics appears in the *Almagesto* of Tolomeo (150 A.D. ca.) (Kline, pp. 146). This author also compiled some tables to record the relationships between the time and the angular positions of the planets. The Indians and the Arabs also used trigonometric and astronomic tables to transcribe their observations. However, it is important to point out that these tables highlight the relationship between numbers, rather than the variational property of the mathematical objects.

According to Radford (1996, pp. 47), we can find a more elaborate concept of variable in the work of Diophantus entitled *On Polygonal Numbers*. Diophantus shows four deductively connected propositions concerning the arithmetical progressions. Radford (1996, pp. 49-50) thinks that the numbers implicated in the demonstration of every proposition are *abstract set values* and they cannot be considered variables[(11)]. But the situation changes when Diophantus asks to find the polygonal number S_n when the n side is known. Now these numbers S_n and n are no longer *abstract set values*, they become *dynamic quantities* because the values of one quantity depend on the values taken by the other one, that is, S_n and n become *variable mathematical objects*. It is interesting to emphasise that Diophantus does not consider the variable through the concept of function, but through the concept of formula.

Radford (1996, pp. 51) explains in details the differences between the concepts of unknown and of variable (as "thing that varies") in the two works of Diophantus: *Arithmetic* and *On Polygonal Numbers*. The first difference is found in the context in which these two concepts appear. In fact, the goal of *Arithmetic* is to solve problems, for example, to determine the value of one or more unknowns; whereas the aim of *On Polygonal Numbers* is to establish relationships among numbers in terms of propositions organized in a deductive structure. In this way, the variables derive from the passage from the relational problem to the problem dealing with the abstract set values calculations. The second difference is found in the representation of the two concepts. In the book *On Polygonal Numbers* the key concept is the *abstract set values* (which leads to that of variable) and it is represented geometrically or by letters. In *Arithmetic,* instead, the key concept is the unknown (the *arithme*) which is not represented geometrically. Radford (pp. 51) concludes saying: "While both of these concepts deal with numbers, their conceptualizations seem to be entirely different".

The construction of the concept of variable covered a long walk. In the 14^{th} Century Oresme studied the change and the rate of change in quantitative terms and represented graphically some physical laws of motion. In this context the variable is connected to continuous quantities, whereas with Diophantus it was connected to discreet quantities.

During the Middle Ages and the Renaissance the research on the motion of the bodies gains great importance. In this context the principal problem is to describe the relationship among variables. This description precisely leads to the concept of function. Galileo frequently uses it in the *Two New Sciences* (1638), the work in which he founded the classical mechanics. This author expresses the functional relationships verbally and with the language of the proportions, successively these relations will be written in a symbolic form with the expansion of the algebraic symbolism (Kline, pp. 395).

Most of the functions introduced during the 17^{th} Century were studied as curves, before the concept of function was express in a precise way; for example, the transcendent functions: $log\ x$, $sen\ x$ and a^x. It is interesting to underline that, the known and new curves are defined in terms of motions, that is, as the trajectory described by a mobile point.

Gregory (1667) formulated a more explicit definition of the concept of function as “a quantity obtained by other quantities through a succession of algebraic operations or any another imaginable operation”(12). Newton used the term “fluent” to indicate any relationship between variables. Leibniz (1673) initially used the term “function” to indicate any quantity that varies from a point to another of a curve. He considered that the curve could be expressed through an equation and he introduced the terms “variable”, “constant” and “parameter”, the latter in connection with a family of curves. Successively, Leibniz used the term “function” to denote the quantity that depends on a variable. Bernoulli (1697) defined the function as “a quantity formed by variables and constant, in any way”, he adopted the phrase of Leibniz “function of x” to indicate this quantity and then he used the notation fx (Cfr. Kline, pp.397).

Euler introduced the notation *f(x)* in 1764 and defined the “function of a variable quantity as an analytical expression built by a variable quantity and by constants in any way”; he considered also the functions with more variables and he classified the functions as algebraic and transcendent (Kline, pp.471).

Cauchy (1821) considered that the functions were tightly connected to variable quantities (definition already given by his predecessors); Fourier (19th Century), instead, thought that every function could be represented by a trigonometric series, the series of Fourier. Afterwards, the functions have also been defined with the aid of the theory of the series. Particularly, the function is considered as an arbitrary correspondence between two series, not necessarily based on an algorithmic relationship between the variables *x* and *y* (for instance, the function of Dirichlet, 1837) (Gagatsis, 1997).

According to Gagatsis (2000), after Fourier, Cauchy, Dirichlet and Riemann, one considers that the definition of a function *y* of an independent variable *x* as an arbitrary correspondence will contribute to important changes in Analysis.

1.7 CONCLUSIONS

The deep historical analysis on the construction of the algebraic language allows to highlight the principal conceptions, the precursory procedures, the passages from a concept to the other and, particularly, the passages through the linguistic levels of the different phases: rhetoric, syncopated and symbolic. Therefore, from this study it is possible to draw some considerations that result functional to the communication of mathematics and to the didactical research. The following reflections represent a real contribution that the history can give to the research of the epistemological obstacles that the pupils meet in the situations of learning the algebraic language:

1. The development of the symbolic language is very slow: from certain names denoting the unknown and certain relations, to the abbreviations of these words, to the intermediary codes between the rhetorical language and the syncopated one and finally to the symbols. In other words, these abbreviation and these codes are gradually purified up to the elaboration of a syntactically correct and operationally more efficient algebraic symbolism; in this process the progressive abandonment of the natural language as mediator of expression of the algebraic notions is observed.
2. In the syncopated phase it is necessary to have recourse to other languages: natural, arithmetical or geometric, in lack of an adequate symbolic language. These languages - semantically richer of that algebraic one - allow to formulate the rules, to interpret the problems to solve adequately, to obtain its solution and to justify the passages algebraically effected. With the elaboration of a more adequate algebraic language the languages of support are gradually abandoned.
3. The visual representative registers are present in the different resolutive procedures that use the geometric language, for example: in Euclid, al-Khowârizmî, al-Khayyam and in the cut-and-paste geometry; but also in the arithmetical method of the double false position and in the analytical one of al-Tusi.
4. The concept of equality varies according to the adopted resolutive procedures. For example, equality points out the result of an arithmetical operation, obtained substituting any value to the unknown; it designates the equivalence of plain figures (equality between areas); it represents the equivalence between two ways of expressing the same quantity or it points out “the conditioned equality” between two members of an equation.
5. In the phase of transition between the arithmetical thought and the algebraic thought, some obstacles at arithmetical level can delay the development of the algebraic language and the introduction of new strategies and of the new algebraic contents can eclipse the preceding arithmetical knowledge (Cfr. Malisani, 1990 and 1993).
6. The necessity of introducing new objects of a more abstract nature always appears with the impossibility of completing the resolving procedure of a particular problem, that is a computational process.

7. In the process of construction of the algebraic language it is possible to distinguish two levels of conceiving the generality of a method: one concerning the feasibility of applying it to a variety of specific cases and the other one regarding the possibility of expressing it through the language of the symbolic algebra.

8. The historical analysis emphasizes that the notions of unknown and variable as “thing that varies” have a totally different origin and evolution. Even if both the concepts deal with numbers, their processes of conceptualizations seem to be entirely different (Radford, pp. 51).

9. The notion of variable as “thing that varies” is very ancient; but it is difficult to establish exactly the origin of this concept in the history of algebra. Its evolution is very slow: from the relationship among the numbers contained in the tables, to the dynamic, but discreet, quantities reported by the concept of formula; to the variable connected with continuous quantities in the study of the physical laws; to the curves described in kinematical terms; to the description of the relation among variables that leads precisely to the concept of function.

10. The notion of unknown has its origin in the resolution of problems that ask the calculation of one or more quantities. It was introduced by Diophantus with the name of *“arithme”*, that is *the number of the problem*. The preponderance of this notion in the resolutive procedures is notable up to 1600, although some attempts to consider the dependence between variables are recorded in Diophantus and al-Tusi.

NOTES

1. The Babylonians used the words *us* (length), *sag* (width) and *asa* (area) as unknowns. The unknowns did not necessarily represent these geometric quantities, but probably many algebraic problems were originated from geometric situations and thus the geometric terminology became standard. At times the Babylonians used some special symbols to represent the unknown, which corresponded to the ancient Sumerians pictorial symbols, not longer in use in the current language (Cfr. Kline, pp. 14).

2. Leonardo Pisano wrote two works of fundamental importance: the *Liber Abaci* (1202, revised in 1228) and the *Liber Quadratorum* (1225). The aim of the *Liber Abaci*, that is the “book of the abacus”, was to introduce in Europe the Indian-Arabic system of numeration and the methods of Indian calculation. This work was used for long time and it practiced an enormous influence on the people, because it introduced arithmetical procedures simpler than the methods founded on the Roman system. The *Liber Quadratorum*, that is the “book of the square numbers”, contains important results on the Theory of the Numbers. For this reason, some authors (Bortolotti, 1950, pp. 650) think that “ ... for the originality of the method and the importance of the results this work *made Leonardo* the greatest genius of the theory of the numbers, appeared in the fifteen centuries between Diophantus and

Fermat". But, unfortunately this book remained unknown for more than six centuries, some very important results had to wait until the arrival of Fermat.

3. The *Trattato d'Algibra* was written at the end of the 14th Century by an anonymous Florentine abacus master. It represents much more than a classical *"trattato d'abaco"*, it is a wide and organic text of algebra: because it does not only tackle all the merchant matters that characterize this kind of work, but it also contains an entire section dedicated to algebra. It represents an important contribution to the theory of resolution of equations. Franci and Pancanti (pp. VI) think that this work is one of the best mediaeval and Renaissance essays of abacus that they have examined. Particularly they point out that: "... the final chapters dedicated to algebra... are fundamental in reconstructing the history of this discipline between 13th and 14th Centuries".

4. The Egyptians wrote the fractions different from 2/3 and with a denominator different from 1 as sums of unitary fractions (with numerator equal to 1). The Egyptian arithmetic was essentially additive, because they effected the four operations using precisely the decomposition in unitary fractions. Thus, the calculations became complicated and laborious in its execution. A more deep analysis on the topic is found in Loria (pp. 41-47) and Malisani (1996, pp. 27-28).

5. <u>Proposition 28</u>*: Build a parallelogram equal to a given polygon on a straight line, lacking of a parallelogram similar to a given parallelogram. The given polygon has to be not bigger than the polygon built on half the given straight line, and similar to the lacking polygon* (Euclid, pp. 146-147). This theorem is the geometric equivalent of the solution of the equation of second degree: $ax - (b/c)\, x^2 = S$, where a is the straight line, S is the area of the given polygon, b and c are the sides of the given parallelogram. The second part: $S \leq a^2c/4b$ corresponds to the necessary limitation so that the roots of the equation are real.
<u>Proposition 29</u>: *Build a parallelogram equal to a given polygon on a given straight, surplus of a parallelogram similar to another given one.* (Euclid, pp. 148)
In algebraic terms, it corresponds to the equation: $ax + (b/c)\, x^2 = S$ with a, b, c, and S given positive numbers. S is not subject to any limitation (only to be positive) because the equation always admits a real solution.

6. Mohammed ibn Musa al-Khowârizmî (780 ca.- 850 ca.) composed an essay on arithmetic entitled: *Algorithmi de numero indorum*. The word *"Algorithm"* derives from the alteration of the appellative: *al-Khowârizmî* attributed to Mohammed. This term, after having suffered several variations of meaning and denomination, it was used to express a constant procedure of calculation (Loria, pp. 336-337). Al-Khowârizmî also wrote a book of algebra: *Al-jabr w'al muqâbala*. In this title he precisely pointed out the two fundamental operations of the resolution of the equations of first degree: the word *al-jabr* signifies "to restore", that is to restore the equilibrium between the members of an equation through the transport of terms and the word *al muqâbala* means "simplification", that is the reduction of similar terms. The word *al-jabr* was transformed in *algebrista* in Spain, it was translated *algebrae* in Latin and, finally it was shortened in *algebra* to indicate the name of the discipline.

7. The list of the 25 types of equations resolved in the *Trattato d'Algibra* is the following:

1- $ax = b$	9- $ax^3 = bx^2$	17- $ax^4 + bx^3 = c\,x^2$
2- $ax^2 = b$	10- $ax^3 + bx^2 = c\,x$	18- $ax^4 + cx^2 = b\,x^3$
3- $ax^2 = bx$	11- $ax^3 + cx = b\,x^2$	19- $ax^4 = bx^3 + c\,x^2$
4- $ax^2 + bx = c$	12- $ax^3 = bx^2 + cx$	20- $ax^4 + bx^2 = c$
5- $ax^2 + c = bx$	13- $ax^4 = b$	21- $ax^4 + c = b\,x^2$
6- $ax^2 = bx + c$	14- $ax^4 = bx$	22- $ax^4 = bx^2 + c$
7- $ax^3 = b$	15- $ax^4 = bx^2$	23- $ax^3 + bx^2 = c$
8- $ax^3 = bx$	16- $ax^4 = bx^3$	24- $ax^3 = bx^2 + c$
		25- $ax^3 + c = b\,x^2$

8. *L'Algebra* of Bombelli (written around 1550, partially published in 1572 and successively in 1579) is a very important work and it differs from another text of the time. It is composed by five books, in the first three books the author introduces the theory of the resolution of the equations of the first four degrees in a systematic way. In the last two books (not published up to 1929) Bombelli carries out the geometric demonstrations of the results obtained in the first three books and the resolution of geometric problems through the application of algebra. It is interesting to observe that the disposition and the order of the treated matters, the performed constructive and demonstrative procedures and the level of used language represent a notable step toward the construction of the symbolic algebra.

9. The protomathematic notions are the knowledge that the mathematicians use without calling them or defining them in mathematical terms (implicit) (Cfr. Spagnolo, 1995, pp. 17).

10. The equations of the type $x^3 = px + q$, $x^3 + q = px$ are resolved applying the following formula:

$$x = \pm \left[\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^2}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^2}} \right]$$

Precisely when $\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^2 < 0$ the square root of a negative number appears, that is an imaginary number. Nevertheless when the two cubic roots that compose the solution are complex conjugate numbers the solution becomes a real number.

11. Diophantus as well as Aristotle considers the number as composed by discreet unity. In modern notation the numbers that Diophantus used in his work *On Polygonal Numbers* to show the four propositions are: S_n the polygonal number, n the side of the polygonal number and d the difference.

12. Gregory explains the necessity to add to the five operations of algebra a sixth operation (imaginable): the passage to the limit (Bourbaki, pp. 267-268).

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CHAPTER TWO:

THE MAGIC SQUARE. AN EXPERIENCE ON THE TRANSITION BETWEEN THE ARITHMETICAL LANGUAGE AND THE ALGEBRAIC LANGUAGE

2.1 INTRODUCTION

Many students meet serious difficulties when they pass from the arithmetical thought to the algebraic thought. Numerous research study the conceptual changes necessary in this transition, that are related to the equality, the conventions of notation and the interpretation of the concept of variable (Matz, 1982; Kieran & Filloy, 1989; Kieran, 1991). Other works focus their attention on the difficulties of pupils in solving algebraic expressions, equations and algebraic problems (Gallardo & Rojano, 1988; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994, 1996).

Many errors that may be found in the protocols of the students originate from the dialectical relationship, not yet overcome, between procedural aspects and structural aspects, verifiable respectively but not exclusively in arithmetic and in algebra (Arzarello *et alii*, 1994).

Some studies indicate that the introduction of the concept of variable represents the point of critical transition (Matz, 1982; Wagner, 1981, 1983). This concept is complex because it is used with different meanings in different situations. Its management depends on the particular way of using it in the activity of problem-solving. But the multiplicity of aspects is exactly the reason why this notion becomes difficult to define and it is possibly the cause of most of the difficulties that pupils meet in the study of algebra (Wagner, 1981, 1983; Usiskin, 1988).

The notion of variable could take on a plurality of conceptions:

- ***generalized number*** (it appears in the generalizations and in the general methods);
- ***unknown*** (its value could be calculated considering the restrictions of the problem);
- ***"in functional relation"*** (relation of variation with other variables);
- ***totally arbitrary sign*** (it appears in the study of the structures);
- ***register of memory*** (in informatics) (Usiskin, 1988).

The difficulties met by the subject that learns can be very close to those experimented by generations of mathematicians. Some experimental studies (Harper, 1987; Sfard 1992) seem to confirm the thesis of Piaget on the convergence between historical development and individual development (Garcia and Piaget, 1989).
From the thorough historical analysis carried out in the preceding chapter, we drew some important observations on the development of the algebraic language, pointing out evidence the principal conceptions, the precursory procedures, the passages from one concept to the other and, particularly, the passages over the linguistic levels of the different phases: rhetoric, syncopated and symbolic. Beginning from the reflections effected on the epistemological and historical-epistemological representations, we planned the study of the obstacles that the pupils meet in building up and assimilating certain concepts, in the passage from the arithmetical thought to the algebraic thought.
The aim of the present work is to study some characteristics of the period of transition between the arithmetical language and the algebraic language. We want to analyze if the different conceptions of variable are evoked by the pupils in the resolution of problems and if the procedures in natural language and/or in arithmetical language prevail as resolutive strategies, in absence of an adequate mastery of the algebraic language.
This experimentation was effected, thanks to the collaboration of a group of teachers coordinated by the Prof. Teresa Marino and the author, in a few classes of middle school (11-12 years of age) and high school (14-15 years of age) of Piazza Armerina (a provincial town in the province of Enna), during the months of January and February 2002. This work belongs to a project of experimentation on the teaching-learning of Mathematics entitled "*Inferring, conjecturing and demonstrating in the school of all*", coordinated by Prof. Filippo Spagnolo.
The experimental work was divided into three phases: in the first one, the teachers prepared the a-didactical situations and they carried out a-priori analysis of the problem (Cfr. Brousseau, 1986; Brousseau, 1998); in the second one, the experimental data was analyzed qualitatively; and in the last phase, the data was analyzed quantitatively, using the software of inferential statistic Chic 2000 (Classification Hiérarchique Implicative et Cohésitive) and the factorial analysis survey S.P.S.S. (Statistical Package for Social Sciences).
The teachers performed the didactic experimentation on the resolution of the magic square: "complete the square inserting the lacking numbers, so that the sum of the numbers of every line, column or diagonal is always the same".

The use of the magic square has different motivations: it is a problem that can be adapted well enough to the experimentation in the two scholastic levels, because it can be introduced with different modalities and with different degrees of difficulty. But mainly, the magic square allows to study some aspects of the period of transition between the arithmetical language and the algebraic language.

2.2 HYPOTHESIS

H_1: The pupils evoke the different conceptions of variable (constant, unknown, "thing that varies", etc.) also in the absence of an adequate mastery of the algebraic language.

H_2: The procedures in natural language and/or in arithmetical language prevail as resolutive strategies in absence of an appropriate mastery of the algebraic language.

2.3 EXPERIMENTAL REPORTS FOR SCHOLASTIC LEVEL

2.3.1 MIDDLE SCHOOL

Twenty seven pupils attending the first year of middle school in two different classes (11-12 years of age), participated in the experimentation.

2.3.1.1 THE A-DIDACTIC SITUATION AND ITS PHASES

Phase 1: Delivery

The teacher communicates the type of game to the pupils.
A pupil is invited to play with the teacher at the blackboard with a magic square 3×3. She verifies if the assignment has been completely understood by everybody, by asking questions.

Phase 2: Action

The teacher delivers a magic square 3×3 to every pupil for completing. Then she invites the pupils to write on a sheet the type of procedure that they are going to use to solve the problem step by step. The winner will be the first student that succeeds in delivering the solution with the complete description of the procedure.

(The magic squares used in the experience are shown in the Appendix N° 1 at the end of Chapter 2).

Phase 3: Formulation

The class is divided into three heterogeneous groups for logical-mathematical ability.
The teacher gives every group the following magic square 4×4:
"Complete the magic square in way that the greatest number to insert is equal to 92".

Sum 26 + a

14		1	
	9	12	
11		a	10
	16	13	

At this point every group must find a common solution. The resolutive procedure must also be handed in writing by the group, this time. The first group that completes the square and the description of the procedure is the winner.

Phase 4: Validation

The pupils write the resolutive affirmations that all think valid on the blackboard and so, they formulate a theorem.

Time: 50 minutes (an hourly unity) for the action and other 50 consecutive minutes for the formulation. The validation can be treated, instead, in following moments.

2.3.1.2 THE A-PRIORI ANALYSIS

We hypothesize that the pupils can apply to one or more of the following resolutive strategies individualized in the a-priori analysis:

A1: To insert numbers at random

A2: Complementary + to insert numbers in boxes at random

A3: Difference + to insert numbers in boxes at random

A4: Complementary

A5: Complementary and by difference

A6: By difference

A7: Complementary + equation of first degree

A8: Difference + equation of first degree

A9: Equation of first degree

A10: He/she does not attempt any resolutive strategy

A11: He/she has a resolutive strategy, but he/she does not succeed in communicating the procedure in writing.

We compiled a table with a double input "pupils/strategies". For every pupil we indicated the strategies that he used with the value 1 and those that he did not apply with the value 0. The strategies actually used by the pupils, that participated in the experimentation, are those considered in the tabulation of the data. This list is the following:

A1: to insert the numbers at random

A2: complementary effected by inserting numbers in boxes at random

A4: complementary

A6: by difference

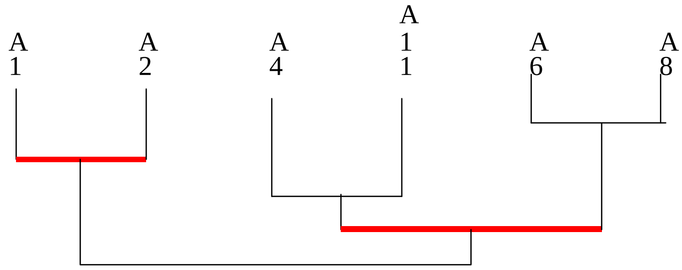
A8: by difference with equation of first degree

A11: complementary without the delivery of a correct written description

The table of the data is presented in the Appendix N° 2 at the end of the Chapter 2.

2.3.1.3 QUANTITATIVE ANALYSES OF THE DATA

Similarity tree



Arbre de similarité : C:\CHIC\chic 2000\CartelEXC CSV (MS-DOS).csv

From the graph we observe a great similarity between the following pairs of strategies:

- A1 e A2: "to insert the numbers at random" and "complementary inserting numbers in boxes at random".

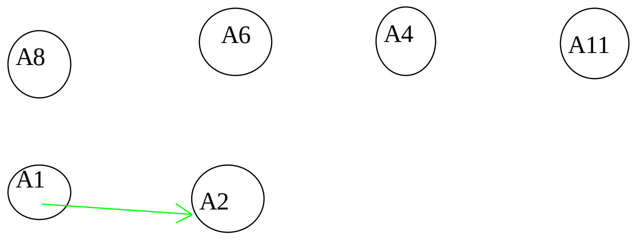
- A4 e A11: “complementary” and “complementary without the delivery of a correct written description”.
- A6 - A8: “by difference” and “by difference with equation of first degree”.

Two groups emerge from the graph.

The big group is made up of the pupils that completed the square, inserting the numbers at random, or that applied the strategies of the complementary, inserting numbers in boxes at random.

The small group, instead, is made up of those students who have chosen a winning strategy, calculating the numbers to insert by difference, by difference with equation of first degree or that have also applied the strategy of the complementary one without the delivery of a correct written description.

Implicative graph

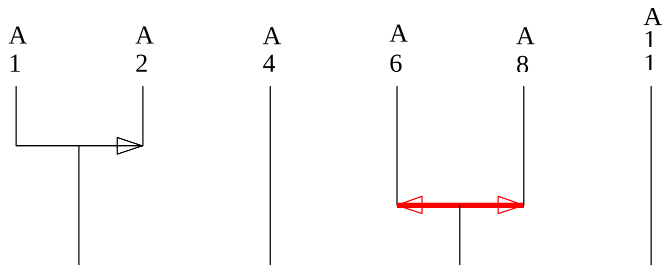


Graphe implicatif : C:\CHIC\chic 2000\CartelEXC CSV (MS-DOS).csv

99 95 92 85

From the implicative graph we observe that only one implication exists. This connects the strategy of inserting the numbers at random and the strategy of the complementary inserting the numbers in boxes at random.

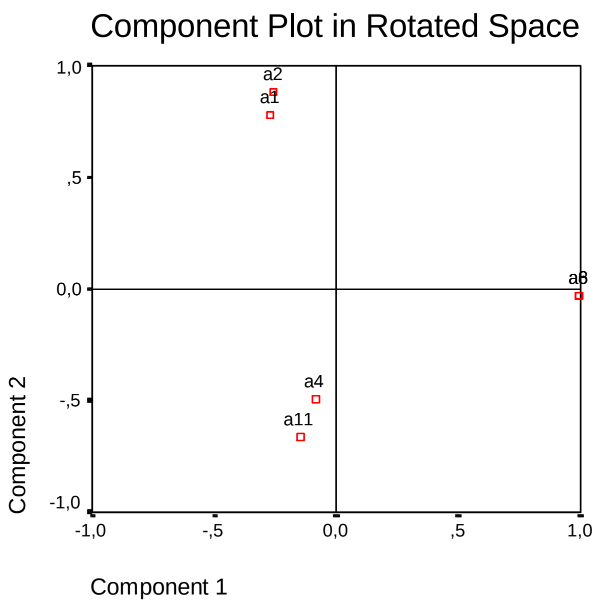
Hierarchical tree



Arbre hiérarchique : C:\CHIC\chic 2000\CartelEXC CSV (MS-DOS).csv

The hierarchical tree shows a marked hierarchy between the strategy A1 and A2, because the pupil that chooses "to insert a number at random", surely can also choose the strategy of "the complementary, inserting the numbers in boxes at random".
Besides we observe an equivalence (double implication) between the strategy A6 "by difference" and the strategy A8 "by difference with an equation of first degree".
There is not hierarchy between the variable A4 "complementary" and A11 "complementary without the delivery of a procedure written in a correct way".

Factorial analysis



From the factorial analysis we survey two pairs of strategies: A1 - A2 "to insert the numbers at random" and "complementary, inserting numbers in boxes at random", and the strategies A4 - A11 "complementary" and "complementary without the delivery of a correct written description". These pairs are contrasted and their antithesis depends on the resolutive strategy A8 "by difference with equation of first degree". This last strategy therefore discriminates the two pairs.

2.3.1.4 QUALITATIVE ANALYSIS

In this section we present the qualitative analysis effected in one class of first year students of middle school. In the phase of "formulation" the pupils were divided into three groups and every group chose a spokesman. The discussions were recorded on audio-cassette and successively it was transcribed. The qualitative analysis is effected on the protocols obtained by the transcription.

Group 1

The group does not attend the delivery that specifies that the greatest number to insert must be equal to 92 and it does not present a written strategy.

Immediately we observed that one student of the group acts as the leader, while the others let themselves be guided. From the beginning the group proceeds silently on project "sum 26". The leader starts from a box at random, only when by chance, he comes upon some "data of fact" (a complete column, a line with an only empty box); he begins to work making hierarchies and generalization with some references of pragmatic type. In this work the *"a"* appears already in the third column with numbers that together give the sum of 26, therefore it is set equal to the value zero. Only after the leader has completed the square unsuccessfully, the pupils of the group seem encouraged to intervene. So the group returns on performed procedure: *"Let's try in all the ways…"*, *"let's make the diagonals"*, *"perhaps we could do this …"*; *"perhaps we have to change this…"*. The group inserts the numbers to be subtracted in some boxes, in two or more occasions to "balance the accounts". Different attempts follow, but the group does not succeed in systematizing the square to the sum 26+a.

Group 2

The group does not deliver a written strategy. From the beginning of the discussion, one of the components affirms that the *"a"* is a number that must be added to all the columns, to all the lines and the two diagonals, therefore the intuition of the concept of variable appears in an unaware way.

Immediately the group works in way that the sum inside the boxes of the different lines, columns and diagonals is 92. Therefore the students proceed with a clear but not completely correct definition of the delivery. However the organizational strategy of the phase of planning is missing, because the pupils proceed at random, in filling in methodically the boxes of the various lines.

The students complete the lines and they think that have arrived at the solution. When the teacher asks them to verify if the sum results also on the diagonals, the group thinks that, perhaps the number 66 was inserted in the wrong boxes. The pupils attempt other strategies, but they do not arrive at the solution.

A "germ" of algebraic thought emerges in the activity of this group altogether.

Group 3

The group does not attend the delivery that specifies that the greatest number to insert must be equal to 92. It delivers a written description of the strategy. The qualitative analysis is made on the base of this description and on the phonic recording performed during the discussion.

The *"a"* present inside the square is considered a symbol, replaceable with a number. The components of the group believe that the sum of the square 4×4 must be 26. The phase of planning is also missing in this group, because the pupils proceed at random, in filling in methodically the boxes of the various lines. Since in the last line, the sum of the numbers already inserted gives a value greater than 26, the group follows the pragmatic suggestion of one of the components: *"...I have made them with a minus..."*. Thus the group inserts the use of the negative number (understood as something to subtract).

The following activity underlines the use of a false justified reasoning, in which the group also works out of the square with the aim of reaching the total sum of 26. That is the pupils add or subtract others numbers to the sum given, by the numbers present in all the boxes of a line or a column, (that however cannot be put in any box). The group systematizes the other boxes of the following columns with this last strategy. The problem remains open when the group discusses the solution of the secondary diagonal.

2.3.1.5 DISCUSSION OF THE RESULTS

The experimenters think that the search of resolutive strategies, related to the magic square 3×3, generally resulted simple enough for the students. Instead the approach to the resolutive strategies of the magic square 4×4 appeared more difficult.

In proposing the resolution of this last square, the intent was to study some aspects of the period of transition between the arithmetical language and the algebraic one.

The delivery of the magic square 4×4 (explained, perhaps on purpose, with a partly ambiguous language) was substantially not very clear to most pupils.

In proposing this a-didactic situation again, the experimenters recommend therefore to replace the text of this query with a phrase that makes the delivery more comprehensible, for instance: "complete the magic square in a way that its sum is equal *26+a*. Replace *"a"* in a way that the greatest number to insert in the boxes is *92*".

From the analysis quantity and qualitative of the data it emerges that the arithmetical thought appears already structured enough in pupils. Even if a numerous group still proceeded by attempts, because in the square 3×3 they inserted the numbers at random or they chosen the boxes at random, this is due surely to the complexity of the exercise. The pupil must understand the mutual dependence that exists between the different lines, columns and diagonals to complete the magic square. Therefore, the student must individualize the boxes from which he can begin and, successively, continue to play. It is interesting to notice that, the students used the negative numbers as “numbers to subtract”, in the case that the partial sum of some boxes was superior to the total sum of the square.

It is possible to underline that the pupils evoked different conceptions of variable during the resolution: some considered the *“a”* a constant equal to 0; for others, instead it was a symbol that could be replaced by a number. For others yet, the *“a”* represented a variable, that is a symbol that had to be added to all the columns, to all the lines and the two diagonals. Although these pupils had not approached the study of algebra yet, they considered the symbol *“a”* under different aspects: constant, numerical value, 0, “thing that varies”. Therefore, the algebraic thought is clearly present, even if is not yet structured, because the pupils did not succeed in operating with the symbolic value.

The qualitative analysis of the protocols, related to the phase of the “formulation”, has shown that the resolutive strategies use fundamentally the natural language and the arithmetical language; the algebraic language is nearly absent.

2.3.2 HIGH SCHOOL

Thirty nine pupils belonging at two first classes of the Psycho-Pedagogical High school (14-15 years of age) participated in this experimentation.

2.3.2.1 THE A-DIDACTIC SITUATION AND ITS PHASES

Phase 1: Delivery (time 30’)

The teacher simulates the game with the pupil and she explain in a clear and comprehensible way the procedure for the compilation of the square 3×3. During the game, she comments and illustrates the phases. Then two pupils, selected at random, continue the game at the blackboard with another magic square 3×3.

Successively, the other students complete other magic squares 3×3, playing in groups of two. They decide of common accord the numbers to insert, choosing therefore a suitable strategy.

The rules spring from the situation, they are not given by the teacher, accordingly the action reduces the ambiguity of the message and introduces the feed-back.
(The magic squares used in this experience are presented in the Appendix N° 3 at the end of the Chapter 2).

Phase 2: Individual work with motivation, phase of action (time 50')

The teacher goes away and the student deals with of the problem. The students must compile individually a square 4×4, writing on the sheet the various phases of the adopted strategy. They must decide between the possible individualized strategies, which is the more convenient and motivate it.
In this phase every pupil becomes responsible, he builds his own knowledge alone.
At the end of this phase, the pupils hands over his own work that will come successively assessed quantitatively through a special grid.

Phase 3: Game of team, group against group (time 20')

All pupils are divided into two groups, the game becomes a team game. Every group has a spokesman. Inside the group the pupils discuss, every student tries to convince the others of his own strategy, thus, he has to communicate: in this phase the deducing and the conjecturing enter in game. Therefore the formulation of knowledge comes about.
This phase and the following one are recorded by the teacher and after they these will be assessed qualitatively.

Phase 4: Situation of validation (the game of the discovery, proof and demonstration) (time 20').

In this phase the pupil must make the feed-back, he must reason, discuss the situation and share or look over his opinions.
The students take conscience of the definite strategy of common accord and then they write the demonstration on a sheet. They are motivated to discuss a situation, their validation of it is so favoured. The "motive" is understood by the students, the knowledge becomes social and not individual.
The game is won by the team that succeeds in completing the square first and convincing the whole class of the strategy used by them and thus formulating a valid demonstration.

2.3.2.2 DESCRIPTION OF THE PHASE OF VALIDATION

The winning group of the 1° "A" used the **arithmetical method**: they considered that at least a box must contain the number *92* and that the sum must be equal to $26+a$. They took in examination the column in which the numbers *1*, *12* and *13* are inserted, their sum is *26*, therefore they substituted to *"a"* the maximum number that is *92*.

Some pupils of the second group of the class 1° "A", instead, began with an arithmetical-algebraic procedure, inserting some values: *5*, $-4+a$, $9+3a$, etc., but they did not complete the square because they did not understand to which box to attribute the value 92. This strategy was abandoned in favour of the arithmetical method.

Both the groups of the class 1° "B" used the arithmetical-algebraic method: they considered that a box of every line and every column must contain the symbol *"a"* and they completed the square. In the boxes they noticed the presence of: a, $a-2$, $a-4$, $a-6$ and they considered that the value 92, the greatest, must replace *"a"*.

(A most exhaustive description on the resolutive methods is introduced in the Appendix N° 4 at the end of the Chapter 2).

2.3.2.3 A-PRIORI THE ANALYSIS

The a-priori analysis of the problem has allowed us to determine all the possible strategies, that the pupils can use for the resolution of the magic square.

- Ø A1: He/she inserts the numbers at random.
- Ø A2: As A1 and he/she abandons.
- Ø A3: Arithmetic calculus considering *"a"* any constant, he/she takes into consideration the question around 92.
- Ø A4: As A3 and he/she abandons.
- Ø A5: Arithmetic calculus considering *"a"* any constant, he/she does not take into consideration the question around 92.
- Ø A6: As A5, but he/she does not justify the obtained values.
- Ø A7: As A5 and he/she abandons.
- Ø A8: Algebraic calculus considering *"a"* any constant, he/she inserts some wrong values but he/she does not justify them, he/she does not consider the question around 92.
- Ø A9: Algebraic calculus considering *"a"* any constant, he/she inserts some wrong values justifying them, he/she does not consider the question around 92.

- Ø A10: Arithmetical calculus considering *“a”* constant, he/she takes into consideration the sum equal to 92.
- Ø A11: As A10 and he/she abandons.
- Ø A12: Algebraic calculus, *“a”* constant but he/she does not consider the question around 92.
- Ø A13: As A12, but he/she does not justify the obtained values.
- Ø A14: Algebraic calculus, *“a”* constant, he/she considers the question around 92.
- Ø A15: As A14 and he/she abandons.
- Ø A16: Algebraic calculus, *“a”* constant, he/she plans a system of 7 equations with 7 unknowns.
- Ø A17: As A16, he/she considers the question around 92.
- Ø A18: As A16 and he/she abandons.
- Ø A19: As A17 and he/she abandons.
- Ø A20: Algebraic calculus, *“a”* variable, he/she plans a system of 8 equations with 8 unknowns, but he/she does not consider the question around 92.
- Ø A21: As A20, he/she considers the question around 92.
- Ø A22: As A20 and he/she abandons.
- Ø A23: As A21 and he/she abandons.
- Ø A24: Algebraic calculation, *“a”* constant, he/she plans a system of 3 equations with 3 unknowns, but he/she does not consider the question around 92.
- Ø A25: As A24, he/she consider the question around 92.
- Ø A26: As A24 and he/she abandons.
- Ø A27: AsA25 and he/she abandons.
- Ø A28: Algebraic calculus, *“a”* variable, he/she plans a system of 3 equations with 3 unknowns, but he/she does not consider the question around 92.
- Ø A29: As A28, he/she considers the question around 92.
- Ø A30: As A28 and he/she abandons.
- Ø A31: As A29 and he/she abandons.

We completed a table with a double input “pupils/strategies”. For every pupil we pointed out the strategies that he used with the value 1 and those that he did not apply with the value 0.

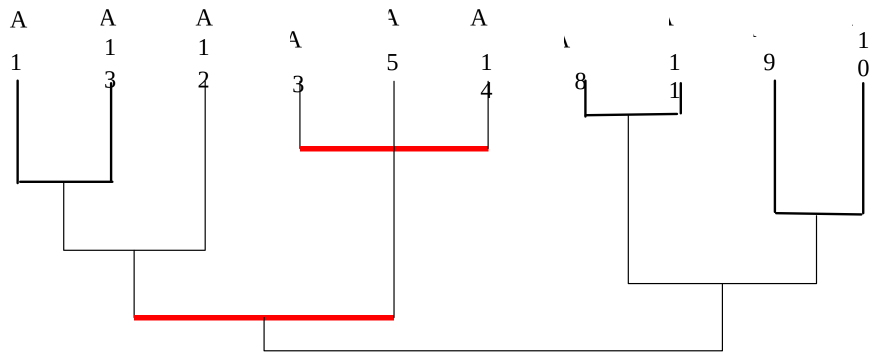
The strategies actually used by the pupils, that participated in the experimentation for resolution of the magic square are those considered in the tabulation of the data. The list is the following:

- A1: He/she inserts the numbers at random.
- A3: Arithmetic calculus considering *"a"* any constant, he/she takes into consideration the question around 92.
- A5: Arithmetic calculus considering *"a"* any constant, he/she does not take into consideration the question around 92.
- A8: Algebraic calculus considering *"a"* any constant, he/she inserts some wrong values but he/she does not justify them, he/she does not consider the question around 92.
- A9: Algebraic calculus considering *"a"* any constant, he/she inserts some wrong values justifying them, he/she does not consider the question around 92.
- A10: Arithmetical calculus considering *"a"* constant, he/she takes into consideration the sum equal to 92.
- A11: As A10 and he/she abandons.
- A12: Algebraic calculus, *"a"* constant but he/she does not consider the question 92.
- A13: As A12, but he/she does not justify the obtained values.
- A14: Algebraic calculus, *"a"* constant, he/she considers the question around 92.

The table of the data is introduced in the appendix N° 5 at the end of the Chapter 2.

2.3.2.4 QUANTITATIVE ANALYSIS OF THE DATA

Similarity tree



Arbre de similarité : C:\CHIC\chic 2000\GRIGRUPPO1.csv

The graph shows two groups similar to one another:

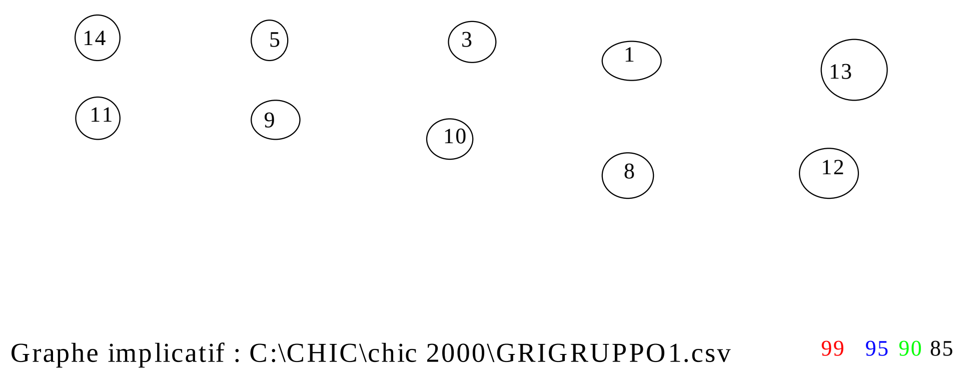
R1 = (A1, A13, A12, A3, A5, A14) and R2 = (A8, A11, A9, A10).

The pupils of the group R1 have used chiefly the algebraic calculus assigning *"a"* a constant value. The strategies A1, A3 and A5 of this group concern the arithmetic calculus, but they have been used only by four students.

Those students of the second group have used, instead, the strategies that are referred to the arithmetic calculus or to wrong algebraic calculus, they do not consider the question around 92.

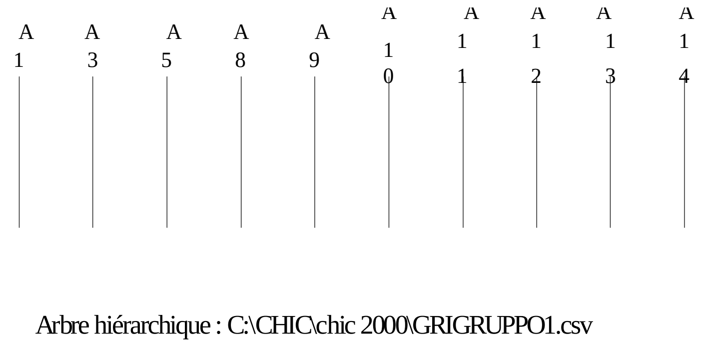
However it is interesting to underline that almost all pupils has effected algebraic calculations.

Implicative graph



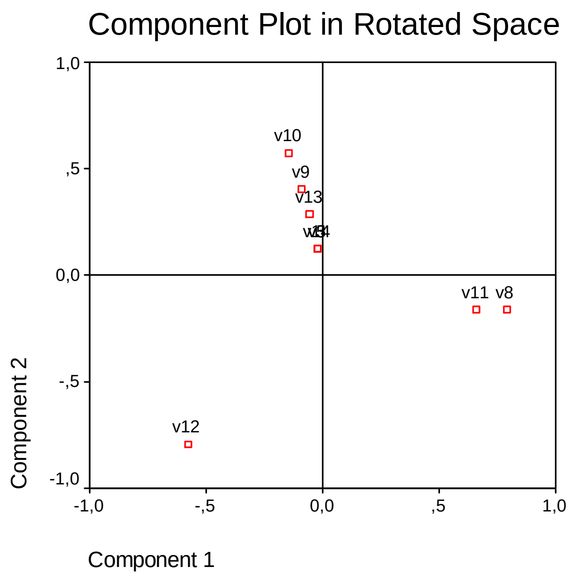
From this graph we deduce that there are no remarkable implications between the variables.

Hierarchical tree



The graph does not introduce meaningful data for a statistic survey.

Factorial analysis



From the two-dimensional projection we observe that the strategies A8 and A11 (wrong algebraic calculations and arithmetical calculus with abandoning of the resolution) are contrasted to the strategy A12 (algebraic calculation), in comparison to the second factor (vertical axis). All the other variables A5, A9, A10, A13, A14 form a cloud on the vertical axis and they correspond mainly to those strategies that do not consider the question around 92 or they do it in the wrong manner, namely, taking 92 as the sum of the magic square.

2.3.2.5 QUALITATIVE ANALYSIS

During all the phases of the game, the pupils have shown remarkable interest and an active participation. The assignment has resulted stimulating and pleasant.

In the **first phase,** when the teacher explained the square 3×3 at the blackboard and when the pupils played in couples, they resolved the assigned task easily and enthusiastically.

In the **second phase**, when the teacher delivered the magic square 4×4, the students worked individually with some difficulties. Many pupils expected a magic square 4×4, that did not contain any variable, but almost all were able to complete the magic square, after they have overcome the first impact. Nobody, however, succeeded in understanding the part of the assignment that said textually: “Complete the magic square so that the greatest number to insert is equal to 92”. On this second part, the pupils discussed broadly in the **third phase**, during the group work, and they proposed various types of strategies.

(The protocols of the experience are introduced in the Appendix 6 at the end of the Chapter 2).

2.3.2.6 DISCUSSIONS OF THE RESULTS

From the qualitative analysis of the protocols we observe that the symbol *"a"* assumes different aspects for the pupils, for example:

1. "Therefore if *a = 92*, we must give a *value to a"*.
2. "…we *must not give any value to a*".
3. "…we *must not attribute anything to a*".
4. "… *a* must be a *value*, because *a is a constant*, *it is not variable*".
5. "If we have said that *a* is a *constant*, how can it be negative?"
6. *"a is an unknown* therefore it must be replaced with a number that does not overcome 92…".
7. If we put 92 and we attribute it to *a*, then the *a* must be considered like a kind of *variable* and we must subtract it for these: -6, -2, -4; the problem is ….".

Therefore the symbol *"a"* is considered like a constant, a numerical value, a variable, an unknown, a symbol without any value. The expression of Felicia is eloquent when she talks of the value that the sum of the square assumes: *" ... it depends on the meaning that we give a"*. Precisely, a difficult characteristic of symbolic values is that their precise nature changes, they can assume different aspects that have one characteristic in common: the fact that they are abstract.

From the qualitative and quantitative analysis we deduce that, almost all pupils effected algebraic calculations, that is they operated with the symbolic value *"a"*; even if in some cases we recorded errors, for example: considering *26+a* like *26a*.

While in one class the arithmetical thought prevailed over the algebraic one, because the pupils transformed the magic square in an arithmetical problem, attributing to *"a"* the value 92; in the other class, instead, the students completed the different boxes with numbers and expressions containing *"a"*, then stimulated by the teacher to complete a feed-back and they deduced that the value 92 was to be attributed to *"a"*. It is opportune to underline that, the pupils did not adopt some of the strategies anticipated in the a-priori analysis, for example: formulation of equations of first degree or systems of equations. Probably this is due to the fact that these matters were not discussed during the scholastic year.

However, it is important to put in evidence the presence of the algebraic thought in different phases of organization that depend on the single pupils.

2.4 CONCLUSIONS

From the analysis of the data, we deduce that the pupils of middle school (11-12 years of age) consider the symbol *"a"* of the magic square like: a constant, a numerical value, 0 and "some thing that varies", even if they had not approached the study of algebra yet. For the pupils of the high school (14-15 years of age), instead, the symbol *"a"* can assume very different aspects: a constant, a not negative constant, a numerical value, a variable, an unknown and symbol no value. These conceptions depend on the particular way of using the symbols within the activity of problem solving and from the individual development of the algebraic thought. Therefore these results allow to falsify the first hypothesis: "The pupils evoke the different conceptions of variable, also in the absence of an adequate mastery of the algebraic language".

The students of the middle school did not succeed in operating with the symbolic value *"a"* and they used essentially resolutive strategies in arithmetical language and/or natural language. Most pupils of high school used algebraic calculations, instead, but in certain cases they made some errors ($26 + a$ like $26a$).

In one class of high school the winning group used the arithmetical procedure, that consists of attributing the value 92 to *"a"*. Some pupils of the other group began with an arithmetical-algebraic procedure, instead, inserting some values: 5, $-4+a$, $9+3a$, etc.; but they did not succeed in completing the square because they did not understand to what box they were to attribute the value 92. Therefore, the students abandoned this strategy and they used also the arithmetical method, too.

In the other class of high school, both groups used the arithmetical-algebraic procedure, considering that a box of every line and every column must contain the symbol *"a"*; thus, they completed the square. In the boxes the pupils noticed the presence of: a, $a - 2$, $a - 4$, $a - 6$; stimulated by the teacher to complete a feed-back and therefore, they deduced that the value 92, the greatest, was to be attributed to *"a"*.

It is interesting to underline that, the pupils did not adopt any algebraic strategies anticipated in the a-priori analysis.

Therefore these results allow to falsify the second hypothesis: "The procedures in natural language and/or in arithmetical language prevail as resolutive strategies in absence of an appropriate mastery of the algebraic language".

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APPENDIX N° 1: Magic square used in Phase II in middle school

Complete, inserting the lacking numbers in the magic square, (the sum of the numbers of every line, column or diagonal is always the same one).

Sum 27

12		10
	9	

Sum 45

	15	
24		12

Sum 18

9		
	6	
5		

Sum 60

		8
	20	
		24

APPENDIX N° 2: Table related to the quantitative analysis of middle school

Legend: CAS 1C1 ÷ CAP 1C13: pupils

A1 ÷ A11: strategy

	A1	A2	A4	A6	A8	A11
CAS 1C1	1	1	0	0	0	0
CAS 1C2	1	1	0	0	0	0
CAS 1C3	1	1	0	0	0	0
CAS 1C4	1	1	0	0	0	0
CAS 1C5	1	1	0	0	0	0
CAS 1C6	1	1	0	0	0	0
CAS 1C7	1	1	0	0	0	0
CAS 1C8	1	1	0	0	0	0
CAS 1C9	1	1	0	0	0	0
CAS 1C10	1	1	0	0	0	0
CAS 1C11	1	1	0	0	0	0
CAS 1C12	1	1	0	0	0	0
CAS 1C13	1	1	0	0	0	0
CAS 1C14	1	1	0	0	0	1
CAP 1C1	0	0	0	0	0	1
CAP 1C2	0	1	0	0	0	0
CAP 1C3	0	0	0	0	0	1
CAP 1C4	1	1	0	0	0	0
CAP 1C5	0	0	0	1	1	0
CAP 1C6	0	0	0	0	0	1
CAP 1C7	0	0	1	0	0	0
CAP 1C8	0	1	0	0	0	0
CAP 1C9	0	0	1	0	0	0
CAP 1C10	1	0	1	0	0	0
CAP 1C11	0	0	0	0	0	1
CAP 1C12	0	1	0	0	0	0
CAP 1C13	1	0	0	0	0	0

APPENDIX N° 3: Magic square used in Phase II in high school

Game: The magic square

Complete, inserting the lacking numbers in the magic square, (the sum of the numbers of every line, column or diagonal is always the same).

Sum 15

	5	
4		8

Other magic squares used in Phase 1

Sum 18

	6	
5		3

Sum 21

	7	
12		6

Sum 24

	8	
11		7

Sum 27

	9	
8		6

Sum 30

	10	
11		13

Sum 33

	11	
14		10

Sum 60

	20	
32		24

Sum 45

	15	
24		12

Sum 48

	16	
19		7

Sum 63

	21	
33		25

Magic square used in Phases 2 and 3

Delivery: Complete the magic square so that the greatest number inserting is equal to 92

Sum 26 + a

14		1	
	9	12	
11		a	10
	16	13	

APPENDIX N° 4: Analysis a-priori of the square 4×4

Arithmetical procedure

We know that at least a number of the magic square must be equal to *92* and that the sum must be *26 + a*.

From the examination of the column in which the numbers *1, 12* and *13* are inserted, we observe that their sum is *26*, therefore *"a"* must be replaced by the maximal number, that is *92*.

Arithmetical-algebraic procedure

In the question the pupil does not take into consideration, in the beginning, that the greatest number to insert is 92.
He/she considers, instead, that a box of every line and every column must contain the symbol *"a"*.
After this premise he/she understands that in the 4° line the sum of the two numbers already introduced is greater than to *26*. Then he/she deduces that it is necessary to operate in the set Z.
The principal diagonal already contains the symbol *"a"*, therefore he/she completes with the lacking number that gives the total sum of *26*. At this point, he/she returns to the fourth line and in the empty box he/she puts the symbol *"a"* plus the negative relative number, that gives the sum of 26.
With the same procedure he/she proceeds moving to those lines or in those columns that have only one empty box to fill. Looking at the complete square, he/she points out the following values: *a; a - 2; a - 4; a - 6*. He/she considers that the value 92, the greatest, will have to replace *"a"*.

Algebraic procedure

1. The pupil considers *"a"* constant. He/she departs from the third line or from the principal diagonal because they contain three elements, he/she attributes an unknown to the lacking value and plans an equation of first degree.

   $11 + x + a + 10 = 26 + a$

   or

   $14 + 9 + a + x = 26 + a$.

   The route to complete the square, is not forced because he/she can proceed considering the lines or columns with three elements.

   At the end of the procedure the pupil understands that three inserted elements are literal terms in which *"a"* appears.

14	a - 4	1	15
7	9	12	a - 2
11	5	a	10
a - 6	16	13	3

From here there are two possibilities:

- the pupil does not consider the question that the greatest number is 92
- he/she realizes the question and he/she attributes to *"a"* the value 92, because the other three values are inferior to *"a"*.

2. Considering *"a"* constant it is possible to plan a linear system of 7 equations with 7 unknowns.
   This strategy is abandoned immediately because it is long and difficult.

3. Considering *"a"* variable it is possible to plan a linear system of 8 equations with 8 unknowns.
   This strategy is abandoned immediately because it is long and difficult.
4. The pupil considers *"a"* variable and plans a linear system of 3 equations with 3 unknowns, of which one is an identity. Therefore, to complete the square he/she introduces other variables. During the resolution he/she realizes that some variables depend on *"a"*. Accordingly it is essential to give to *"a"* constant value that will have to be necessarily 92, because the other introduced variables result inferior to 92.

$$11 + x + a + 10 = 26 + a$$

$$1 + 12 + a + 13 = 26 + a$$

$$z + 9 + x + 16 = 26 + a$$

14	z	1	p
m	g	12	l
11	x	a	10
g	16	13	t

$z = a - 4$

$g = a - 6$ $\qquad a = 92$

$l = a - 2$

APPENDIX N° 5: Table related to the quantitative analysis of the High School

Legend: b1 ÷ a17: pupils,
A1 ÷ A14: strategy

	A1	A3	A5	A8	A9	A10	A11	A12	A13	A14
b1	0	0	0	1	0	0	0	0	0	0
b2	0	0	0	0	0	0	0	1	0	0
b3	0	0	0	0	0	0	0	1	0	0
b4	0	0	0	1	0	0	0	0	0	0
b5	0	0	0	0	0	0	0	0	1	0
b6	0	0	0	0	0	0	0	1	0	0
b7	0	0	0	0	0	0	0	1	0	0
b8	0	0	0	0	0	0	0	0	1	0
b9	0	0	0	0	0	0	0	1	0	0
b10	0	0	0	0	1	0	0	0	0	0
b11	0	0	0	0	0	0	0	1	0	0
b12	0	0	0	0	0	0	0	1	0	0
b13	0	0	0	0	0	0	0	1	0	0
b14	0	0	0	0	0	0	0	1	0	0
b15	0	0	0	1	0	0	0	0	0	0
b16	0	0	0	0	0	0	0	1	0	0
b17	0	0	0	0	0	0	0	0	0	1
b18	0	0	0	0	0	0	0	0	1	0
b19	0	0	0	0	1	0	0	0	0	0
b20	0	0	0	0	0	0	0	1	0	0
b21	0	0	0	0	0	0	0	1	0	0
b22	0	0	0	0	0	0	0	1	0	0
a1	0	0	0	0	1	0	0	0	0	0
a2	1	0	0	0	0	0	0	0	0	0
a3	0	0	0	0	0	0	1	0	0	0
a4	0	0	0	0	1	0	0	0	0	0
a5	0	0	1	0	0	0	0	0	0	0
a6	0	1	0	0	0	0	0	0	0	0
a7	0	0	0	1	0	0	1	0	0	0
a8	0	0	0	1	0	0	0	0	0	0
a9	0	0	0	0	0	1	0	0	0	0
a10	1	0	0	0	0	0	0	0	0	0
a11	0	0	0	0	0	1	0	0	0	0
a12	0	0	0	0	0	0	0	1	0	0
a13	0	0	0	0	0	0	1	0	0	0
a14	0	0	0	0	0	1	0	0	0	0
a15	0	0	0	1	0	0	0	0	0	0
a16	0	0	0	0	0	1	0	0	0	0
a17	0	0	0	0	0	1	0	0	0	0

APPENDIX N° 6: Protocols of the High School

Protocolli dei gruppi

Nella terza fase gli alunni sono stati divisi in due gruppi e ogni gruppo aveva un portavoce. Sono state registrate le discussioni. Dal lavoro di sbobinatura si sono ottenuti i seguenti protocolli:

CLASSE: 1° A

Protocollo del Gruppo A **(portavoce Giusy Quaceci)**

- **Giusy:** "Prima di tutto ho cercato il valore di a, in modo che 26+a=92 nella colonna dove si trova a. Ho trovato il valore mancante per ottenere 92, poi ho fatto 92-26=66 che è il valore di a, ed ho continuato a compilare le altre colonne avendo come riferimento il valore di a e la somma 92".
- **Ilenia:** "Leggendo l'intestazione ho visto che il numero più grande da inserire era 92, allora ho fatto 26+92 ma non risultava, allora ho aggiunto 91. Poi ho fatto 26+91=117, ho sommato i numeri ed il risultato del quadrato magico era 117, tranne una colonna che non mi risultava".
- **Valentina:** "Io ho provato ma non ci sono riuscita".
- **Marisa:** "Anche io ho tentato, ma non ci sono riuscita".
- **Erika:** "Ho fatto lo stesso procedimento di Giusy, ho fatto valere a=66 e ho sommato per ogni numero".
- **Francesca:** "Io ho fatto 92-26=66 e poi ho trovato i numeri adatti per completare il quadrato magico".
- **Giusy:** "Dai ragazzi iniziamo a svolgerli, via!".
- **Ilenia:** "Il testo qua dice che nel quadrato magico il numero più grande sia 92, però dobbiamo inserire almeno un numero che arrivi 92, in modo che si completi il quadrato".
- **Giusy:** "Secondo me, abbiamo sbagliato perché noi abbiamo messo come somma 92, invece non è la somma, ma il numero da inserire. Quindi, se a=92, dobbiamo dare un valore ad a".
- **Ilenia:** "Bisogna fare 26+92?".
- **Marisa:** "No ma che c'entra".
- **Giusy:** "Ma in sostanza cosa si deve fare?".
- **Valentina:** "Allora si deve fare 92-66? Anzi 92-26=66?".
- **Ilenia:** "Ma se noi facciamo 66+26=92, non possiamo fare 92 come somma totale".
- **Giusy:** "92 lo abbiamo considerato come somma totale, invece 92 è il numero da inserire".
- **Marisa:** "Allora non si fa 92-26?".
- **Giusy:** "No, 92 bisogna metterlo al posto di a. Proviamoci. Se mettiamo 92 al posto di a, abbiamo altri due numeri nella colonna: l'11 e il 10 quindi 92+10=102;102+11=113 e poi bisogna fare la somma".
- **Ilenia:** "Allora si fa così: nella terza colonna c'è 1, 12, a, 13, se al posto di a mettiamo il valore 92 si fa 92+1=93+12+13 risulta 118 e secondo me bisognava sommare gli altri numeri in modo che la somma dava 118".
- **Giusy:** "Attribuito ad a il valore di 92, il testo ci dice che la somma è 26+a, quindi dobbiamo fare la somma 26+92=118. Nella colonna abbiamo 11, il valore di a che è 92, 10, 13 e quindi la somma = 118. (Nelle altre colonne) dobbiamo trovare il valore mancante che sommato agli altri numeri ci dia 118. Stiamo riuscendo, dai 92+9=101; 101+14=115+3=118".
- **Giusy e Ilenia:** "3+13+16=32. Ora dobbiamo trovare un numero che sommato a 32 mi dia 118, è 86. Ora si fa 86+5=91+12=103, 103 per arrivare 118 troviamo 15. Ora facciamo 15+1=16, 16+14=30, per arrivare a 118 si fa 118-30=88. Poi 88+9=97+5=102, 102+16=118. Ora bisogna fare 15+10=25+3=28, 118-28=90. Proseguiamo:

90+9=99+12=101, 101+17=118. Ora facciamo 11+7=18, 18+14=32, 32+86=118. Evviva! Abbiamo vinto!”.

Validazione

Giusy: “Abbiamo visto che ci sono molti procedimenti, tra i quali uno che ha trovato la mia compagna Francesca che aveva attribuito ad a il valore 66. Però leggendo bene il testo abbiamo capito che potevamo attribuire ad a il valore numerico 92, visto che il testo ci diceva che il valore da inserire era 92. La somma data è 26+a, quindi 26+92=118. Nella tabella abbiamo sostituito a con 92 e risolvendo la terza colonna verticale, il risultato era 118, quindi la somma del quadrato magico è 118. Facendo lo stesso procedimento per le altre colonne, il risultato era sempre quello. Possiamo concludere che il valore a che è 92 è costante”.

Protocollo del Gruppo B: (portavoce Giusy Martello)

- **Giusy:** “Ho risolto questo quadrato magico, trovando il numero che mancava per arrivare alla somma 26+a della terza linea orizzontale dove i numeri che avevo erano 11, a, 10 e per arrivare a 26+a ho aggiunto 5. Così risulta 26+a. Considerando a una lettera, ho completato la seconda linea verticale. Avendo il 16, il 9 e il 5, ho messo -4+a e così sono giunta al risultato. Dopo ho risolto la prima linea orizzontale dove avevo 14, -4+a , +1 e sono giunta al risultato 9+3a.In alcune caselle ho messo alcuni numeri relativi come ad esempio -2a, +3a. Dopo ho risolto anche le altre linee però non riflettendoci, ho sbagliato un calcolo, ma ho trovato una strategia per risolverlo”.
- **Lorena:** “Il numero più grande da inserire era 92”.
- **Giusy:** “Infatti il dubbio che abbiamo tutti, e che mi sono posta pure io quando l'ho fatto, è che questo numero si dovrebbe inserire, però non ho trovato in quale posto”.
- **Igea:** “Secondo me l'ho dobbiamo inserire nella a”.
- **Giusy:** “Però questo 92 si deve sottrarre, si deve togliere per arrivare alla a”.
- **Igea:** “Sempre 92 resta”.
- **Giusy:** “Non resta 92, perché la soluzione è 26+a”.
- **Igea:** “26+92 quanto fa? Fa 118. Se tu fai la somma 92+12+1+13 fa 118, e gli altri devono risultare pure 118”.
- **Lorena:** “Secondo me ha ragione Igea perché qua c'è un legame, perché è 26+a, tu devi attribuire il valore ad a, quindi se a = 92 la somma deve venire 118”.
- **Giusy:** “Non può essere un quadrato magico, perché il risultato deve venire uguale per tutti infatti dovrebbe risultare 26+a. Se noi facciamo come dice Igea, non risulterebbe 26+a, risulterebbe 118+a”.
- **Felicia:** “Però dipende dal significato che diamo ad a”.
- **Lorena:** “La somma vale 118 e basta, se a vale 92 il risultato non viene più 118+a ma solo 118”.
- **Felicia:** “Giusy, praticamente la a devi far finta che è 92”.
- **Igea:** “Attribuiamo ad a il valore 92, poi fai la somma 26+92 che fa 118”.
- **Giusy:** “Quindi abbiamo trovato la soluzione: dobbiamo attribuire ad a il valore di 92 e trovando questo valore dobbiamo riuscire a risolvere tutto il quadrato magico”.
- **Lorena:** “Sin dall’inizio avevamo detto di attribuire un valore ad a che non cambiava, una volta si diceva 3, una volta 5 e così via. Allora abbiamo risolto il dubbio, grazie ad Igea!”.
- **Eugenia:** “Perché avete messo il 92 proprio nella a ?”.
- **Giusy:** “Perché a è senza valore e il testo ci dice che il numero più grande da inserire è uguale a 92”.
- **Eugenia:** “Secondo me è sbagliato, invece ad a non si deve attribuire niente”.
- **Igea:** “Secondo me abbiamo messo 92 al posto di a perché altrimenti il valore 92 dove lo metti? Così se tu fai la somma 26+92 uguale 118”.
- **Lorena:** “Secondo me ha ragione Eugenia perché ad a non si deve dare alcun valore”.
- **Eugenia:** “Secondo me si deve trovare il valere 26 e poi aggiungere a, non dare il valore ad a”.

- **Felicia:** “Invece secondo me no, a deve avere un valore, perché a è una costante, non è variabile”.
- **Giusy:** “Nella terza linea orizzontale il 5 è giusto. Il risultato deve essere 118, ora proviamo a risolvere anche le altre. Sappiamo che a=92, quindi procediamo. Facciamo 12+5=17, dobbiamo arrivare a 118, possiamo fare 90 …..”.
- **Vanessa:** “Secondo me ha ragione Igea e non Eugenia perché qui il testo dice che si deve inserire un numero uguale a 92 . Quindi in qualche modo a 92 lo dovete fare entrare da qualche parte e l’unico elemento che non ha valore è proprio a. Quindi a=92 secondo me, secondo Eugenia invece no”.
- **Giusy:** “Anche secondo me hanno ragione Igea e Lorena”.
- **Felicia:** “E allora tu che stai svolgendo la linea trasversale, dove la metti la a, dove lo metti il 92?”.
- **Giusy:** “Il 92 non si deve ripetere tante volte!”.
- **Felicia:** “Perché non si deve ripetere?”.
- **Giusy:** “Perché nel quadrato magico non si possono ripetere gli stessi numeri anche se hai fatto errori di calcolo per risolverlo. Dobbiamo fare 92+17 dove 17=5+12 poi 92+17=109…”.
- **Eugenia:** “Secondo me ci sono numeri negativi, oltre a quelli positivi”.
- **Lorena:** “Secondo me non è come dici tu”.
- **Felicia:** “Se abbiamo detto che a è una costante, come può essere negativa?”.
- **Giusy:** “Sto risolvendo la prima linea orizzontale. Quindi 14+1+92=107 per arrivare a 118 abbiamo 11. Perché non riesce? 9+5+12+il valore di a che è 92 da 118, fino a qua è giusto. Risolviamo la prima linea verticale dove manca un solo numero”.
- **Lorena:** “Secondo me hai sbagliato ad inserire il 92, perché lo hai messo qui sopra”.
- **Felicia:** “Perché questa linea trasversale risulta 118; 26+92 uguale 118”.
- **Igea:** “Secondo me potrebbe essere così ma non ne sono sicura”.
- **Lorena:** “Ma a vale 92, non c’è un altro valore che è 92. Il 92 è qua oppure qua, non deve ripetersi due volte lo stesso valore. C’è qualcosa che non quadra. L’altro gruppo ha già finito, ma non prediamoci d’animo “.
- **Giusy:** “Siamo riusciti a risolverlo, trovando numeri tutti diversi”.
- **Lorena:** “Hai ripetuto due volte 92”.
- **Felicia:** “Non è cosi. Lorena dov’è il numero che si ripete”.
- **Igea:** “Secondo me non si può mettere cosi”.
- **Giusy:** “Si può mettere 90+2 che non è 92”.

Validazione

Parla Giusy portavoce del gruppo:
“Quando abbiamo discusso, oltre la mia strategia, ne abbiamo trovata un’altra che consiste nell’attribuire ad a il valore di 92, e alla fine abbiamo deciso di utilizzare questa. Quindi abbiamo attribuito ad a il valore di 92 e sommandolo al 26 doveva dare il risultato di 118 e così abbiamo messo i numeri e completato la tabella che così è risultata. Ci siamo accorti che la strategia è giusta ma ci sono degli errori di calcolo”.

CLASSE: 1° B

Protocollo del Gruppo A (portavoce Rita Di Martino)

- **Prof: “**Avanti, forza riempite il quadrato”.
- **Rita Di Martino:** “Allora lo abbiamo risolto tutti no!! Allora iniziamo da qua perché abbiamo due numeri e una lettera, quindi viene praticamente 21+a per arrivare a 26+a gliene mancano 5. Ora facciamo questo perché abbiamo tre numeri e ci va la a, per forza, e facendo la somma viene 30 quindi dobbiamo sottrarne quanto? 4, quindi viene a-4. Adesso facciamo questo abbiamo altri tre numeri, nella prima riga in alto, dobbiamo aggiungere 5. Ora facciamo la diagonale che inizia con 14 abbiamo due numeri e una lettera, la lettera a, quindi dobbiamo aggiungere un numero, dobbiamo aggiungere 3. Giusto, siete convinti?!”.

- **Componenti del gruppo:** "Si!!".
- **Rita: "**Allora, praticamente ……"
- **Componenti del gruppo: "**Qua ci va la a!".
- **Prof.:** "Vi ricordo che il numero più grande da inserire è uguale a 92".
- **Rita:** "Allora facciamo l'altra diagonale viene…".
- **Anna Matranga: "**Viene 10, 15, 18….quindi 15".
- **Rita:** "Si certo 15!! Qui viene 20, 32 quindi ne dobbiamo sottrarne 6 ; viene 24-6, quindi ne dobbiamo mettere 7".
- **Luana Romano: "**19 e 9, 36 No!?".
- **Silvia:** "Quella da i numeri!!".
- **Rita:** "Quindi qua viene 11, no aspetta…".
- **Anna:** "19".
- **Rita:** "28, quindi –2 ; l'ho detto già io, qua viene 28-2, giusto?! Ora dobbiamo ragionare sul 92. Qui dice: completa il quadrato magico in modo che il numero più grande da inserire sia uguale a 92".
- **Prof.: "**Anna tu devi pensare!"
- **Anna:** "Si io sto pensando".
- **Luana:** "Nella terza colonna orizzontale".
- **Rita:** "Nella terza riga!!".
- **Prof.:** "No! nella terza riga".
- **Rita:** "Abbiamo messo…".
- **Prof.:** "Prima cosa hai messo?!".
- **Rita:** "Nella terza riga il numero 5".
- **Prof.:** "Perché…".
- **Rita:** "Perché già avevamo due numeri e la lettera a…".
- **Silvia:** "Quella che parla è Rita Di Martino".
- **Rita:** "Poi abbiamo inserito nella seconda colonna da sinistra…".
- **Silvia:** "E la scrittrice è Anna Matranga".
- **Rita:** "…da sinistra verso destra a-4 perché avevamo già tre numeri esatti e poiché…".
- **Prof.: "**Dovete convincere anche i vostri compagni. Tu Ciofalo lo stai capendo?".
- **Rita: "**…e poiché la somma veniva maggiore di 26 abbiamo sottratto il numero 4. Poi, quindi, dove siamo arrivati? Si poi abbiamo inserito il numero 15 nella prima riga poiché avevamo già due numeri e la lettera a".
- **Rita: "**Poi abbiamo inserito nell'ultima riga il numero 3 poiché, nella diagonale che inizia con il numero 14 avevamo già due numeri esatti e la lettera. Poi abbiamo inserito nell'ultima riga a-6 perché nella diagonale che inizia con il numero 15 avevamo già tre numeri corretti. Poi, alla fine, abbiamo inserito nella prima colonna il numero 7 perché già avevamo due numeri e la lettera a e alla fine, nell'ultima colonna , a-2".
- **Silvia:** "Adesso viene il bello il 92!! Chissà dove lo dovremo mettere!!".
- **I componenti:** "Sommiamo tutto quanto!!".
- **Rita:** "Tutto quanto? Io non penso che sia tutto quanto perché altrimenti avrebbero messo la somma, invece, qui dice: completa il quadrato magico in modo che il numero più grande da inserire sia uguale a 92".
- **Valeria Passarello:** "Io ho provato a fare 92:4, per 4*4 no!? E mi è venuto 23".
- **Rita:** "Ma potrebbe essere, perché…..se noi non, magari non addizioniamo questi –6 –2 in ogni colonna, in ogni riga…".
- **Valeria:** "Io ho fatto l'addizione di tutti i numeri normali più la sottrazione e mi veniva 104".
- **Rita:** "Aspetta 4*4 fa 16; se noi facciamo 92-16 ,92:16… Per vedere questo 92 in ogni quadratino a quale numero corrisponde".
- **Anna:** "Fa 23".
- **Rita:** "Fa 23; no devi fare diviso16, per vedere… se noi facciamo 4*4=16 e poi diviso 92 otteniamo il numero che in ogni buco ci deve stare"
- **Anna:** "Scusa, non può essere che…".

- **Rita:** “No, ma perché…”.
- **Anna:** “No, qui infatti dice, un attimo solo…”.
- **Rita:** “No, ma io ho fatto così perché qui si sta parlando del numero più grande da inserire”.
- **Anna:** “E appunto, quindi, lo dobbiamo inserire nella tabella”.
- **Valeria:** “Ma ci sarà un nesso logico per mettere questo 92, quindi io per questo ho pensato di fare…”.
- **Rita:** “Ma una calcolatrice non ce l’abbiamo? Per fare subito i calcoli”.
- **Anna:** “Ma questo 92 lo dobbiamo inserire nella tabella”.
- **Rita:** “Fai 92:16; si vediamo, perché io non penso che comunque vada inserito il 92 in questo modo, a meno che…”.
- **Valeria:** “2,75”.
- **Rita:** “Dunque…”.
- **Anna:** “Qui dice il numero più grande da inserire, quindi dobbiamo inserirlo”.
- **Silvia:** “Un momento di suspance!”.
- **Rita:** “Il numero più grande da inserire è uguale a 92; a meno che, forse con questo metodo che noi abbiamo fatto…., se utilizziamo un altro metodo con i numeri più grandi…”.
- **Valeria:** “Facciamo la somma e poi dividiamo?”.
- **Rita:** “Fare tutta la somma ! Sottraendo pure questi numeri che noi abbiamo sottratto?”.
- **Valeria:** “Prima…ehh…”.
- **Rita:** “Proviamo”.
- **Valeria:** “Io credo che viene 104”:
- **Rita:** “Scusa facciamolo con la calcolatrice, dato che…”.
- **Anna:** “Scusa ma volete sommarli?”.
- **Rita:** “Ma io non riesco a capire, perché, secondo me, questa somma non centra”.
- **Luana:** “I numeri 26?! 26,26,26,26”.
- **Rita:** “No! vuole sommare…”.
- **Anna:** “Rita ma se dobbiamo inserirlo questo 92 lo metti dentro la tabella”.
- **Rita:** “Si, ma secondo me, sto inserire è una cosa, cioè inserire in un modo particolare, perché altrimenti sarebbe troppo semplice inserire il 92 nella tabella, secondo me ce sotto qualcosa, un ragionamento”.
- **Valeria:** “104 no, giusto viene!”.
- **Rita:** “104”.
- **Jessica Oliva:** “Ma perché 16?”.
- **Anna:** “4*4=16”.
- **Rita:** “Fai :16”.
- **Jessica:** “Ma perché 16?”.
- **Anna: “**Perché 4*4; ma se si fa 92 meno tutti questi numeri?! Avete provato a farlo”.
- **Jessica:** “Ma già dava 104 , 92-104 non si può fare”.
- **Anna:** “…dico la somma di tutti i numeri che abbiamo fatto da…”.
- **Rita:** “Dici quelli che abbiamo inserito?”.
- **Anna:** “Si, tutti i numeri che abbiamo inserito”.
- **Prof.:** “Anna…”.
- **Anna:** “Dobbiamo provare a sommare tutti i numeri che abbiamo inserito e poi a dividerli per 92 o…”.
- **Prof.:** “No! Perché, leggi bene”.
- **Anna:** “Ma qui dice il numero da inserire!”.
- **Prof.:** “Dobbiamo inserire un numero, e questo numero, il più grande deve essere 92”.
- **Anna:** “Quindi dobbiamo inserirlo”.
- **Valeria:** “Ma quindi dobbiamo rifare il quadrato?”.
- **Prof.:** “No, è già fatto il quadrato”.
- **Anna:** “Perché qua dice: completa il quadrato magico, quindi dobbiamo farlo, ed è fatto, in modo che il numero più grande sia uguale a 92; è un enigma!”.

- **Prof.:** "Il numero più grande da inserire è 92. Già questi ce li hai 15, 3, li puoi cambiare? No!!".
- **Anna:** "No!".
- **Prof.:** "La somma è sempre 26+a".
- **Anna:** "No, ormai è fatto".
- **Prof.:** "Allora dove lo devi mettere questo 92? La somma sempre 26+a deve dare".
- **Giusy Gangemi:** "Una lettera può essere".
- **Prof.:** "Quale lettera?".
- **Componenti del gruppo:** "La a!!".
- **Prof.:** "Provate!".
- **Luana:** "Quindi 92-6".
- **Prof.:** "Cosa hai detto Giusy? Dillo di nuovo".
- **Luana:** "Sostituire le lettere a con 92".
- **Prof.:** "Messe al posto di?".
- **Giusy: "**Di a mettere 92".
- **Anna:** "Aspetta un attimo, dobbiamo sottrarre per tutto: quindi fai –6…".
- **Valeria:** "Cosa?!".
- **Anna:** "86".
- **Valeria:** "86 meno cosa?".
- **Anna:** "Meno 5, 12-15".
- **Rita:** "Se noi mettiamo 92 e lo attribuiamo alla a, allora la a si deve considerare come una specie di variabile e dobbiamo sottrarla per questi: -6,-2,-4; il problema è…".
- **Valeria:** "Dobbiamo sottrarre tutti i numeri fino ad arrivare a 26+a".
- **Prof.:** "+a, ma " a ", a cosa è uguale avete detto?".
- **Componenti del gruppo:** "A 92".
- **Giusy:** "La a diventa 92".
- **Prof.:** "La a diventa….?".
- **Giusy:** "92".
- **Prof.:** "Quindi?".
- **Giusy:** "Fa 26+92, viene".
- **Prof.:** "Brava Giusy!!".
- **Silvia:** "Giusy ha trovato la soluzione!".
- **Rita:** "La a=92; 92+26=118".
- **Enza Alessandro:** "Fa 118? Ah, finisce così?"
- **Anna:** "E' così punto e basta. Abbiamo sostituito alla a il numero 92".
- **Rita:** "Sottolinea questo passaggio e lo ripeti dopo; sottolinea e riporta sotto a parole….; quindi, abbiamo sostituito alla a il numero 92 e lo abbiamo addizionato al numero 26 ed abbiamo ottenuto il numero 118".
- **Componenti del gruppo:** "Prof. abbiamo finito!!".
- **Silvia:** "La prof. ci deve fare una domanda?".
- **Prof.:** "Avete spiegato tutto?".
- **Anna:** "Si, abbiamo sostituito alla a il numero 92 e sommato il 26 fa 118, perché a è una variabile".
- **Prof.:** "Si, ma perché proprio 92 ad a?".
- **Rita:** "Perché già avevamo dei numeri, non si possono sostituire dei numeri".
- **Prof.:** "Questo mi sta bene, però c'è un'altra risposta; perché il numero più grande, qua dice, è 92?".
- **Giusy:** "Perché tutti i numeri sono minori di 92".
- **Prof.:** "Tutti e anche..? Giusy, e anche….?".
- **Giusy:** "E anche il 26".
- **Prof.:** "Questi si vedono che sono più piccoli di 92, e anche…".
- **Anna:** "E anche la somma".
- **Prof.:** "No, la somma non è una variabile? Anche chi ?".

- **Giusy:** “I numeri che abbiamo inserito!”.
- **Prof.:** “Questo si vede che sono più piccoli”.
- **Giusy:** “I numeri che già c’erano”.
- **Prof.:** “a-4 ; avreste potuto scrivere a=92 pensateci; che dici?”
- **Stefania Mattia:** “Perché facendo la sottrazione da 92 meno il numero che abbiamo inserito ci da un numero minore di 92”.
- **Prof.:** “Brava! Perché qua, a-4 viene più piccolo di 92; a-6 ci da un numero più piccolo. Scrivete questo”.
- **Rita:** “Abbiamo sostituito alla lettera a il numero 92 e lo abbiamo addizionato al numero 26 e abbiamo ottenuto il numero 118. Abbiamo fatto questo perché sottraendo il numero 92 a quei numeri che accompagnavano la lettera a abbiamo ottenuto un numero minore di 92”.

Validazione

“Sono la rappresentante del gruppo A Rita Di Martino. Allora, abbiamo inserito nel quadrato magico il numero 5 perché nella terza riga avevamo già due numeri e la lettera a. Poi abbiamo inserito la somma dei numeri che si trovavano nella seconda colonna poiché già avevamo tre numeri ed abbiamo inserito a-4 perché andava inserita la a e perché eseguendo la somma dei tre numeri si otteneva un numero che era maggiore di 26 e più precisamente di 4 numeri e quindi abbiamo sottratto il numero 4 agli altri numeri, quindi a-4. Poi, successivamente, abbiamo eseguito tutti gli altri calcoli”.
“Per quanto riguarda il 92, il test diceva di completare il quadrato magico in modo che il numero più grande da inserire fosse 92; poiché avevamo già inserito tutti i numeri e il quadrato magico lo avevamo già risolto, potevamo attribuire il 92 solamente alla a considerandola, quindi, una variabile. Tutti i numeri che abbiamo inserito sono minori di 92 e, in oltre, eseguendo la sottrazione tra il 92 e i numeri che accostano la variabile a si ottiene anche in questo caso un numero minore di 92; ad esempio a-4,a-6,a-2 quindi necessariamente a deve essere 92”.

<u>Componenti del gruppo</u>:
Mattia Stefania, Anna Matranga, Rita Di Martino, Ciofalo Davide; Romano Luana, Silvia Rausa, Jessica Oliva , Alessandro Rosa, Giusy Gangemi, Valeria Passatello, Alessandro Enza.

Protocollo del Gruppo B (portavoce Ester Sanalitro)

- “Mi chiamo <u>Ester Sanalitro</u> abbiamo iniziato dalla terza colonna orizzontale perché mancava un solo termine per completare la somma, e quindi abbiamo raggiunto la somma 26 +a facendo (11+a+10) che verrebbe 21a, poi abbiamo fatto (26+a-21a) e il risultato è cinque e quindi il termine mancante è 5. Poi abbiamo fatto la prima colonna obliqua facendo (14+9+a) e verrebbe 23a, quindi abbiamo sottratto da 26 +a il 23+ a e verrebbe 3.
  Poi abbiamo continuato con la seconda colonna verticale e abbiamo sommato 16+5+9 che fa 30 e così abbiamo messo a-4 che viene 26a; ora continuiamo con l’ultima colonna orizzontale facendo 16+13 che viene 29 che sommato al 3 fa 32 e quindi mettiamo a-6.
  Stiamo continuando con un'altra colonna obliqua e abbiamo i numeri a-6; 5 e 12 quindi facciamo 12+5 che fa 17, poi sottraiamo a-16 e viene 11 e quindi il termine mancante è 15.
  Continuiamo con l’ultima colonna verticale facendo 15+10 che viene 25, che sommato al 3 fa 28, quindi il termine mancante è a-2 ora rimane l’ultima colonna, che è la seconda orizzontale, con i numeri (9-12-a-2) facciamo la somma sottraiamo 26 a e troviamo 7.
  Ora possiamo verificare se l’operazione eseguita è stata corretta facendo la somma e vedendo se in tutte le colonne otteniamo 26+a”.
  Ora dobbiamo risolvere il problema del 92 perché dice “completa il quadrato magico in modo che il numero più grande da inserire sia uguale a 92” quindi per inserire il 92 potrebbero anche esserci altri metodi…”
- **Ester:** “L’esercizio dice: completa il quadrato magico in modo che il numero più grande da inserire sia uguale a 92”.
- **Prof.:** “Pensate”.

- **Ester:** “Ma in pratica, professoressa dobbiamo inserire il 92?”.
- **Prof.:** “Si, che sia più grande”.
- **Ester:** “Oltre il 92?”.
- **Prof.:** “No. Così c’è scritto? Concetta cosa stavi pensando?”.
- **Concetta Oste:** “Forse fare la somma di tutti i termini, se è più grande di 92 sottrarlo con 26 a .
- **Prof.:** “Proviamo!”.
- **Irene:** “Io dicevo, sommiamo solo quelli che abbiamo scritto noi”.
- **Concetta:** “Non ci arriviamo a 92”.
- **Simona:** “Oppure sommare tutti i numeri di una colonna e moltiplicarla per 26”.
- **Ester:** “Forse sommare tutti i numeri scritti e sottrarli con tutti quelli che abbiamo scritto noi.
- **Morena:** “Oppure fare la somma di tutte le colonne per 4”.
- **Prof.:** “Il massimo numero ….Già voi avete dei numeri. Si! E sono 14, 11, 7, 10, in qualche posto questi già ci sono. Dobbiamo metterci il 92”.
- **Concetta:** “Forse il 92 metterlo al posto della lettera a”.
- **Ester:** “E se aggiungiamo un'altra colonna?”.
- **Jessica Grisaffi:** “Io dicevo di sottrarre il 92 per il risultato di ogni colonna”.
- **Irene:** “Non può risultare!”.
- **Ester:** “Secondo me non ha senso…”.
- **Serena:** “Io dicevo di sommare tutti i numeri e sottrarli a quelli con cui noi abbiamo sottratto a-4, a-2...”.
- **Jessica Giunta:** “Prima stiamo sommando tutti i numeri che erano già presenti nel quadrato magico e quelli che abbiamo inserito noi. Dopo di che il risultato intendiamo sottrarlo a 26 solo che non ci risulta perché ci viene 82 e invece deve risultare 92”.
- **Serena:** “Sommiamo prima tutti i numeri positivi tranne quelli negativi, cioè a-2, a-6 e cosi via.
- **Jessica Giunta:** “Quelli che hanno a-2, a-4…..E sottrarli a 26”.
- **Serena:** “Eh ! no a 26. E perché a 26?”.
- **Jessica Giunta:** “Non si può sottrarre a 26”.
- **Serena:** “No, va bene si può sottrarre”.
- **Jessica Giunta:** “No perché è un numero troppo grande”.
- **Serena:** “Allora il risultato della somma dei numeri negativi dà –12 più quelli positivi dà 82; ma non può essere perché dovrebbe risultare 92. Si ma facciamola questa somma”.
- **Jessica Giunta:** “Abbiamo provato a fare la somma dei numeri positivi meno la somma dei numeri negativi (116-12)”.
- **Serena:** “E questo volevo dire io”.
- **Morena:** “Viene un numero con lo zero”.
- **Jessica:** “Verrebbe 90, ma forse ho sbagliato a fare il calcolo”.
- **Prof.:** “Fatemi capire!”
- **Jessica:** “Se facciamo la somma di tutti i numeri positivi tranne a-4 e gli altri numeri negativi”.
- **Ester:** “No stiamo sbagliando, il massimo numero da inserire è 92, giusto prof.?”.
- **Prof.:** “Si , il massimo numero da inserire è 92”.
- **Ester:** “Ma si può inserire un'altra colonna?”.
- **Jessica:** “Se le colonne sono tutte occupate, come facciamo a inserire un altro numero?”.
- **Prof.:** “Ci sono alcune colonne che non sono ben definite”.
- **Ester:** “Ad esempio dove c’è scritto a-2…”.
- **Jessica:** “16+5+9 no e se facciamo il 92 meno la somma di alcune colonne”.
- **Prof.:** “Il massimo numero da inserire sia 92”.
- **Concetta:** “Se facciamo la somma 26 * 4 (numero delle colonne). Il risultato verrebbe 104 meno la somma dei numeri negativi (12) e dà come risultato 92”.
- **Jessica Grisaffi:** “Ma non è il risultato 92, lo devi usare!”.

- **Concetta:** "Io ho fatto la somma 26+a per 4, il numero delle colonne che dà 104 meno la somma dei numeri negativi, cioè –12, risulta 92".
- **Prof.:** "Dove lo metti il 92?".
- **Jessica:** "Ma se noi lo inseriamo, poi non risulta più 26 a".
- **Prof.:** "Ma non è 26a, è 26+a".
- **Serena:** "Se facciamo 92 –26 si trova a".
- **Simona:** "Ma che centra!"
- **Prof.:** "E' sbagliato ma centra!".
- **Ester:** ""a" è un incognita quindi deve essere sostituito con un numero che non supera il 92, quindi per esempio "a-4" se noi mettiamo un numero che sottratto ad a-4 poi sommato con tutta la colonna dobbiamo trovare il massimo numero che sia 92 e quindi vediamo se risulta. Stiamo cercando di sostituire "a" con un numero, però non più grande di 92".
- **Irene:** "Ma se facciamo 16+5+9=30-4=26+92 (che è la "a") e viene 26+92 ed ecco risultato il problema "a" è 92".
- **Morena:** "Penso che: il numero deve essere per forza 92 perché sottratto con "a-4" dà –92 quindi è un numero minore di 92 e anche a-6 ed a-2 perché tutti i numeri sarebbero minori di 92".

Validazione:

Parla Ester Sanalitro, portavoce del gruppo.
"Faccio parte del secondo gruppo e mi chiamo Sanalitro Ester, noi abbiamo risolto nel quadrato magico per prima cosa la terza riga orizzontale perché mancava un solo termine quindi dopo averla eseguita trovando 5 abbiamo continuato con la diagonale obliqua ed abbiamo trovato 3, in seguito abbiamo svolto la seconda colonna verticale trovando "a-4", perché dovevamo mettere per forza la "a" dato che la somma è 26+a , abbiamo sottratto da "a" un numero che era quattro, perché la somma dava un numero maggiore di "26 +a".
Poi abbiamo riflettuto, dopo aver completato il quadrato inserendo altri numeri, sul fatto di inserire nella tabella il 92 e siamo arrivati alla conclusione che dato che la "a" è un incognita, abbiamo provato a sostituirla con 92 sottraendola in questo caso con 4 viene un numero che è risultato più piccolo di 92 e quindi abbiamo fatto la stessa operazione con "a-2" ed "a-6" e vengono sempre numeri più piccoli di 92.
Poi abbiamo provato a fare anziché "a-4"; "a+4" ed abbiamo visto che il numero era maggiore di 92 e quindi non poteva risultare".

Componenti del gruppo:
Ester Sanalitro, Jessica Giunta, Concetta Oste, Morena Costanzo, Serena Costa, Jessica Grisaffi, Gabriele Perra, Federica Guaietta, Irene Trommino

CHAPTER THREE:

THE NOTION OF VARIABLE IN DIFFERENT SEMIOTIC CONTEXTS

3.1 INTRODUCTION

The notion of variable could take on a plurality of conceptions: generalized number, unknown, “thing that varies”, totally arbitrary sign, register of memory, etc. In the preceding chapter we verified that these conceptions are evoked spontaneously by the pupils, even in absence of an adequate mastery of the algebraic language.

In high school the first three conceptions are chiefly privileged: general number, unknown and functional relation, but the notion of register of memory is also used in informatics.

The pupils meet many difficulties in the study of algebra. It is possible that these derive from the inadequate construction of the concept of variable. An opportune approach to this concept should consider its principal conceptions, the existing inter-relationships between them and the possibility to pass from one to the other with flexibility, in relation to the requirements of the problem to solve.

Kücheman (1981) showed that most of the pupils between 13 and 15 years of age treat the letters in expressions or in equations as specific unknowns more than generalised numbers or variables in a functional relation. Trigueros, M. *et alii* (1996) demonstrated that the beginner university students have a fairly poor conception of variable in its aspects of generalized number and functional relation. They have difficulty, chiefly, in understanding the variation in a dynamic form, that is the relation of variation with other variables. The obstacles are greater when the resolution of the questions does not take place by manipulation, but through interpretation and symbolization.

Panizza *et alii* (1999) showed that the linear equation in two variables is not recognized by the pupils as an object that defines a set of infinite pairs of numbers. The notion of unknown would not be effective to interpret the role of the letters in this type of equations. Instead, if the pupil uses the concept of function, he can calculate different solutions more easily.

The present chapter intends to study the relational-functional aspect of the variable in problem-solving, considering the semiotic contexts of algebra and analytical geometry. We want to analyze if the notion of unknown interferes with the interpretation of the functional aspect, and if the natural language and/or the arithmetical language prevail as the symbolic systems in absence of an adequate mastery of the algebraic language.
To effect this research we chose the linear equation in two variables for two reasons: firstly, because it represents a nodal point from which the pupils derive the conceptions of the letters as unknowns or "things that varies". We anticipate that the students will find some difficulties in treating the equations that have a plurality of solutions, in the context of concrete problematic situations[(1)]. Secondly, this type of equation is well known by the pupils, studied under different viewpoints: linear function, equation of a straight line and component of the linear systems.
Even if from the mathematical point of view these three terms (linear function, equation of a straight line and component of the linear systems) represent the same object, for the pupil it means evoking different (external) mental models[(2)]. According to Bagni (2001), the expression $ax + by + c = 0$ could be situated in a geometric context (to evoke, for instance, models of the concept of straight line in analytical geometry), or in a purely algebraic context (that is speaking of equations of first degree or, improperly, of polynomial). But this choice reflects a quite different attitude that has interesting motivations (they are also tied to the didactic contract) and remarkable didactic consequences.

3.2 METHODOLOGY OF THE RESEARCH

One hundred eleven students between the ages of 16-18 of the Experimental High School of the city of Ribera (AG)- Italy have participated in this research. They were thus distributed: 23 Fifth year pupils of Classic High School and 88 of the Scientific High School: 37 in the Third year, 20 in the Fourth year and 31 in the Fifth.
We want to explain that all the pupils knew the matters relative to: equations and inequations of first and of second degree, systems of equations, analytical geometry and functions. Particularly, the students of the Fifth's year had already effected the graphic study of functions within the mathematical analysis.
The questionnaire presents four questions (Appendix N° 1). In the **first** of them, the variable takes on the relational-functional aspect in the context of a concrete problematic situation. We also ask them to think over the solution set. With this

question we want to analyze the resolution strategy used and if the unknown notion interferes with the interpretation of the functional point of view.

The **second question** asks for the formulation of a problem. This must be resolved by means of a given equation, namely, the student must translate from the algebraic language into the natural language. We consider that this activity represents a fundamental point. It reveals the difficulties that pupils meet in interpreting the variable under the relational-functional aspect.

The **third question** asks to interpret, with a "short answer", the following relations of equality: $ax + by + c = 0$ and $y = mx + q$. We try to understand to which model and context, these equations are associated by the students. The purpose of this question is to compare the models evoked from these expressions with those activated by the problems 1 and 4.

In the **fourth question**, the variable takes on its relational-functional aspect in the context of a concrete problematic situation. We also ask the students to think about the solution set. While in the first problem the pupil was free to choose the resolutive context, in this one, instead, we force him to operate within the analytical geometry.

We effected a-priori analysis for each query of the questionnaire. The aim was to determine all the possible strategies that the pupils could use. Some mistakes that students might possibly make in the application of these strategies were also individualised.

We assigned the questionnaire during the last week of April 2002. The pupils worked individually, we did not allow them to consult books or notes. The given time was sixty minutes.

In the table we filled in with a double input "pupils/strategy", we have indicated for every pupil the strategies that he used with the value 1 and those that he didn't apply with the value 0.

The data was analysed in a quantitative way, using the implicative analysis of the variable of Regis Gras (1997, 2000) and with the help of the CHIC 2000 and the factorial statistical survey S.P.S.S. (Statistical Package for Social Sciences).

3.3 A-PRIORI ANALYSIS

We effected a very detailed a-priori analysis for every question of the questionnaire, because we wanted to favour a close qualitative and quantitative examination of the experimental variables. The aim was to determine all the possible strategies that the pupils could use and to individualize the errors that they could made in applying these strategies.

In the **first question** the variable takes on the relational- functional aspect in the context of a concrete problematic situation. Beginning from the a-priori analysis we have determined the principal experimental variables. They are the followings:

AL1: The pupil answers the question.

AL2: He/she shows a procedure in natural language.

AL3: He/she shows a procedure by trial and errors in natural language and/or in half-formalized language.

AL4: He/she adds a datum.

AL5: He/she translates the problem into an equation of first degree with two unknowns.

AL7: He/she translates the problem into an equation of first degree with two unknowns and he/she uses the algebraic method of "substitution into the same equation" (3).

AL9: He/she abandons the pseudo-algebraic procedure and he/she tries with another method.

AL11: He/she considers, in an explicit or implicit way, that the problem represents a functional relation.

AL13: He/she makes some errors in the resolution of the equation and he/she finds (or he/she tries to find) the only solution.

AL14: He/she considers that a relation of proportionality exists between x and y.

AL15: He/she has insufficient mastery of the algebraic language (AL4 + AL7 + AL9 + AL13 + AL14 + ~AL5).

AL16: He/she uses the natural language as means of expressive, both in the resolutive procedure and to motivate the answers (AL16 includes AL2).

AL17: He/she uses the arithmetical language in a not purely algebraic context in an explicit (because he/she does some operations) or implicit way (because he/she makes reference to the results of calculations effected mentally or with a calculator).

ALb1: The pupil calculates the solution set.

ALb2: He/she shows a particular solution that verifies the equation.

ALb3: He/she shows several solutions that verify the equation.

ALb4: He/she considers the infinite solutions expressly.

ALb5: He/she explicitly considers that the data are insufficient to determine only one solution.

ALb6: He/she considers a plurality of solutions (it includes ALb4 and ALb5)(4).

The **second question** asks them to translate from the algebraic language into the natural language. The a-priori analysis has allowed to individualize the more representative experimental variables; they are:

IAL1: The pupil answers the question.

IAL2: He/she transforms the equation to its explicit form.

IAL3: He/she resolves the equation applying the method of "substitution into the same equation"(3).

IAL4: He/she shows a particular solution that verifies the equation.

IAL5: He/she shows several solutions that verify the equation.

IAL6: He/she adds another equation and forms a system.

IAL7: He/she produces a text that considers only constants.

IAL7.1: The question refers to the second member of the equation, that is to 18.

IAL9: He/she produces a meaningful text for the given relation, but he/she does not formulate the question.

IAL10: He/she produces a meaningful text for the given relation and he/she formulates the question, but with some mistakes.

IAL11: He/she answers correctly.

IAL12: He/she produces a text that considers 2 variables, but that it does not translate the given equation exactly.

IAL14: He/she translates the algebraic language with difficulty.

The **third question** wants to know which mental model and context the pupil associates the two algebraic expressions. The principal experimental variables are the followings:

MMa1: The pupil answers the question.

MMa2: He/she associates the expression to the equation of a straight line.

MMa4: He/she associates the expression to the equation of a parable or a circumference.

MMa5: He/she associates the expression to an equation of first degree with two unknowns.

MMa6: He/she associates the expression to a polynomial.

MMb1: The pupil answers the question.

MMb2: He/she associates the expression to the equation of a straight line.

MMb3: He/she associates the expression to the equation of a sheaf of straight lines.

In the **fourth query** the variable takes on its relational-functional aspect in a concrete problematic situation in the context of analytical geometry. The a-priori analysis has allowed to individualize the more representative experimental variables, they are:

GAa1: The pupil answers the question.

GAa2: He/she shows a procedure in natural language.

GAa3: He/she shows a procedure by trial and errors in natural language and/or in half-formalized language.

GAa4: He/she translates the problem into an equation of first degree with two unknowns.

GAa5: He/she represents the relation graphically, but with some mistakes.

GAa6: He/she represents the relation graphically in a correct way, but he/she does not consider the bonds.

GAa7: He/she represents graphically in correct way.

GAa8: He/she considers that a relation of proportionality exists between *x* and *y*.

GAa9: He/she considers the bonds in an explicit way.

GAa12: He/she motivates the plurality of solutions, considering that the equation represents a functional relation.

GAa14: He/she represents it graphically and he/she abandons.

GAbc1: The pupil calculates the solution set.

GAbc2: He/she shows a particular solution that verifies the equation.

GAbc3: He/she shows several solutions that verify the equation.

GAbc4: He/she explicitly considers the infinite solutions.

GAbc5: He/she considers explicitly that the data are insufficient to determine only one solution.

GAbc6: He/she considers a plurality of solutions (it includes GAbc4 and GAbc5)(4).

The complete table of the experimental variables is in Appendix N° 2 at the end of Chapter 3.

3.4 THE HYPOTHESES AND THE A-PRIORI TABLE

H_1: The conception of variable as an unknown interferes with the interpretation of the functional aspect.

If the conception of variable as an unknown prevails in the context of a problematic situation	**then** the relational-functional aspect is not evoked
AL4, ALb2	~AL3, ~AL11, ~AL14, ~ALb3, ~ALb4, ~ALb5, ~ALb6

In the context of the first problem, the conception of variable as an unknown is highlighted by the experimental variable AL4 *"he/she adds a datum"*. Precisely, "to add a datum" is equivalent to introducing a new equation and thus forming a system of two linear equations with the equation of the problem or part of it. The solution of the system is a *"particular solution that verifies the equation of the problem"* (ALb2).

The relational-functional aspect of the variable is evoked when the pupil exhibits a *"procedure by trial and errors"* (AL3), through which he recalls the notion of dependence between the variables. In this way, the pupil *"considers implicitly or expressly that the problem represents a functional relation"* (AL11) or he manifests, incorrectly, that *"a relation of direct proportionality exists between the variables"* (AL14). Therefore the pupil *"shows some solutions that verify the equation"* (ALb3) or *"he considers that the problem has a plurality of solutions"* (ALb6). The experimental variable ALb6 includes the variable ALb4 *"he/she explicitly recognises the existence of infinite solutions"* and ALb5 *"he/she thinks that the data are insufficient to determine only one solution"*(4).

Accordingly *"not to evoke the relational-functional aspect of the variable"* is equivalent to the negation of the experimental variables above described: ~AL3, ~AL11, ~AL14, ~ALb3, ~ALb4, ~ALb5, ~ALb6.

H_2: The natural language and/or the arithmetical language prevail as symbolic systems in absence of an adequate mastery of the algebraic language.

If the pupil uses the natural language and/or the arithmetical language predominantly	**then** he has an insufficient mastery of the algebraic language
AL16 o AL17	AL15 (AL4, ~AL5, AL7, AL9, AL13)

The experimental variable AL17 considers *"the use of the arithmetical language in a context that is not purely algebraic, in an explicit way because the pupil effects some operations, or in an implicit manner, because he refers to the results of calculations made mentally or with a calculator"*.

The *"insufficient mastery of the algebraic language"* (AL15) is pointed out when the pupil *"does not translate the problem into an equation of first degree with two unknowns"* (~AL5) and *"he adds a datum"* (AL4) or when *"he translates, but he resolves the equation using the algebraic method of substitution into the same equation" (AL7).* This method consists in writing a variable in function of the other, then, replacing it in the original equation and in this way obtaining an identity.

Since the pupil does not know how to interpret the identity, *"he changes the resolving procedure, abandoning the pseudo-algebraic one"* (AL9) or he starts again *"resolving the equation and he makes some wrong algebraic transformations to find only one solution"* (AL13).

H_3: The translation of a functional relation from the algebraic language into the natural one does not "happen" spontaneously.

If the pupil carries out a purely syntactic manipulation of the algebraic formula or he produces the text of a problem that is not meaningful for the given relation	**then** he translates the algebraic language with difficulty
IAL2, IAL3, IAL4, IAL5, IAL6 or IAL7, IAL7.1, IAL12	IAL14

In the context of the second question, we consider that the pupil carries out a purely syntactic manipulation of the formula when he makes some algebraic transformations, instead of producing the text of a problem. For example, when the student *"finds one or several solutions that verify the equation"* (IAL4 - IAL5), or *"he transforms the equation into the explicit form"* (IAL2) and *"he resolves it applying the method of substitution into the same equation"* (IAL3) or *"he adds another equation and forms a system"* (IAL6).

We consider that the student produces a text that is not meaningful for the given relation, when he formulates *"the text of an arithmetical problem with specific numerical values"* (IAL7) and/or *"he ask the question related to the second member of* *the equation, that is 18"* (IAL7.1) or when he writes *"a text with two variables, but that does not exactly translate the given equation"* (IAL12).

The experimental variable IAL14: *"the student translates the algebraic language with difficulty",* has been defined as the negation of IAL11, that is ~IAL11 *"he/she answers the question but not correctly"*.

H_4: "If in a problematic situation we place the visual representative registers at the pupil's disposal then he understands more easily the relational-functional aspect of the variable".

	If in a problematic situation we place the visual representative registers at the pupil's disposal	**then** he understands more easily the relational-functional aspect of the variable
PROBLEM 1	**NO**	**NO (preponderance of GAbc2 = only one solution)**
PROBLEM 4	**YES**	**YES (preponderance of GAbc6 = plurality of solutions)**

The variable takes on its relational-functional aspect in a concrete problematic situation in the first and in the fourth question. While in the first one the pupil is free to choose the resolutive context, in the fourth one, instead, we force him to operate within the analytical geometry and to use visual representative registers. We consider that the pupil understands the relational-functional aspect of the variable more easily when he is able to consider the existence of a plurality of solutions.

3.5 **QUANTITY ANALYSIS**

We observe in the table of frequencies that the highest percentages of answers are obtained in the first problem with 95% and in the two questions of the third query with 97% and 99% respectively. We find 76% of the answers in the fourth problem, while only 60% in the second question. [Cfr. the experimental variables AL1, MMa1, MMb1, GAa1 and IAL1 in Appendix N° 3 at the end of Chapter 3].

In the third query, 76% of the students interpret the expression $ax + by + c = 0$ within the analytical geometry (equation of a straight line 49%, a circumference or parable 26% and a sheaf of straight lines 1%), while 26% recall the algebraic context (linear equation with two variable 21% and polynomial 5%). For the expression $y = mx + q$, instead, the totality of the pupils refer to analytical geometry (equation of a straight line 68%, sheaf of straight lines 30% and parable 1%).

[Cfr. the experimental variables MMa2, MMa4, MMa3, MMa5, MMa6, MMb2, MMb3 and MMb4 in Appendix N° 3 at the end of Chapter 3].

It is interesting to observe the exact coincidence of the expressions $ax + by + c = 0$ and $y = mx + q$ with the explicit and implicit equations of the straight line (generally presented in this way by the textbooks and by the teachers). This coincidence directs the interpretation toward the context of analytical geometry. These results agree with those of Bagni (2001).

3.5.1 First implicative analysis and comments of the first problem

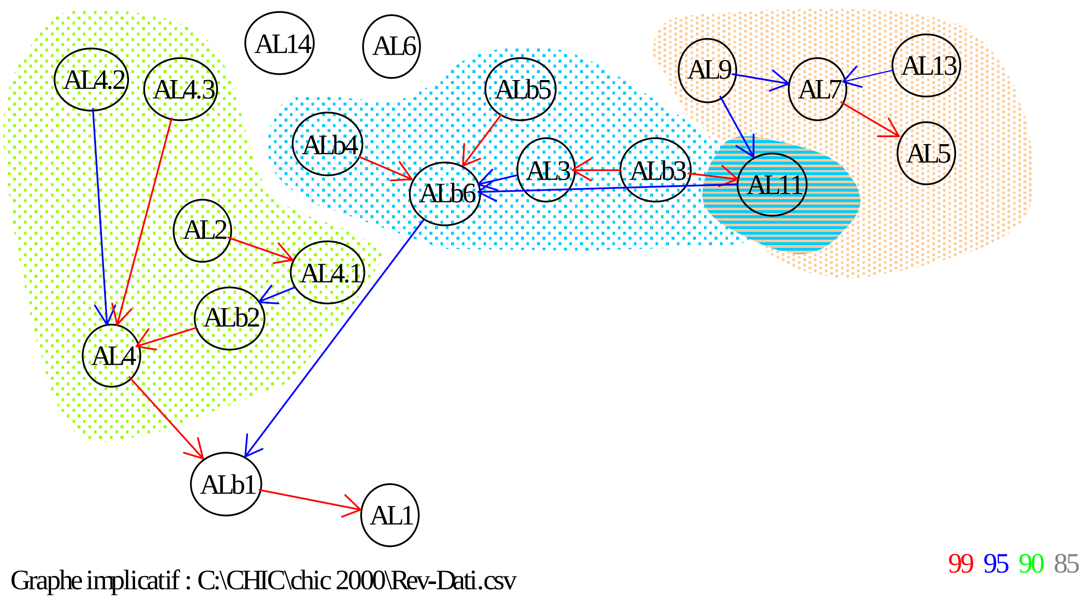

Figure 8

The implicative graph (carried out with the software Chic 2000) shows, with statistical percentages of 95 % and 99 %, three well defined groups of the experimental variables. They are pointed out through the green, yellow and celestial clouds (the yellow-celestial ruled cloud indicates the intersection between these two clouds). The three groups are directly or indirectly connected with the variable ALb1 *"the pupil calculates the solution set"* and AL1 *"the pupil answers to the question"*. Every group corresponds to a different kind of strategy used by the students:

- Ø ***Procedure in natural language*** (green cloud): the pupil adds a datum considering that the wins are equal (generally dividing it in half) or that the bets are equal(5). In this way, the student transforms the question into a typical arithmetical problem and he resolves it finding only a particular solution that verifies the equation. This result is confirmed by the implicative links between the experimental variables AL2, ALb2 and AL4 (with its variations AL4.1, AL4.2 and AL4.3).

The procedure in natural language is the most used by the pupils, it leads to the oneness of the solution and therefore the predominant conception of variable is that of unknown.

Ø ***Method by trials and errors in natural language or in half-formalized language*** (yellow cloud): the pupil that applies this strategy generally assigns several values to one of the variable (for example, the sum betted by Charles) and he finds the corresponding values in the other variable (the sum played by Lucy). In this way, the student shows some solutions that verify the equations and/or he considers that it has a plurality of solutions. That is, he generally considers in an implicit way that the problem represents a functional relation. The described result is obtained by the implicative links between the variables AL3, ALb3, AL11 and ALb6. This method leads to many solutions, allows to evoke the dependence between the variables, but a strong conception of the relational-functional aspect does not appear yet.

Ø ***Pseudo-algebraic strategy*** (celestial cloud): the pupil translates the text of the problem into an equation of first degree with two unknowns and applies the method of "substitution into the same equation", that is the incorrect procedure that consists in writing one variable in function of the other and then replacing it in the original equation thus obtaining an identity(3). Since the pupil does not succeed in interpreting the identity, he either changes his resolving procedure abandoning the pseudo-algebraic one or resumes the resolution of the equation and makes some errors to try to find only one solution. The described result is deduced by the implicative links between the experimental variables AL9, AL13, AL7 and AL5.

It is interesting to observe that, if the pupil abandons this strategy then he considers, in an implicit or explicit way, that the problem represents a functional relation. This result is confirmed by the implicative link between the experimental variables AL9 and AL11. This link allows the connection between the two procedures: by trials and errors and pseudo-algebraic (celestial-yellow ruled cloud). This result is strengthened by the double implication AL9 ↔ AL11 of the hierarchical tree [Cfr. Appendix N° 4 at the end of Chapter 3].

However, the pseudo-algebraic strategy is rarely used and it leads to the correct solution of the problem only in some cases.

3.5.2 Falsication of H_1:

We consider

p: in the context of a problematic situation the conception of variable as unknown (experimental variables AL4 and ALb2) prevails;

q: the relational-functional aspect is evoked (experimental variables AL3, AL11, AL14, ALb3, ALb4, ALb5 and ALb6).

The hypothesis 1 is equivalent to:

$p \rightarrow \sim q$ that, from the logical point of view, is equivalent to

$\sim(p \wedge \sim (\sim q))$ or $\sim (p \wedge q)$

Therefore, to falsify this hypothesis it is sufficient to demonstrate the empty intersection between the experimental variables of ***p*** and ***q***, in other words:

p corresponds to the procedure in natural language in which the conception of variable as unknown prevails;

q is equivalent to the method by trials and errors in which the relational-functional aspect of the variable predominates.

From Fig. 1 we deduce that the sets of experimental variables, corresponding to ***p*** (green cloud) and to ***q*** (yellow cloud), are disjoined. This result allows to falsify the first hypothesis.

3.5.3 Profile of the pupils

The possible profiles of the pupils that resolve the first problem clearly emerge from the previously effected analysis. They are:

Ø **NAT:** this profile corresponds to the pupil that performs a procedure in natural language. Then he adds a datum considering that the wins are equal (generally dividing it in half) or that the bets are equal (5) and he resolves the problem finding only a particular solution that verifies the equation. This profile is characterized by the presence of the followings experimental variables: AL1, AL2, AL4, AL15, ALb1 e ALb2.

Ø **FUNZ:** it corresponds to the pupil that applies a strategy by trials and errors in natural language and/or half-formalised language. He generally assigns several values to one of the variable and he finds the corresponding values in the other variable. Therefore the student shows some solutions that verify the equations and/or he considers that it has a plurality of solutions. The experimental variables that describe this profile are: AL1, AL3, AL11, ALb1, ALb3, ALb4, ALb5 e ALb6).

- **PALG1**: it corresponds to the pupil that uses the pseudo-algebraic procedure. He translates the text of the problem into a linear equation with two unknowns and he applies the method of "substitution into the same equation"(3). When the student reaches the identity he does not succeed in interpreting it. Then he resumes the resolution of the equation, he makes some errors of syntactic kind trying to find only one solution. This profile is characterized by the presence of the experimental variables: AL1, AL5, AL7, AL13, AL15, ALb1 e ALb2.
- **PALG2:** it is a variation of the profile PALG1. In this case, when the pupil arrives at the identity he changes resolutive procedure abandoning the pseudo-algebraic one. The experimental variables that describe this profile are the followings: AL1, AL3, AL5, AL7, AL9, AL11, AL15, ALb1 and ALb3.
- **ALG:** it corresponds to the pupil that applies an algebraic procedure. He translates the problem into an equation of first degree with two unknowns, he considers, in an implicit or explicit way, that it represents a functional relation and therefore that it is verified by a plurality of solutions. The experimental variables of this profile are the followings: AL1, AL5, AL11, AL15, ALb1, ALb4 e ALb6.

3.5.4 The hierarchical tree

In the hierarchical tree of Fig. 2 we observe a very meaningful group of variables that implicate the NAT profile, pointed out in red. To the right of NAT we see a small set of variables that belong to the same group. It corresponds to the pupils that followed the procedure described in NAT and found one solution to the problem. But, afterwards, when they were questioned about the possible solutions, they considered that the equation can be satisfied by a plurality of solutions.

We also observe two small groups that indicate the profiles PALG1 (pointed out in fuchsia) and PALG2 (in electric blue). There are not meaningful implications, instead, for the supplementary variable ALG (in turquoise) and FUNZ (in bright green).

Therefore the profile described in NAT is the most meaningful because it represents the strategy that the pupil used most.

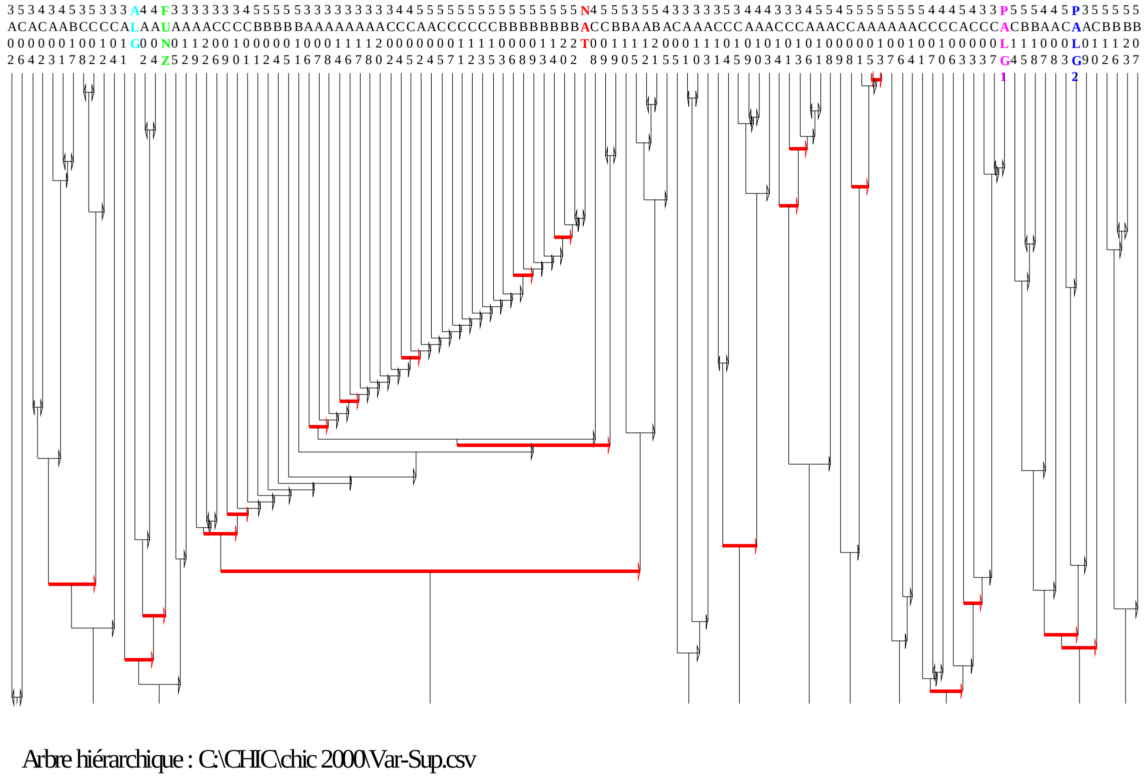

Figure 2

3.5.5 The factorial analysis by S.P.S.S.

The graph shows that the first component (horizontal axis) is strongly characterized by the pair of supplementary variables: NAT (hooped in red) and PALG1 (pointed out in fuchsia).

The profiles ALG (hooped in turquoise), PALG2 (in blue electric) and FUNZ (in green) form a cloud that strongly characterises the vertical component. The supplementary variable PALG2 is very near to FUNZ because the student that abandons the pseudo-algebraic procedure generally adopts the profile described in FUNZ.

The winning strategies are precisely those described in the profiles ALG, PALG2 and FUNZ that lead to the plurality of solutions, while NAT and PALG1 lead to the oneness of the solution. This finds a strong correspondence with the different conceptions of the concept of "variable". Therefore, the horizontal axis represents the conception of variable as *unknown*, the vertical axis, instead, reproduces its relational-functional aspect. These results allow us to falsify the first hypothesis again.

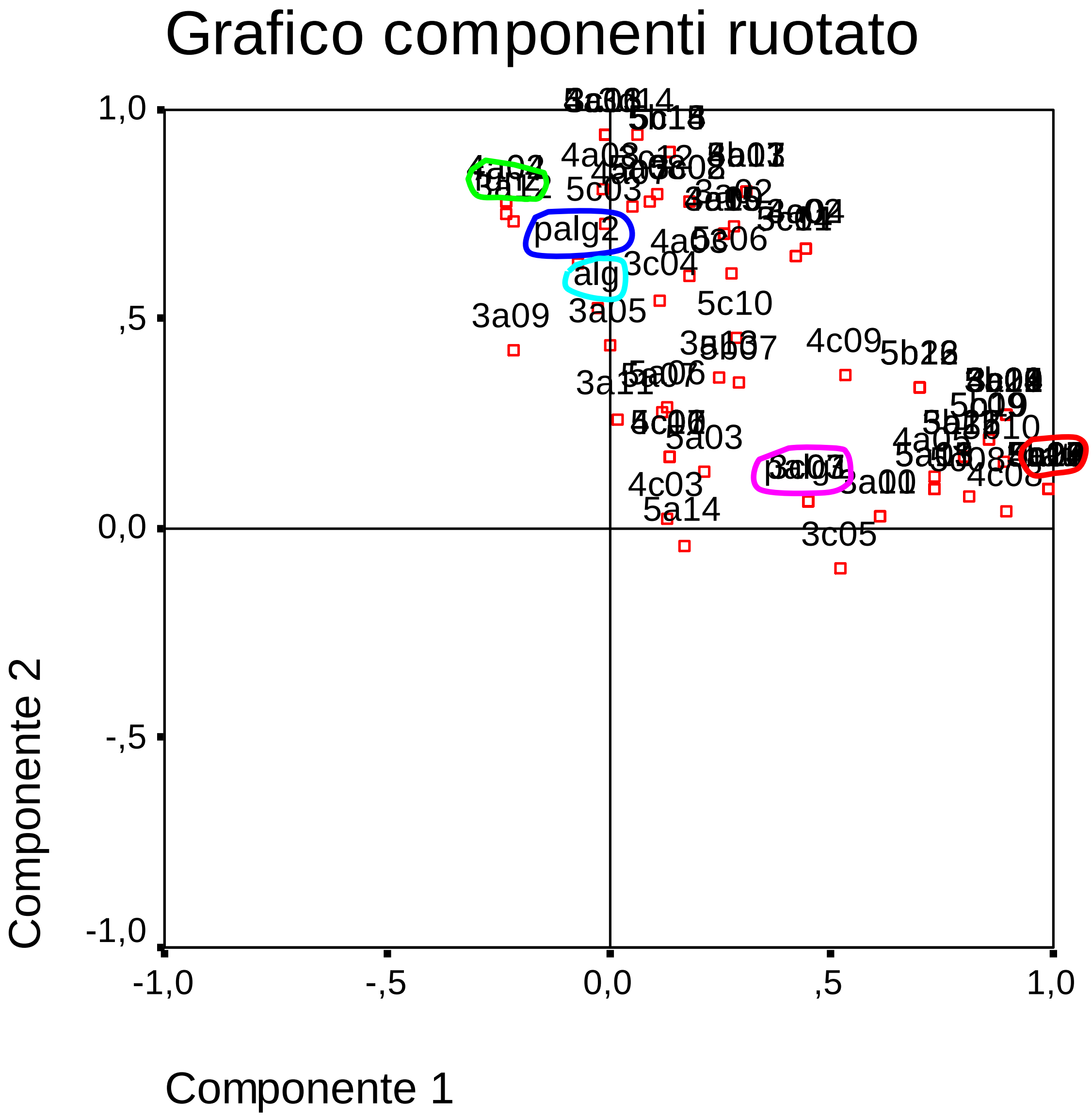

Figure 3

3.5.6 Second implicative analysis and comments of the first problem

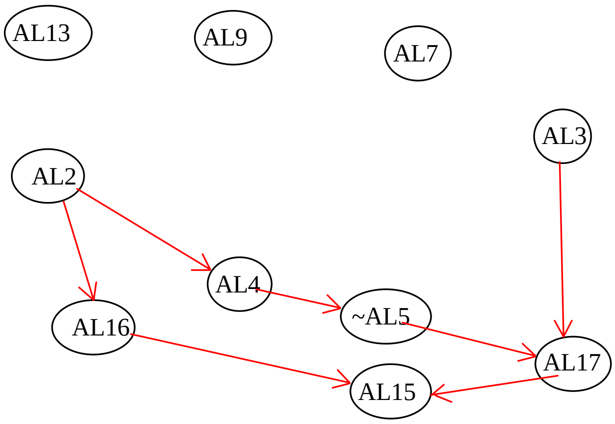

Figure 4

Graphe implicatif : C:\CHIC\chic 2000\Ipot-2.csv 99 95 90 85

The implicative graph of Fig. 4 shows three well defined groups of experimental variable:

- Ø ***First implicative group*** (with statistical percentage of 99%): it is represented by the link AL2→AL16→AL15. The first implication is obvious because the experimental variable AL16 (*the pupil uses the natural language*) contains the variable AL2 (*he/she shows a procedure in natural language*). The second implication, instead, is very important: if the student uses the natural language as an expressive means, then he has an insufficient mastery of the algebraic language.
- Ø ***Second implicative group*** (with statistical percentage of 99%): if a pupil shows a procedure in natural language, then he adds a datum; he does not translate the problem into an linear equation with two unknowns, thus he uses the arithmetical language and as a consequence he shows a scarce mastery of the algebraic language (Link AL2→AL4→ ~AL5→AL17 →AL15).
- Ø ***Third implicative group*** (with statistical percentage of 99%): if the student uses a procedure by trials and errors, then he utilises, in an implicit or explicit way, the arithmetical language in a context that is not purely algebraic and therefore, he shows an insufficient mastery of the algebraic language (Link AL3→AL17→AL15).

It is interesting to observe that the variables AL7 (*he/she applies the method of "substitution into the same equation"*), AL13 (*he/she resolves the equation and he effects some wrong algebraic transformations to find only one solution*) and AL9 (*he/she abandons the pseudo-algebraic procedure and he/she tries with another method)* are connected among themselves with a statistical percentage of 95% (to see Fig. 4). These variables do not result, instead, linked with AL15 (*he/she has an insufficient mastery of the algebraic language*), probably because these variables register a low percentage of answer and in some cases they lead to the correct solution of the problem.

3.5.7 Falsication of H_2:

From the prior analysis we can deduce that:

- Ø AL16 → AL15: if the pupil uses the natural language as an expressive means, then he has a scarce mastery of the algebraic language.

Ø AL17 → AL15: if the student uses the arithmetical language in a not purely algebraic context, in an explicit or implicit way, then he has an insufficient mastery of the algebraic language.

This result allows to falsify the hypothesis 2.

3.5.8 Third implicative analysis and comments of the second problem

The second query asks the formulation of a problem that has to be resolved by a given equation; in other words, the pupil must carry out the translation from the algebraic language into the natural one. This exercise turned out to be difficult for the pupils, because of 60% that has answered the question only 7% did it correctly [Cfr. the experimental variables IAL1 and IAL11 in Appendix N° 3 at the end of Chapter 3].

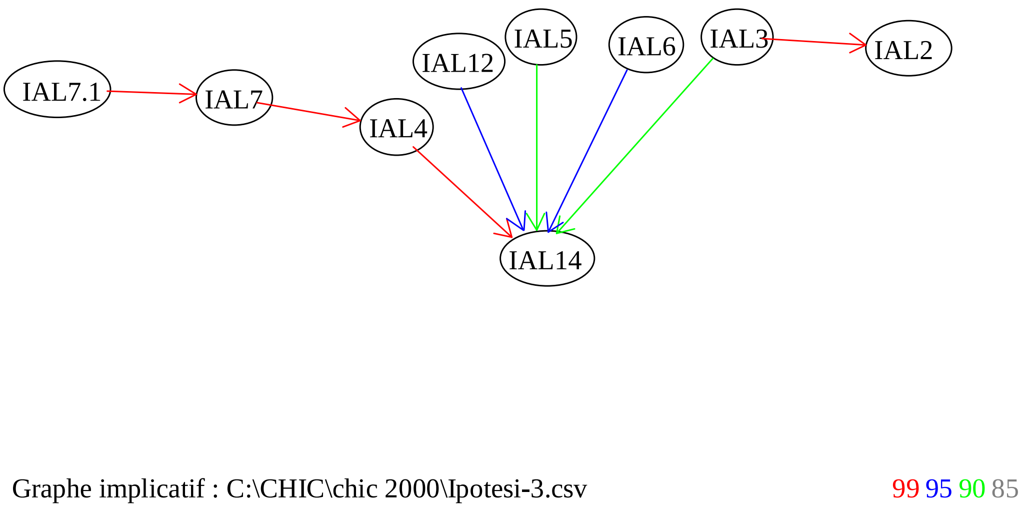

Figure 5

Different implicative paths are drawn in Figure 5, but all have the same consequent: the experimental variable IAL14 *"the pupil translates the algebraic language with difficulty"*. The most important implicative path, with a statistic validity of 99%, is the following: if a student formulates the question of the problem related to the second member of the equation, that is 18, then he has produced the text of a classical arithmetical problem with specific numerical values, therefore he has calculated a particular solution that verifies the equation and consequently he meets difficulties in translating the algebraic language (Implicative links IAL7.1, IAL7, IAL4 and IAL14).

Other implications, with statistical percentages of 95% and 90%, have as antecedents the following experimental variable: IAL12 *"he/she produces the text of a problem that considers the two variables, but that he/she does not translate the given equation* *exactly"*, IAL6 *"he/she adds another equation to the given equation and he/she forms a system"*, IAL5 *"he/she shows several solutions that verify the equation"* and IAL3 *"he/she resolves the equation applying the method of substitution into the same equation"*. Moreover, if a pupil uses this method then he has transformed the equation into its explicit form (IAL3 →IAL2).

3.5.9 Falsication of H_3:

From the prior analysis we can extract two groups of experimental variables that correspond to two different strategies to resolve the question. The first implicative group is constituted by the variables IAL2, IAL3, IAL5 and IAL6 that characterize the activity of purely syntactic manipulation of the formula. The second group contains the variable IAL12 and the route IAL7.1 → ILA7 → IAL4 that corresponds to the production of the text of a problem not meaningful for the given relation.

The two groups indicate the variable IAL14 *"the pupil translates the algebraic language with difficulty"*. This result allows to falsify the third hypothesis.

3.5.10 Fourth implicative analysis and comments of the fourth problem

Some interesting particularities emerge from the implicative graph of Fig. 6:

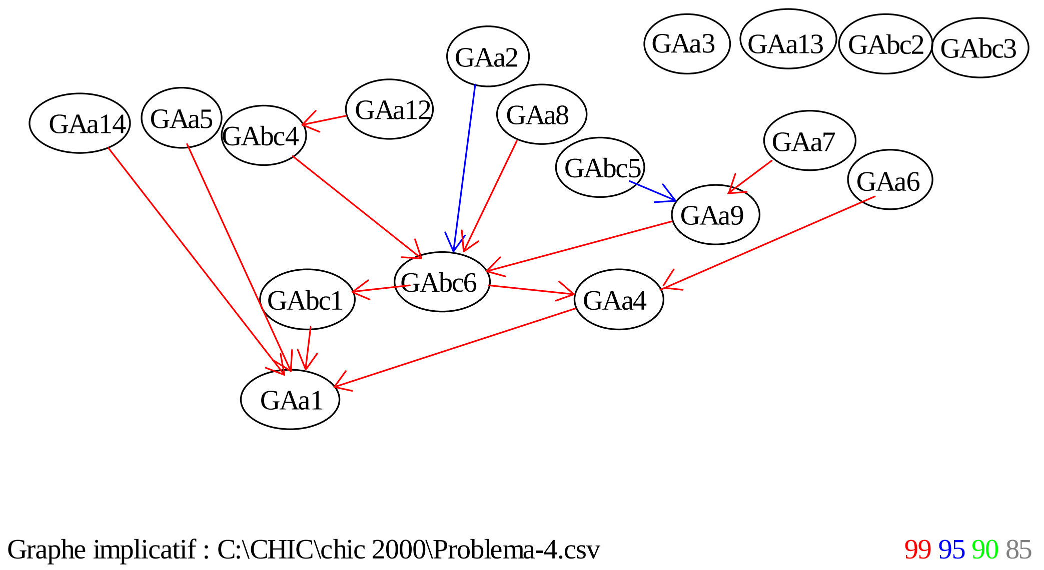

Figure 6

- All the implications arrive, directly or for transitive property, at the experimental variable GAa1 *"the pupil answers the question"*, pointing out, with statistical percentage 99%, a varied range of possible answers: *"the student translates the problem into an equation of first degree with two unknowns"* (GAa4), *"he/she represents the relation graphically with some error or he/she does not consider the bonds"* (GAa5 and GAa6), *"he/she immediately abandons after the graphic representation"* (GAa14), the pupil resolves the problem and *"he/she answers on the possible solutions"* (GAbc1).

- The experimental variable GAbc6 represents an important implicative knot because the variables GAa2, GAa7, GAa8, GAa9 and GAa12 converge in it with statistical percentages of 99% and in some case of 95%. That is, if *"the pupil shows a procedure in natural language"* (GAa2), if *"he/she represents graphically in a correct way"* (GAa7), if *"he/she considers the bonds in an explicit way"* (GAa9), if *"he/she motivates the plurality of solutions considering, that the equation represents a functional relation"* (GAa12) or if *"he/she thinks wrongly that a relation of proportionality exists between x and y"* (GAa8) then, in every case, the student *"considers that the problem has a plurality of solutions"* (GAbc6). The convergence of the experimental variables GAbc4 and GAbc5 in GAbc6 are obvious because *"the plurality of solutions"* (GAbc6) includes the cases of *"infinite solutions"* (GAbc4) and those of *"lack of data to determine only one solution"* (GAbc5).

- The pupil that answers on the possible solutions basically considers that the problem has a plurality of solutions (implicative link between the experimental variables GAbc6 and GAbc1). This result is strengthened by the lack of connection between the experimental variable GAbc2 *("he/she shows a solution that verifies the equation"*) and GAbc3 *("he/she shows several solutions that verify the equation")* with GAbc1, but chiefly from the implication GAbc6 $\rightarrow$ GAbc1 to the first level of the hierarchical tree (Cfr. Appendix N° 6 at the end of Chapter 3). On the other hand, the percentages of answers confirm the preceding result: 3% for GAbc2, 4% for GAbc3 and 57% for GAbc6 (Cfr. Table of Frequencies of Appendix N° 3 at the end of Chapter 3).

3.5.11 Fifth implicative analysis and comparison between the first and the fourth problem

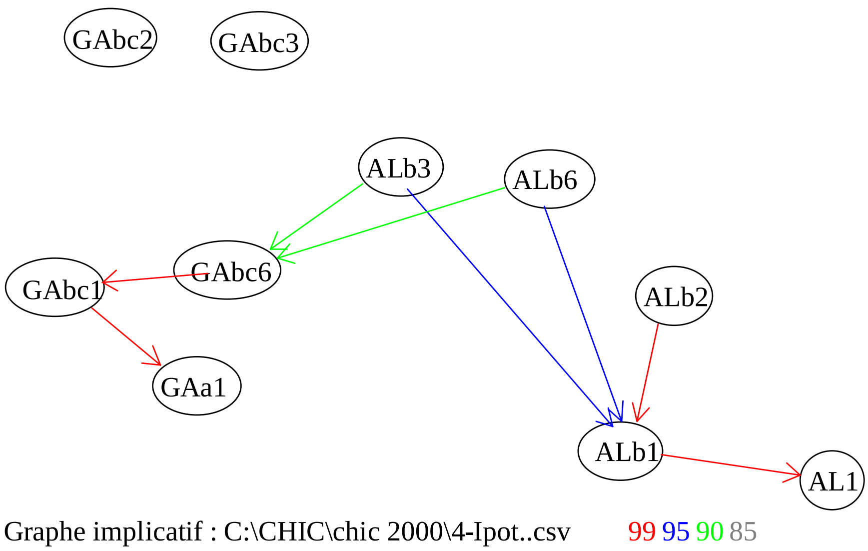

Figure 7

To falsify the fourth hypothesis we must compare the answers of the first problem and those of the fourth one. Even if in both problems the variable takes on the relational-functional aspect, in the first question the student is free to choose the resolutive context, while in the fourth one, we force him to operate within the analytical geometry and use visual representative registers.

- Ø In the first problem most of the pupils that answer on the quantity of solutions consider that the equation has only one solution (implicative link ALb2 → ALb1, with statistical percentage 99%). The students, instead, that show more solutions that verify the equation or that consider that it has a plurality of solutions (implicative links ALb3 → ALb1 and ALb6 → ALb1, with statistical percentage 95%) are less numerous than those of the preceding case.

- Ø In the fourth problem the pupil who answers on the possible solutions basically considers that the problem has a plurality of solutions (implicative link GAbc6 GAbc1 with statistical percentage 99 %). The experimental variables GAbc2 and GAbc3 *("he/she shows one or several solutions that verify the equation")*, instead, are not connected to the variable GAbc1 (*"he/she calculates the solution set"*). These results have been discussed in the preceding section.

- Ø It is interesting to notice two implications (with statistical percentage 90%) that link the first problem to the fourth one: ALb3 → GAbc6 and ALb6 → GAbc6. In other words, if the pupil considers that the equation of the first question is verified by several solutions or by a plurality of solutions, then he allows for a manifold of solutions also in the fourth problem.

3.5.12 Falsification of H_4:

We consider that the pupil understands the relational-functional aspect of the variable more easily when he is able to allow for the existence of a plurality of solutions. This can be verified especially in the fourth problem in the presence of visual representative registers and with the pupils with insufficient mastery of the algebraic language. This result allows to falsify the fourth hypothesis.

3.6 CONCLUSIONS

From the analysis of the data we observe the use of three types of strategy to solve the first problem. They are the followings:

Ø ***Procedure in natural language***: it corresponds to the NAT profile, it turned out to be the most used by the pupils and it leads to the oneness solution. The predominant conception of variable is that of unknown.

Ø ***Methods by trials and mistakes in natural language and/or in half-formalized language*** (generally arithmetical): it corresponds to the FUN profile, it conducts to several solutions. The dependence of the variables is evoked, but a strong conception of the relational-functional aspect does not appear yet.

Ø ***Pseudo-algebraic strategy*** (corresponding to the PALG1 and PALG2 profiles): it is little used by the pupils and it leads to the correct solution of the problem only in some cases.

The exhaustive study effected, with the implicative and the factorial analysis, allows to falsify the first hypothesis, that is: "if in the context of a problematic situation the conception of variable like unknown prevails, then its relational-functional aspect is not evoked".

From the effected analysis we observe that the pupils predominantly use the natural language as an expressive means to resolve the first problem. They also use, in an explicit or implicit way, the arithmetical language in a not purely algebraic context. These results allow to falsify the second hypothesis: "the natural language and/or the arithmetical language prevail as symbolic systems, in absence of adequate mastery of the algebraic language".

It is interesting to observe that no pupil uses visual representative registers to solve the first question, and that many students consider the problematic situation to have only one solution (variable as unknown). The fourth problem presents a concrete situation similar to the preceding one, but in the context of the analytical geometry. For this question the students that answer on the possible solutions consider direct the plurality of solutions (variable in functional relation).

These results show that: the pupils with insufficient mastery of the algebraic language could consider more easily the plurality of solutions, in the presence of visual representative registers, by evoking the mental model of the equation of the straight line.

For the third question almost all the pupils have interpreted the expressions $ax + by + c = 0$ and $y = mx + q$ within the analytical geometry, but the model of straight line has not been recalled with the equation of the first problem. Thus the graphic representation is totally absent in the resolution of the problem. This behaviour called "avoidance of visualization" was already found in the didactic research (Cfr. Eisenberg & Dreyfus, 1991; Vinner, 1989; Furinghetti & Somaglia, 1994; Chiarugi, I. *et alii*, 1995).

In this situation we think that the "avoidance of visualization" is linked to a matter of didactic contract. Usually, the problems with equations given in school are solved in an algebraic context where the variable engages the unknown aspect. The concrete problematic situations generally are never solved within the analytical geometry, recalling visual representative registers. The problems of analytical geometry given at school are different. In the fourth problem, the pupil is forced to use the model of straight line with its Cartesian representation. Therefore the equation becomes "perceivable" through the graph and the student can "visualize" more easily the plurality of solutions. These results allow to falsify the fourth hypothesis: "If in a problematic situation we place the visual representative registers at the pupil's disposal then he understands more easily the relational-functional aspect of the variable".

From the preceding analysis we can confirm that the student is more inclined to consider the variable under the unknown aspect (searching the oneness of the solution of the linear equation) in the context of a concrete situation and in absence of representative graphic registers. As a consequence, we can affirm that there is a certain interference of the conception of unknown in the functional aspect (Hypothesis 1).

However, we believe that the matter must still be deepened analyzing in detail the resolutive strategies used. We should investigate how the conceptions of unknown and the functional relation are activated and how the passage from a conception to the other could occur without interference, in the process of resolution of a concrete problematic situation.

The translation from the algebraic language into the natural language results a difficult exercise for the pupils (third hypothesis). Some students carry out only a purely syntactic manipulation of the formula; others, instead, are able to produce the text of a problem that does not result meaningful for the given relation. Some interesting particularities emerged; for example, the students who generate the text of a classical arithmetical problem with specific numerical values, they produce a question referred to the second member of the equation, that is 18. Therefore, to these pupils, the expression is an unidirectional relation that put the answer on the right side. Thus, a return to the primitive perceptions that the 12-13 year old students have on the equations of first degree with an unknown is shown (Kieran, 1981). Some pupils needed to know the values of the unknown before they involved it in the elaboration of the problem. This shows an obstacle of the language at a purely syntactic level that should be analysed more thoroughly.

Other salient questions that emerge from this research are: the importance of the visualization in problem-solving and of the coordination of different representative registers (Duval, 1999).

NOTES:

(1) In the present study we prefer to use the term "plurality of solutions" rather than the word "infinite solutions", because we have not considered the possible connotations of the word "infinite". However, we defined two experimental variables ALb4, (for the first problem) and GAbc4 (for fourth), to take into account the cases in which the pupil explicitly considers the existence of infinite solutions.

(2) We consider that it is opportune to explain the used terminology in this research; to this purpose we will follow D'Amore & Frabboni (1996). We call mental image what is elaborate by the pupil, even unintentionally, before any request (interior or external): it is an interior image, therefore not express, at least initially. All the mental images of a concept constitute the mental model relative to this concept (Johnson-Laird, 1988). Thus, the built conceptions must often be expressed, communicated by means of a specific translation; therefore, an external model is created and frequently this is expressible in a well determined language. Thus, every form of communication of a content, of a mathematical message occurs with the use of external models (Shepard, 1980).

(3) We have called "procedure of substitution into the same equation" the incorrect method that consists in writing one variable in function of the other, then replacing it in the original equation thus obtaining an identity. That is, the pupil applies the method of substitution used to solve the systems of equations to a single equation.

(4) The experimental variables ALb6 and GAbc6 *"he/she considers a plurality of solutions"* admit answers like: "so many solutions", "a lot of solutions" and "infinite solutions". The latter corresponds to the variables ALb4 and GAbc4. The plurality of solutions also includes the cases in which the pupil explicitly considers the data to be insufficient to determine only one solution (ALb5 and GAbc5).

(5) The experimental variable AL4 *"he/she adds a datum"* considers two possibilities: equal wins or equal bets (AL4.3). The first case takes into account the two other alternatives: the win is divided in half (AL4.1) or both the teenagers win 300 € (AL4.2).
"To add a datum" is equivalent to introducing a new equation and to forming (with the equation of the problem *3 x + 4 y = 300* or part of it) a system of two equations with two unknowns. Therefore a system corresponds to each case:

- *"The win of 300 € is divided in half"* (AL4.1): it is equivalent to the system:
$$\begin{cases} 3x + 4y = 300 \\ 3x = 4y = 150 \end{cases}$$
- *"The win is equal to 300 € for both the teenagers"* (AL4.2): it corresponds to the system:
$$\begin{cases} 3x = 300 \\ 4y = 300 \end{cases}$$
- *"The bets are equal"* (AL4.3): is equivalent to the system:
$$\begin{cases} 3x + 4y = 300 \\ x = y \end{cases}$$

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APPENDIX N° 1: Questionnaire

1- Charles and Lucy win the lottery the total sum of € 300. We know that Charles wins the triple of the betted money, while Lucy wins the quadruple of her own.

   a) Determine the sums of money that Charles and Lucy have betted. Comment the procedure that you have followed.

   b) How many are the possible solutions? Motivate your answer.

2- Invent a possible situation-problem that could be solved using the following relation of equality: $6x - 3y = 18$.

   Comment the procedure that you have followed.

3- What is it? Interpret by a "short answer" the following expressions:

   a) $ax + by + c = 0$ …………………………………………………………………….

   b) $y = mx + q$ ………………………………………………………………………

4- To use the telephone of another person, a man arranges to pay a monthly fee of 5 € and in addition 2 € for hour for the phone calls he actually makes.

   Said: x the number of monthly hours of phone calls made and y the total sum that he pays monthly

   a) Establish which type of relation intervenes between x and y and represent it graphically in the Cartesian plain.

   b) Determine the total sum that he pays monthly and the number of monthly hours of phone calls made. Motivate your answer.

   c) How many are the possible solutions? Motivate your answer.

APPENDIX N° 2: Complete tables of the experimental variable

We have effected a very detailed a-priori analysis for each query of the questionnaire. We report the tables with all the experimental variables. They are the followings:

FIRST QUERY

AL	1	2	3	4	4.1	4.2	4.3
1	The pupil answers	He/she shows a procedure in natural language	He/she shows a procedure by trial and errors in language natural and/or in half-formalized language	He/she adds a datum	As AL4, but he/she considers that the win is divided in half	As AL4, but he/she considers that the win of the two teenagers is equal to 300 €	As AL4, but he/she considers that the bets are equal.
0	NO	NO	NO	NO	NO	NO	NO

AL	5	6	7	8	9	10
1	He/she translates the problem into an equation of first degree with two unknowns	He/she explicitly considers the bonds of the problem	As AL5, but he/she uses the algebraic method of "substitution into the same equation" (3).	He/she adds other equation and forms a system	He/she abandons the pseudo-algebraic procedure and he/she tries with another method.	He/she considers that the equation represents a straight line
0	NO	NO	NO	NO	NO	NO

AL	11	12	13	14	15
1	He/she considers, in an explicit or implicit way, that the problem represents a functional relation	He/she considers that an indeterminate equation has infinite solutions	He/she makes some errors in the resolution of the equation and he/she finds (tries to find) only one solution	He/she considers that a relation of proportionality exists between *x* and *y*	He/she has insufficient mastery of the algebraic language (AL4 + AL7 + AL9 + AL13 + AL14 +~AL5)
0	NO	NO	NO	NO	NO

AL	16	17
1	He/she uses the natural language as an expressive means, in the resolutive procedure or to motivate the answers (AL16 includes AL2).	He/she uses the arithmetical language in a not purely algebraic context, in an explicit or in implicit way
0	NO	NO

ALb	1	2	3	4	5	6
1	The pupil answers	He/she shows a particular solution that verifies the equation	He/she shows several solutions that verify the equation	He/she expressly considers the infinite solutions	He/she explicitly considers that the data are insufficient to determine only one solution	He/she considers a plurality of solutions (it includes ALb4 and ALb5)(4).
0	NO	NO	NO	NO	NO	NO

ALb	2*	3*
1	He/she considers only a particular solution that verifies the equation	He/she considers only some solutions that verify the equation
0	NO	NO

SECOND QUERY

IAL	1	2	3	4	5	6
1	The pupil answers	He/she transforms the equation into its explicit form	He/she resolves the equation applying the method of “substitution into the same equation”[3]	He/she shows a particular solution that verifies the equation	He/she shows several solutions that verify the equation	He/she adds another equation and forms a system
0	NO	NO	NO	NO	NO	NO

IAL	7	7.1	8	9	10
1	He/she produces a text that considers only constants	The question refers to the second member of the equation, that is 18	He/she produces a text that considers only a variable	He/she produces a meaningful text for the given relation, but he/she does not formulate the question	He/she produces a meaningful text for the given relation and he/she formulates the question, but with some mistakes
0	NO	NO	NO	NO	NO

IAL	11	12	13	14
1	He/she answers correctly	He/she produces a text that considers two variables, but he/she does not translate the given equation exactly	He/she formulates a problem in the context of the analytical geometry	He/she translates the algebraic language with difficulty
0	NO	NO	NO	NO

IALb	1	2	3	4
1	The pupil answers	He/she comments the resolution of the equation	He/she comments the assignment of particular objects to the variable	As AL4, but he/she explains the formulation of the question
0	NO	NO	NO	NO

THIRD QUERY

First equation

MMa	1	2	3	4	5
1	The pupil answers	He/she associates the expression to the equation of a straight line	He/she associates the expression to the equation of a sheaf of straight line	He/she associates the expression to the equation of a parable or a circumference	He/she associates the expression to an equation of first degree with two unknowns
0	NO	NO	NO	NO	NO

MMa	6	7
1	He/she associates the expression to a polynomial	He/she associates the expression to an equation of second degree
0	NO	NO

Second equation

MMb	1	2	3	4	5
1	The pupil answers	He/she associates the expression to the equation of a straight line	He/she associates the expression to the equation of a sheaf of straight line	He/she associates the expression to the equation of a parable or a circumference	He/she associates the expression to an equation of first degree with two unknowns
0	NO	NO	NO	NO	NO

MMb	6	7
1	He/she associates the expression to a polynomial	He/she associates the expression to an equation of second degree
0	NO	NO

FOURTH QUERY

The a-priori analysis correspondent is the following:

GAa	1	2	3	4	5
1	The pupil answers	He/she shows a procedure in natural language	He/she shows a procedure by trial and errors in language natural and/or in half-formalized language	He/she translates the problem into an equation of first degree with two unknown	He/she represents the relation graphically, but with some mistakes.
0	NO	NO	NO	NO	NO

GAa	6	7	8	9	9.1
1	He/she represents the relation graphically in correct way, but he/she does not consider the bonds	He/she represents in correct graphically way	He/she considers that a relation of proportionality exists between *x* and *y*	He/she considers the bonds in an explicit way	He/she considers the inferior bonds in an explicit way
0	NO	NO	NO	NO	NO

GAa	9.2	10	11	12	13	14
1	He/she considers the superior bonds in an explicit way	He/she uses the algebraic method of “substitution into the same equation” (3).	He/she abandons the pseudo-algebraic procedure and he/she tries with another method.	He/she motivates the plurality of solutions, considering that the equation represents a functional relation	He/she motivates the plurality of solutions, considering that the equation represents a straight line	He/she represents graphically and he/she abandons
0	NO	NO	NO	NO	NO	NO

GAbc	1	2	3	4	5	6
1	The pupil answers	He/she shows a particular solution that verifies the equation	He/she shows several solutions that verify the equation	He/she considers the infinite solutions expressly	He/she explicitly considers that the data are insufficient to determine only one solution	He/she considers a plurality of solutions (it includes GAbc4 e GAbc5)
0	NO	NO	NO	NO	NO	NO

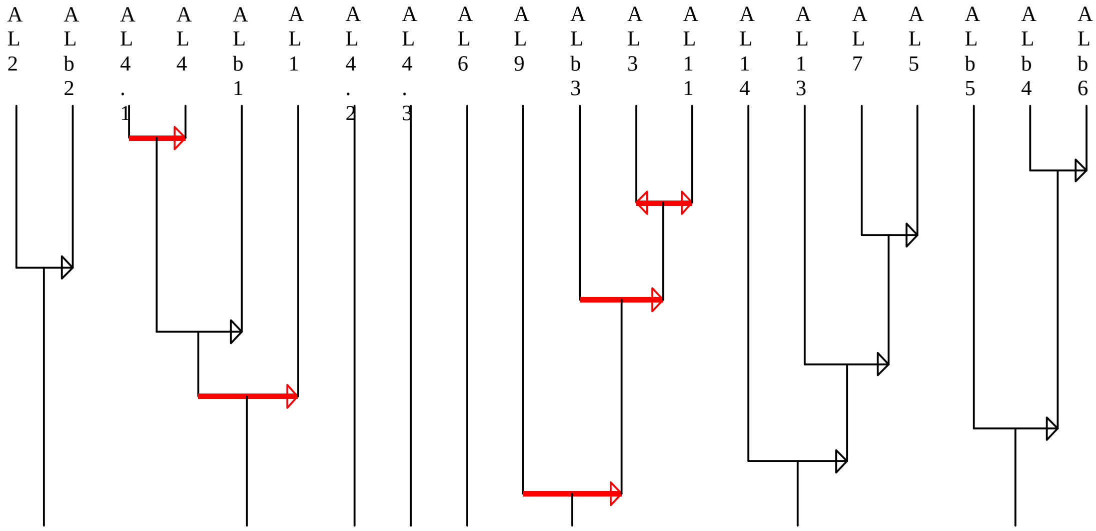
GAbc	2*	3*
1	He/she considers only a particular solution that verifies the equation	He/she considers only some solutions that verify the equation
0	NO	NO

APPENDIX N° 3: Table of frequencies

Variable	Absolute frequency	Relative frequency	Percentage	Spread
AL1	106.00	0.95	95	0.21
AL2	44.00	0.40	40	0.49
AL3	34.00	0.31	31	0.46
AL4	71.00	0.64	64	0.48
AL4.1	53.00	0.48	48	0.50
AL4.2	11.00	0.10	10	0.30
AL4.3	13.00	0.12	12	0.32
AL5	27.00	0.24	24	0.43
AL6	4.00	0.04	4	0.19
AL7	13.00	0.12	12	0.32
AL8	1.00	0.01	1	0.09
AL9	9.00	0.08	8	0.27
AL10	1.00	0.01	1	0.09
AL11	34.00	0.31	31	0.46
AL12	1.00	0.01	1	0.09
AL13	8.00	0.07	7	0.26
AL14	3.00	0.03	3	0.16
AL15	98.00	0.88	88	0.32
AL16	82.00	0.74	74	0.44
AL17	91.00	0.82	82	0.38
ALb1	99.00	0.89	89	0.31
ALb2	63.00	0.57	57	0.50
ALb3	33.00	0.30	30	0.46
ALb4	25.00	0.23	23	0.42
ALb5	13.00	0.12	12	0.32
ALb6	36.00	0.32	32	0.47
ALb2*	45.00	0.41	41	0.49
Alb3*	18.00	0.18	18	0.38
IAL1	67.00	0.60	60	0.49
IAL2	8.00	0.07	7	0.26
IAL3	5.00	0.05	5	0.21
IAL4	27.00	0.24	24	0.43
IAL5	5.00	0.05	5	0.21
IAL6	7.00	0.06	6	0.24
IAL7	20.00	0.18	18	0.38
IAL7.1	18.00	0.16	16	0.37
IAL8	1.00	0.01	1	0.09
IAL9	3.00	0.03	3	0.16
IAL10	7.00	0.06	6	0.24
IAL11	8.00	0.07	7	0.26
IAL12	11.00	0.10	10	0.30
IAL13	1.00	0.01	1	0.09
IAL14	59.00	0.53	53	0.50
MMa1	108.00	0.97	97	0.16
MMa2	55.00	0.50	50	0.50
MMa3	1.00	0.01	1	0.09
MMa4	29.00	0.26	26	0.44
MMa5	23.00	0.21	21	0.41
MMa6	5.00	0.05	5	0.21
MMb1	110.00	0.99	99	0.09
MMb2	76.00	0.68	68	0.46
MMb3	33.00	0.30	30	0.46

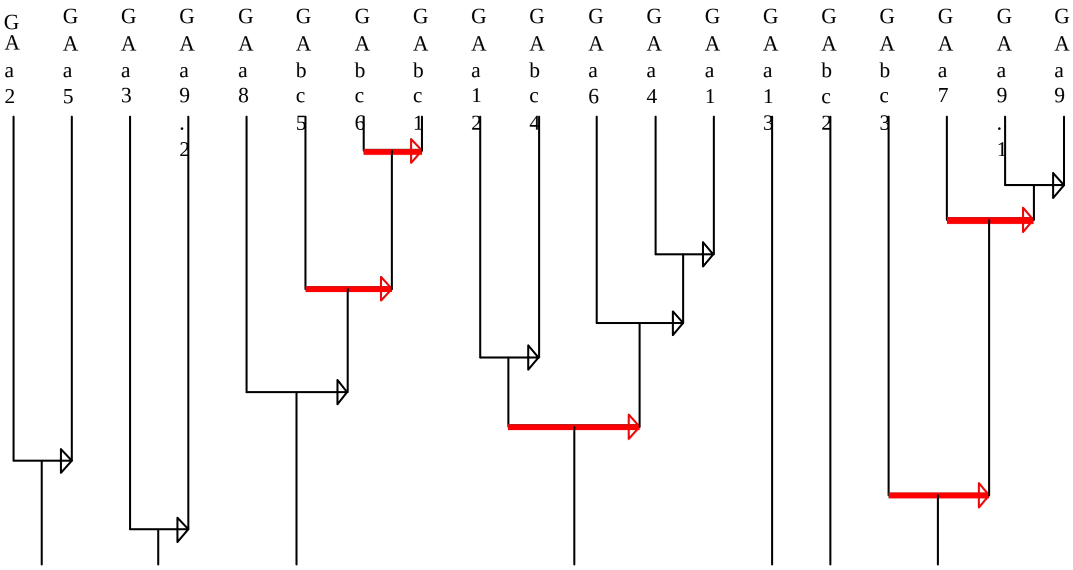
Variable	Absolute frequency	Relative frequency	Percentage	Spread
GAa1	84.00	0.76	76	0.43
GAa2	6.00	0.05	5	0.23
GAa3	3.00	0.03	3	0.16
GAa4	61.00	0.55	55	0.50
GAa5	20.00	0.18	18	0.38
GAa6	34.00	0.31	31	0.46
GAa7	25.00	0.23	23	0.42
GAa8	19.00	0.17	17	0.38
GAa9	42.00	0.38	38	0.48
GAa9.1	35.00	0.32	32	0.46
GAa9.2	17.00	0.15	15	0.36
GAa10	1.00	0.01	1	0.09
GAa11	1.00	0.01	1	0.09
GAa12	18.00	0.16	16	0.37
GAa13	2.00	0.02	2	0.13
GAa14	22.00	0.20	20	0.40
GAbc1	61.00	0.55	55	0.50
GAbc2	3.00	0.03	3	0.16
GAbc3	4.00	0.04	4	0.19
GAbc4	24.00	0.22	22	0.41
GAbc5	31.00	0.28	28	0.45
GAbc6	57.00	0.51	51	0.50
GAbc2*	3.00	0.03	3	0.16
GAbc3*	1.00	0.01	1	0.09

APPENDIX N° 4: Hierarchical tree of the first problem



Arbre hiérarchique : C:\CHIC\chic 2000\Rev-Dati.csv

Appendix N° 5: Hierarchical tree of the fourth problem



Arbre hiérarchique : C:\CHIC\chic 2000\Rev-Dati.csv

CHAPTER FOUR:

THE VARIABLE BETWEEN UNKNOWN AND "THING THAT VARIES". SOME ASPECTS OF THE SYMBOLIC LANGUAGE

3.1 INTRODUCTION

The experimental work of Chapter 3 shows that in the context of a problematic situation, if the conception of variable as unknown prevails, then its relational-functional aspect is not evoked. We have also demonstrated that the resolutive procedures are supported predominantly by the natural language and/or by the arithmetical language as symbolic systems, when the pupils do not have an adequate command of the algebraic language.

From our study we have observed that in the resolution of problems the pupil understands the relational-functional aspect more easily than the variable in presence of visual representative registers. The pupil is more inclined, instead, to consider the variable under the unknown aspect, searching the oneness of the solution of the linear equation, in absence of the graphic representation.

We have also seen that the translation from the algebraic language to the natural language is difficult for the pupils. Some of them succeed in producing the text of a problem that does not result meaningful for the given equation. Others limit themselves, instead, to carry out a purely syntactic manipulation of the formula showing a particular solution that verifies the equation.

A series of questions emerge from this research, for which we have not yet found any answers. For example, in the process of resolution of a problematic situation:

- Ø How are the conceptions of the unknown quantity and of functional relation set going?
- Ø Is the passage from one conception to the other one possible? If yes, how does it happen?

- Ø Does the passage from the single solution to a plurality of solutions of the linear equation necessarily coincide with the passage from one conception to the other one?
- Ø Is the symbolic language present? If yes, is it used for resolving the problem or only in the verbal description as a way of communicating?
- Ø How does the process of translation from the algebraic language to that natural come about?
- Ø How is the syntax-semantics relation represented within the algebraic code?

To study this topic in depth we have effected a new experimentation submitting the same questionnaire to two pairs of pupils.

4.2 METHODOLOGY OF THE RESEARCH

Two pairs of pupils of 16-17 years of age of the Scientific Experimental High School of Ribera (AG) have participated voluntarily in the experimentation. They had not participated in the first investigation.

The proposed questionnaire is the same one used in the preceding experimentation and it was submitted during the first week of May 2003.

The team that conducted the interview was composed of two teachers: an interviewer and an observer. The first one had the assignment to explain the problems and to conduct the interview, the second to take note of all the elements that he thought important.

The whole interview was recorded on audio cassette and after it was transcribed. It was carried on in the following way:

- Ø The pupils had to reach an agreement in their discussion before they could write.
- Ø The interviewer tried to stimulate the pupils, only when they were in difficulty and in a neutral way.

4.3 ANALYSIS OF THE PROTOCOLS OF THE FIRST PROBLEM

4.3.1 FIRST PAIR: Serena and Graziella

4.3.1.1 Types of language: the natural language prevails. They use the arithmetical language, but the algebraic language is completely absent.

4.3.1.2 Resolutive procedure

AL1, AL2, AL4, AL4.3, (AL4.1), ALb1, ALb3, ALb4, ALb6

Serene and Graziella solve the first problem using a procedure in natural language. They add a datum because they consider that the bets are equal and they find a solution. Then they ask themselves by what criterion Charles and Lucy divided the sum of 300 €, because they could have betted different sums. On suggestion of the interviewer, they divide the win in half and they find the solution. They conclude that: *“not knowing how they divided the sum and in what manner they had played, the possible solutions are infinite”* (Line 11).

<table>
<tr><th>Conception of variable</th><th colspan="2">Resolution</th><th colspan="2">Interpretation of the performed procedure</th></tr>
<tr><td rowspan="7">Unknown</td><td colspan="2">AL2: They show a procedure in natural language
AL4.3: They add a datum considering that the bets are equal.</td><td colspan="2">System of two equations with two unknown:
$\begin{cases} 3x + 4y = 300 \\ x = y \end{cases}$</td></tr>
<tr><td colspan="2">They find a solution</td><td colspan="2">They resolve the system</td></tr>
<tr><td>The search of a criterion to divide the sum</td><td colspan="2">is equivalent to</td><td>the search of an equation to form a system.</td></tr>
<tr><td colspan="2">On suggestion of the interviewer
They add a datum and they consider that the win is divided in half (AL4.1)</td><td colspan="2">System of two equations with two unknown:
$\begin{cases} 3x + 4y = 300 \\ 3x = 4y \end{cases}$</td></tr>
<tr><td colspan="2">They find a solution</td><td colspan="2">They resolve the system</td></tr>
<tr><td colspan="2">“We do not know how they divided the sum or in what manner they played” (Line 8)</td><td colspan="2">Another relation between $3x$ and $4y$, or between x and y is not known.</td></tr>
<tr><td>The impossibility to find the criterion</td><td colspan="2">is equivalent</td><td>to the impossibility to form a single system</td></tr>
</table>

⇩

Unknown	They conclude that: *“the possible solutions are infinite”* They do not determine the solution set

4.3.1.3 Comments

An important matter is to analyze the passage from the single solutions, obtained through the resolutions of the systems of equations, to the infinite solutions of the problem considered in the final conclusion. The phrase of Graziella is eloquent: *“Therefore… it does not depend on how many parts win or on the sum that they have played…”* (Line 8); that is, the resolution of the problem is independent from the assumptions that we can make on the wins or the bets. And she continues: *"... we do not know how they have divided the money, or what sum they have played…"* (Line 8); in other words, the problem does not establish with which criterion Charles and Lucy have divided the wins or the bets, therefore it is impossible to form a system of equations. And they conclude: *"the possible solutions are infinite"* (Line 11). The predominant conception of variable in this protocol is that of unknown.

4.3.2 SECOND PAIR: Vita e Alessandra

4.3.2.1 Types of language: the natural language prevails. They use the arithmetical language. The algebraic language is used only in the final part of the resolution, that they looked over after they began the discussion of the second query, which requires exactly the translation from algebraic language to the natural one.

4.3.2.2 Resolutive procedure

AL1, AL3, ALb1, ALb4, ALb2, AL11, ALb6, AL6, AL4, AL4.3, AL5, ALb3, ALb6

The pupils discuss animately about what criterion to adopt for determining the bets and then they decide to proceed by attempts: *"There are not the data of the bets…; we must give one…"* (Line 22) and they continue formulating some hypotheses: *"... We admit that Charles has betted 10 €…"* (Line 13 and 31), *"Let's suppose, if Charles bets 50 €…"* (Line 30), *"If (the bet) is 30 €…"* (Line 37).

The pupils realize that the problem has got infinite solutions and as an example they fix one of the bets and they determine the other one using inverse arithmetical operations:

Natural language	Translation	
	Arithmetical l.	**Algebraic l.**
Alessandra: - *If it is 30 €, the triple ... should win 90…. Therefore from 90 to 300, correct? There are 210…* (Line 37 and 41).	$3 \times 30 = 90$ $300 - 90 = 210$	$y = (300 - 3x) : 4$ *for* $x = 30$
Vita: - *Lucy is the quadruple of that…, but it is divided by 4, isn't it?* (Line 46) ……… **Vita:** *If she wins 210 which is the total sum ...* (Line 50) **Alessandra: -** *Therefore and this must be the quadruple…* (Line 51). **Vita: -** *It is divided by 4* (Line 52).	$210 \div 4 = 52{,}5$	

It is interesting to observe that the idea of linear dependence between the two bets, that is between the two variables, appears implicitly in this discussion.

Alessandra believes that it is necessary to determine the solution set, *"... for me, there is a limit, there are some solutions that go..."* (Line 63), *"according to me... the possible solutions... go from tot to tot..., but we must see..."* (Line 67). In order to do so, she considers the bonds of the problem in an explicit way: *"...If it is speaking of bets, this means that it cannot be a negative number"* (Line 86).

Vita supposes that the bets are equal and she finds the solution using a procedure by successive approximations, but Alessandra insists on the fact that the problem has got a plurality of solutions and on the necessity of determining the solution set.

The students determine the minimal bet equal to 0 without difficulty, but they discuss animatedly on calculating the maximal bet of Charles and Lucy, in a context where the variable takes on the relational-functional aspect.

Vita and Alessandra look over the resolution of the first problem spontaneously after beginning the discussion of the second query. They have found some similarity among them, (the second requires the translation from the algebraic language to that natural): *" ... I think that (the second problem) will be something ...like the first one, it will be similar in some way"* (Line 339).

Therefore they translate the first problem to an equation of first degree with two unknown:

"... I am making other hypotheses, that is, instead of having a possible solution I have a general equation as this one (for the second problem; since 300 is the total sum and we can suppose that 3 for I do not know how much has he betted and 4 for I do not know how much has he betted ..." (Line 351). They write $3x + 4y = 300$.

The pupils discuss animatedly to determine the solution set of the two variables. Even if they write the equation correctly, when they must calculate these sets, they use the same letter *"x"* to designate the two variables. They are confused and they do not succeed in resolving this problem of designation, therefore they only indicate the solution set of the variable x.

Conception of variable	**Resolution**	**Interpretation of the performed procedure**
Relational-functional	AL3: They show a procedure for trial and errors in natural language.	
	ALb4: They considers that the problem has got infinite solutions.	
	They fix one of the bets and they determine the other one using inverse arithmetical operations	$y = (300 - 3x): 4$ for $x = 30$
	Alessandra expresses the necessity to determine the solution set considering the bonds.	$x \geq 0$ and $y \geq 0$
Unknown	*Vita* adds a datum. AL4.3: she considers that the bets are equal.	$\begin{cases} 3x + 4y = 300 \\ x = y \end{cases}$
	She finds the solution using the arithmetical method of successive approximations	Resolution of the system
Relational-functional	ALb6: They consider a plurality of solutions They determine the maximum bets.	$3x + 4y = 300$ *If* $y = 0$, $x = 100$ *If* $x = 0$, $y = 75$
	AL5: They translate the problem to an equation of first degree with two unknown.	$3x + 4y = 300$
	They use the same letter *"x"* to designate the two variable and they succeed in determining only the solution set of it.	$0 \leq x \leq 100$ *(*$0 \leq y \leq 75$ *it is not explained)*

⇩

Relational-functional	They conclude that the solutions are infinite. They determine the solution set with only one variable.

4.3.2.3 Comments

Vita and Alessandra show a long resolutive procedure, they use the natural language predominantly enriched by the numerical language.

In the first part of the resolution, Alessandra considers that the problem has got infinite solutions in a context where the variable takes on the relational-functional aspect. Successively she deems necessary to determine the solution set. Thus she considers that the bonds of the numerical universe are imposed by the context of the problem (bets are not negative). The natural language offers a good semantic control of the quantities in relation to the situation; therefore we think that this control allows to take more easily into account the importance of the bonds.

Even if the aim of the problem appears clear enough from the beginning, the resolutive procedure is long, twisted and it shows a labyrinth of hypothesis and against-hypothesis. We think that the discussion is redundant and not very clear in some aspects; all the ambivalence of the natural language in expressing certain relations between the elements in play becomes apparent. For example, first Alessandra considers that the problem has got infinite solutions, and then she confuses the number of solutions with the maximal win: *"The number of solutions is 300"* (Line 133). Vita considers, instead, that the possible solutions are obtained by the sum of the greatest bets that Charles and Lucy can effect: *"The solutions are these, 175, ... because then at the end it is added…"* (100, the maximum bet of Charles plus 75, the maximum bet of Lucy) (Line 187).

It is interesting to observe that the use of the symbolic language appears only in the final part of the resolution. This had been resumed after the students had begun the discussion of the second problem. Since this query requires the translation from the algebraic language to the natural one, the pupils find certain symmetry with the first problem formulated in natural language; thus they translate it to an equation of first degree with two unknown. But they immediately point out the necessity to give the variables *x* and *y* a meaning in relation to the context of the problem*: "Because x and y represent... the betted money…"* (Line 354 and 451). In other words, the students manifest the need to connect the "original story of the problem" (word problem) with the "story reported in symbols" or the *symbolic narrative*, using the terms of Radford (2002a)[(3)].

Alessandra and Vita use the symbols in the verbal description as a way to communicate: *"maybe because x and y were different ..."* (Line 352), but not to resolve the problem. That is, the symbolic language is used in a rather superficial way because the stream of reasoning is fundamentally supported by the natural language. It is interesting to notice that the language used produces some interferences in determining the solution set. During the discussion the students are able to calculate the solution set of *x* and *y*, but they succeed only in expressing in writing that of *x*: $0 \leq x \leq 100$. They point out the solution set of *y* as: $0 \leq x \leq 75$ and therefore they consider it included in the first one, motivating their choice in this way: *"I wanted to find all the possible solutions. Because if it was not present, here 75 can work at the most and 75 is already included. The most that can be betted is 100, to arrive to a total* (300) *that is the triple of 100"* (Line 403). It becomes thus apparent that the pupils do not possess a good representation of the relation between semantics and syntax within the algebraic code.

The predominant conception of variable in this protocol is the relational-functional one. Even if in certain passages of the resolution Vita considers the variable as unknown because she adds a datum and forms a system, immediately after she abandons this conception in favour of the functional aspect.

4.4 ANALYSIS OF THE PROTOCOLS OF THE SECOND PROBLEM

4.4.1 FIRST PAIR: Serena and Graziella

4.4.1.1 Resolutive procedure

IAL1, IAL4, IAL12, IAL11

Serena and Graziella begin the resolution of the query searching for a pair of values that satisfies the equation: *"...I instinctively search for some numbers ..."* (Line 21). They write the solution they found: *"x = 3, y = 0"* and they comment*: "... we found some numbers that make the equality true ..., therefore we can also build a problem on these two numbers..."* (Line 27). But actually, their discussion is based fundamentally on the resolution of the equation: *"... we invent a problem in which we decide on two numbers and we have to find others two numbers..."* (Line 29) and, successively, Graziella still continues: *"It is necessary to find a pair of numbers, for example 6 and 3; now in this pair of numbers the first one must be multiplied by the first one, the second one by the second one..., and the second one is subtracted from the first one ..."* (Line 35). At a certain moment Graziella asks: *"Find the solution to a problem, but what does it mean to find the problem?"* (Line 39).

The interviewer explains that to invent a problem means to effect a translation from the algebraic language to the natural language.

The pupils paraphrase the text of a problem similar to the first one, but starting from an equation whose coefficients are the pair of solutions found previously, that is: $3x - 0y = 18$. Then Graziella corrects herself: *"It seems me that we must make the inverse procedure, and put these two numbers..."* (Line 51) and she points out the coefficients 6 and 3. Finally the pupils re-phrase the following text:

60. **Graziella:** - *"There are two persons that play these two different sums of money. The first one wins six times the money that they betted, the second wins three times the money that they betted, the difference..."*

61. **Serena:** -*"... between the wins..."*

62. **Graziella:** -*" ... between the wins is equal to 18. Find how much they have betted."*

4.4.1.2 Comments

From the preceding analysis it is deduced that the pupils confuse the activity of solving an equation with that of inventing a problem beginning from an equation. Graziella's question is eloquent: *"... but what does it mean to find the problem?"* (Line 39). Probably this confusion is due to a question of didactic contract: at school pupils usually resolve problems, they do not invent problems.

When the interviewer suggests they translate the equation into natural language, to endow it with semantics, the pupils abandon the preceding syntax. Namely, they leave the resolution, but they invent a problem in which they consider as data the solutions found previously. In other words, they make an interchange of roles between the coefficients of the equation and the pair of solutions. We think that this misunderstanding is due to a certain difficulty in comprehending the meaning of the expression $6x - 3y = 18$.

The pupils correct themselves and they finally succeed in formulating the text of a meaningful problem for the given relation, but paraphrasing an identical situation to the first question. That is, they are not able to endow the equation with different semantics than that of "money and bets". Neither do they make any attempt of changing the context. This observation could be interpreted as the lack of inventiveness. We think that it is due to an insufficient domain or control of the symbols that should guarantee an autonomous life within the problematic situation. The difficulty arises because the pupils have an inadequate representation of the relation between semantics and syntax inside the algebraic code.

4.4.2 SECOND PAIR: Vita e Alessandra

4.4.2.1 Resolutive procedure

IAL1, IAL6, IAL4, IAL7, IAL8, IAL11

The pupils tackle the query asking: *"What does it mean to invent a possible problematic situation? Should we invent a problem?"* (Line 193). The interviewer explains the difference between the resolution of an equation and the inventing of a problem that originates from an equation.

Vita proposes to invent a system of two equations that has got as a solution the equation proposed in the query: $6x - 3y = 18$. Alessandra thinks that it is impossible to build a similar system and she motivates the answer in this way: *"This (the equation) cannot* *come from the problem. Do you know why? It cannot be resolved because we have two different unknowns. Two different unknowns can never be added"* (Line 218).

The discussion becomes animated because Vita insists in forming a system, but she does not make any concrete proposal. It seems that she does not understand what it means "to invent a possible problematic situation". Alessandra also appears quite confused and she persists in the initial idea: *"...It (6x and 3y) can never give an equality because it cannot be added, therefore, even if we state a problem, it will never be a problem, right?"* (Line 235).

Afterwards there are some attempts by Alessandra at inventing a problem and by Vita at finding a particular solution that could verify the equation. The last proposal goes is pursued; they replace the symbols with numbers and they determine a pair of solutions. Vita's comment is eloquent: it expresses the necessity of knowing the solutions *"... can't we help ourselves in making this problem by giving, for example, the solutions?"* (Line 317).

Finally they begin formulating a problematic situation and here we can point out some important phases:

PHASES	PROTOCOL
1. The production of a text that only considers constants, that is the coefficients of the equation	*"... there are 18 apples at the market, that were already there, then someone has taken for example Mark has taken 6 of them and they remained here. How many..., (then for example you must put therefore another datum, eh... then you must put it, eh?). There were 3 in the wardrobe, how many have remained altogether?"* (Line 308).
2. The connection between the first query and the second one	*"But here we have the triple and the quadruple and here we have 6 and 3, 6 minus 3. Therefore, according to me, these things will be similar to develop, then I don't know ... Because there is 18 and here there is 300 the total sum, Charles wins, for example, the triple of the money, instead, the unknown x has 6 here ... The total sum..."* (Line 342).
3. The necessity to allow the variable in the text of the problem to emerge	**Alessandra: -** And 18 €… Then, Charles loses 6 € of the betted money, while Lucia… (Line 450). **Vita: -** *This x is the betted money. x and y are the unknowns of what they have betted.* We have got 3 that should be the triple perhaps, someone loses the triple of it, someone wins it… (Line 450). …………….. **Alessandra: -** Then Charles and Lucy win the total sum of 18 €. We know that Charles wins 6 € of the betted money, while Lucy loses the triple of it… (Line 453). ……………… **Vita: -** It is not 6 €, because 6 is sextuple… (Line 460). **Alessandra: -** Oh! Yes… (Line 461). **Vita: -** *...because x and y are the unknown that represent the betted money.* The sextuple wins, for example, the sextuple of it, … (Line 462).

4. Interpretation of the minus sign	**Alessandra: -** We have a subtraction … (Line 477). **Interviewer: -** Yes (Line 478). **Vita: -** Perhaps, it can be that someone wins and another one loses… (Line 479). . . . . . . **Vita: -** *No, for example, Charles has won a sixth of the sum that has put out and, instead, Lucy has lost the triple of it, perhaps …* (Line 485)
5. Formulation of a text that considers only one variable	*"Charles and Lucy win at the lottery the total sum of 18 €. First they win 6 € of the betted money... After they lose 3 € of it. Determine the sums of money that Charles and Lucy have played. How many are the possible solutions?"* (Line 513).
6. Formulation of a text with two variables	*"Charles and Lucy win at the lottery the total sum of 18 €. First they win six times the betted money, then they lose 3 times of it (of the betted money). How many are the possible solutions?".* (Riga 516).

These six phases do not follow one another in progressive form because the pupils proceed gropingly, going forward and backward. They produce the following temporal sequence: 1, 5, 2, 1, 4, 3, 4, 6, 5, 6:

		TEMPORAL SEQUENCE									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
P	**1**	■			■						
H	**2**			■							
A	**3**						■				
S	**4**					■		■			
E	**5**		■							■	
S	**6**								■		■

With this ambiguous and not quite clear resolving process, Alessandra and Vita only succeed in formulating an identical problematic situation which is the same as the first query.

4.4.2.2 Comments

From the analysis of this protocol we infer that the pupils show a notable difficulty in reasoning on variables. In the first part of the resolution they remain on the syntactic level; they effect a purely syntactic manipulation of the formula as if it were a game of signs without sense. It would seem that the students see the equation like a string of arbitrary symbols, a string governed by arbitrary rules (Linchevski and Sfard, 1991).

We observe this precisely in some passages of the discussion, for example:

- Vita proposes to invent a system of two equations that has got the equation $6x - 3y = 18$ as solution. This is possible only by building a system equivalent to $6x - 3y = 18$, that is a system formed from this equation and from another equation with the coefficients of x and y and the known term directly proportional to 3, 6 and 18 respectively[(1)]. Vita does not consider this requirement and she does not carry out any coherent proposal for proceeding in forming the system.

- Alessandra thinks that such a system cannot be built. In the attempt to motivate her answer, she makes some errors. They are similar to those found by other authors in the abundant literature on the difficulties and misconception in the learning of algebra:

  - She asks to form a system of the type (Line 202):

    $$\begin{cases} 6x - 3y = 0 \\ 18 = 0 \end{cases}$$

    and she does not notice that this arithmetical equality has no sense (Lee & Wheeler, 1989)[(2)].

  - She answers: *"Two unknowns* (6x and 3y) *cannot be added up and, therefore …, they cannot give an equal result, a correct result…"* (Line 239). In this case we observe a certain reluctance in producing a solution that is not a number. This misconception is called by Collis (1974): "difficulty to accept the lack of closing".

We notice the necessity to manipulate the formula syntactically, replacing the symbols with numbers also. This manipulation points out an important loosening between the symbolic language and the possibility of finding a context which gives meaning to the formula. The students have attempted to formulate the problem in a context of "apples and market". They create a typical text of an arithmetical problem, in which only constants appear. Then they succeed only in inserting one variable. Thus they abandon this route and they take the context of the first problem back, after having effected the connection between the first query and the second one.

The approximation to the final formulation happens gradually, by small steps. We thought that to elaborate a text in the context of "money and bets" would have meant to paraphrase the first problem; the students have shown, instead, the necessity to interpret the minus sign and to make the variables emerge in the text, using one or two alternatively.

According to Radford (2002b), in some occasions the symbols produced by the pupils (in this case the minus sign) constitute simplified writings (scripts) that tell important parts of the original story(4). Therefore the exigency to effect the interpretation is associated to the possibility of conferring the correct algebraic sense to the expression and not that of the scripts. On the other hand, the phrase: *"The betted money is this x. x and y are the unknowns of what they have betted"* (Line 451) represents the connection between the equation and the text of the problem, that is between the symbolic narrative and the story of the problem. After having individualised the objects of the context, it is necessary to work on the expressed relation in the equation.

The first text with two variables is the following: *"... they have won 18 € altogether, Charles, for example, has won, for example, the sixth* (instead of sextuple) *of the sum that he had put out , instead, Luigi, for example***, minus 3, for example, y should have to be minus 3,** *that they won""* (Line 512). In this formulation the natural language and the symbolic language are interwoven, the translation is not complete because it still does not endow *3y* of a semantics appropriate to the problematic situation. Then the students go back, formulating an ambiguous text that considers only one variable and finally they correct it adequately to let it include two variables.

From this analysis we observe that Vita and Alessandra have serious difficulties managing and checking the symbols within a problematic situation. For example, if the context is the "market" and there are 18 apples, *x* and *y* could be the quantities: either of two different varieties, either owned by two persons or contained in two cassettes of different dimensions. If, instead, 18 represents money, *x* and *y* could be the prices of two different articles or two prices (purchase and sale) of the same article. The pupils do not succeed in grasping this difference between the two variables in the selected context, their formulations seem a vicious circle around sentences that only consider the constants and those that introduce a variable. Two variables appear in the final text, but it results an imitation of the first problem.

4.5 FINAL CONCLUSIONS

From the analysis of the protocols of the first problem we notice that the resolutive procedures are based on the natural language and they follow the pace of spoken thought in which the semantic control of the situation is developed and takes place.

On the one hand, the second pair exploits the semantic control of the quantities in relation to the problem in determining the bonds of the numerical universe. On the other hand, they are mixed up in a not very clear and redundant discussion that only brings them to develop a long and twisted procedure with a labyrinth of hypothesis and against-hypothesis. So the ambivalence of the natural language in expressing certain relations between the elements in game becomes evident.

The symbolic language is completely absent in the first protocol. Even though in the second one it appears in the final part of the resolution, the pupils use it in a superficial way, only to communicate. Therefore the control that the formula can operate on the flow of the verbal reasoning is missing.

In the first protocol the predominant conception of variable necessary to resolve the first problem is that of unknown. The students calculate particular solutions by resolving two linear systems. Since they do not know the criterion by which the sums of money can be divided, they conclude that the solutions are infinite. The impossibility to find this criterion is equivalent to the impossibility to form a single system. Therefore the passage from the single solutions to infinite solutions is produced through the systems of equations. In other words, for this pair of students the infinite solutions constitute a set of single solutions coming from the resolution of different linear systems that contain the given equation. Accordingly they do not state the problem of the bonds imposed by the context in which the expression is considered.

The relational-functional aspect of the variable prevails, instead, in the second protocol. So the infinite solutions constitute a set of pairs of values that are obtained by varying one of them and calculating the other, beginning from the linear dependence between the variables.

The two couples of students begin the second question by effecting a purely syntactic manipulation of the equation to find some solutions. From the study carried out we deduce that the pupils actually confuse the activity of solving an equation with that of inventing a problem which originates from an equation. We think that this difficulty is due to a matter of didactic contract: at school usually the students resolve problems, they do not invent problems.

The formulation of a problem from an equation implicates fundamentally three activities:

- Choosing an adequate context to give meaning to the equation
- Identifying the objects of the context that represent the variables
- Individualising the properties of the objects that are pointed out by the relation expressed in the equation

We believe that the critical stage is precisely: “to individualize the elements of the context to be associated to the variables”. In the second protocol we assist at the attempt to choose a context of “market and apples”, but the students do not succeed in identifying *x* and *y* with the quantities of apples of two different subject-objects: two shopkeepers, two different varieties, two different cassettes, etc. Thus they formulate the text of a classical arithmetical problem with specific numerical values (the coefficients of the equation); in the attempt of bettering the statement, they succeed only in inserting a variable and therefore they abandon this context.

The two couples of students resolve the query producing a similar text to the first problem. This means to deal with the context “money and bets” and the elements “two persons that play”. They must only adapt the properties of the objects to the new relation that the equation expresses. We thought that this activity would have brought about the paraphrasing of the text of the first problem, but it was not so obvious, especially in the second protocol. One couple felt the need to make the variable emerge in the text of the problem and to interpret the minus sign; their final formulation is the consequence of a gradual elaboration.

In the two protocols we clearly observe an important loosening between the symbolic language and the possibility of finding a different context from “money and bets”, to give meaning to the equation. We think that this is not the consequence of the lack of a certain amount of creativeness, but the result of an insufficient control on the symbols. This is revealed in the impossibility to associate the variables to some elements of the context.

From the study carried out it results evident that an equation alone does not activate forms of productive thought, it is not considered absolutely like the interpretative model of a problem or better still as a class of problems.

To study these conclusions in depth it would be interesting to analyze the existing relation between the variables of an equation and the objects of the context that represent them, from a semiotic perspective of the discourse. It would be important to study how the construction of the sense of a symbolic expression takes place in the space in which the dominion of the symbolic narrative still has not been achieved completely and the story of the problem is just outlined.

NOTES

(1) In other words, a system should be formed in which the matrix of the coefficients and the complete matrix have the same rank r (less than the number of the unknown), for example:

$$\begin{cases} 6x - 3y = 18 \\ 2x - y = 6 \end{cases}$$

The rank r is 1 and the number of the unknown is 2, therefore the solution is not unique. In fact the system is equivalent only to the second equation.

(2) Lee & Wheeler (1989) have observed that some students have arrived at results with no sense in arithmetic, for example, 20 = 4, when they worked syntactically with algebraic expressions. Almost all the students have motivated the development made by a "rule" (also invented) and they have not considered the problem that these "rules" brought to impossible arithmetical results.

(3) Radford (2002a) prefers to speak of symbolic narrative to point out the translation of a given problem into an equation. According to the author this term allows to indicate that a story is still told, but in mathematical symbols.

(4) Radford (2002b) considers that, for some pupils, the minus sign in the expression x - 2, does not always indicate a subtraction on the unknown, sometimes it represents the sign of a simplified writing in relation with the original story.

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APPENDIX N° 1: Synthesis of the analysis of the protocols of the third and fourth query

FIRST PAIR: Serena and Graziella

In the third query the pupils interpret the expressions $ax + by + c = 0$ and $y = mx + q$ in the ambit of the analytical geometry like the generic equations of a straight line in implicit and explicit form, respectively.
The resolutive procedure followed in the fourth problem is the following: GAa1, GAa4, GAa7, GAa9.1, GAbc1, GAbc4, GAbc6.
Graziella immediately realizes that the graphic representation is a straight line. Then the pupils translate the problem to an equation of first degree with two unknowns, they represent it through a table of values and, beginning from it, they sketch the Cartesian graph.
Graziella considers that the equation has infinite solutions and motivates her answer in this way: *"…if the degree of the equation is inferior to the number of unknowns, ... the equation has infinite solutions"* (Line 104). We expected an explanation derived by the graph, instead, Graziella formulates a correct motivation of algebraic nature in the context of analytical geometry.
The students do not consider the bonds to the numerical universe that involves the contextualized sense of the expression. The interviewer asks if the straight line could be prolonged indefinitely in the context of the problem and if x could take on negative values. To answer these questions, Graziella only considers the inferior bond, that is $x \geq 0$.

SECOND PAIR: Vita e Alessandra

The students interpret the expression $ax + by + c = 0$ like an equation of first degree with two unknowns and the expression $y = mx + q$ like the equation of a sheaf proper of straight lines.
The resolutive procedure followed in the fourth problem is the following: GAa1, GAa2, GAa5, GAa9, GAa9.1, GAa9.2, GAbc1, GAbc4, GAbc6.
In this long procedure the pupils used predominantly the natural language, enriched by the arithmetical language. They do not effect the translation of the problem to an equation of first degree with two unknowns, they begin directly with the sketch of the Cartesian axis and the representation of the points (1, 7) and (2, 14). The second point does not belong to the straight line $y = 2x + 5$, but to the straight line $y = 7x$: *"For an hour you pay 7 €, for two hours you pay 14, right?* (Line 615). We think that this error is due to the procedure applied that follows the pace of the spoken thought and it lacks the comparison with the equation of the problem.
Then Vita and Alessandra correct the error and they represent the straight $y = 2x$ on the Cartesian plane, that is the paid sum for the monthly hours of phone calls effected, without considering the fixed fee. They discuss animatedly to calculate the expenses that could be paid in one month, then they determine that greatest sum in one month made up of 30 days, phoning 24 hours daily. They add 5 € of fee to this sum. In this way they calculate correctly the maximum monthly values of x and y. For the minimal values, instead, they consider the amount paid for an hour of phone calls, they do not point out the minimal monthly sum.

APPENDIX N° 2: PROTOCOL OF SERENA AND GRAZIELLA

FIRST QUERY

1. **Graziella:** - E… allora, innanzitutto ci sono due persone Carlo e Lucia che vincono una somma complessiva di 300 E, e sappiamo che uno dei due, Carlo, vince il triplo del denaro scommesso mentre Lucia il quadruplo, e siccome…
2. **Serena:** - Se loro vincono 300 E ...
3. **Graziella:** - Allora se loro si dividono…, se Carlo ha vinto il triplo di quello che ha scommesso, quindi i 3/4, invece Lucia i 4/4, possiamo dividere 300 E in 7 parti, cioè la divisione viene 4,8.
4. **Serena: - No 42,8**
5. **Graziella:** - 42,8; se però loro avessero diviso la somma di 300 E con un criterio stabilito, tipo per esempio..
6. **Serena:** - Stavo pensando se loro avevano giocato la stessa somma di denaro, oppure avevano giocato una somma di denaro diversa, perché uno ha vinto il triplo, l'altro il quadruplo, quindi non sapendo in che modo hanno diviso la somma Carlo e Lucia, non possiamo stabilire…
7. **Sperimentatore:** - Se erano uguali, per esempio le vincite che succedeva?
8. **Graziella:** - Se ad esempio Carlo vinceva 150 E, e anche Lucia 150 E, voleva dire che Carlo aveva giocato 50 E …, invece Lucia aveva giocato di meno. Quindi … non dipende da quante parti vincono oppure dalla somma che hanno giocato complessiva, cioè non sappiamo in che modo loro abbiano diviso il denaro, oppure quale cifra hanno giocato…
9. **Serena:** - E' la stessa cosa, lo stesso ragionamento di prima…
10. **Sperimentatore:** - E' allora a che conclusione siete arrivate…
11. **Graziella:** - Che non sapendo in che modo dividono la somma e in che maniera hanno giocato, cioè se hanno giocato la stessa parte di denaro non possiamo stabilire quant'è questo denaro che hanno giocato. Quindi le soluzioni possibili sono infinite.

SECOND QUERY

12. **Graziella:** - Allora nel secondo problema dobbiamo inventare un problema, alla cui soluzione si possa arrivare tramite un'uguaglianza; l'uguaglianza $6x - 3y = 18$. Allora… questo problema…
13. **Serena:** - Questo è un problema qualunque? Che genere di problema deve essere?
14. **Sperimentatore:** - Voi siete abituate che vi danno il problema…
15. **Serena:** - Infatti…
16. **Sperimentatore:** - …e lo iniziate a risolvere, qui il gioco è diverso.
17. **Serena:** - Cioè, la soluzione dobbiamo cercare…
18. **Sperimentatore:** - Qui dovete inventare il problema.
19. **Graziella:** - Cioè, un problema…, un quesito normale?
20. **Sperimentatore:** - Si, un problema…
21. **Graziella:** - Allora, quindi…, fa 18…, mi viene come l'istinto di cercare dei numeri…
22. **Serena:** - Infatti noi abbiamo al posto di…, che danno un'uguaglianza vera
23. **Graziella:** - Quindi… questo fa 18, quindi faccio come risultato un'uguaglianza vera, quindi 6 per, 6 per 3…
24. **Serena:** - 18, alla y $\longrightarrow$ 0
25. **Graziella:** - Alla x $\longrightarrow$ 3 e alla y $\longrightarrow$ 0 …
26. **Sperimentatore:** - Cosa avete scritto?
27. **Graziella:** - Cioè, abbiamo trovato dei numeri che facevano l'uguaglianza vera…, quindi possiamo anche costruire tipo un problema su questi due numeri in modo per arrivare alla…
28. **Sperimentatore:** - Allora mi dite cosa ne pensate?
29. **Graziella:** - Allora un problema, vabbé può darsi che… non ci siano altre soluzioni quindi, questo non centra niente…, e allora se mettiamo caso, per esempio dobbiamo…, facciamo un problema in cui si danno due numeri e si devono trovare altri due numeri…
30. **Serena:** - Una coppia di valori che deva soddisfare…
31. **Graziella:** - Si una coppia di valori che…

32. **Serena: -** Però era la somma…
33. **Graziella: -** Trovare una coppia di valori che…
34. **Serena: -** Due numeri…
35. **Graziella: -** Bisogna cercare una coppia di valori di cui il primo moltiplicato per il primo numero dato deve essere sottratto al secondo…Bisogna trovare una coppia di numeri, per esempio il 6 e il 3, ora questa coppia di numeri il primo deve essere moltiplicato per il primo, il secondo per il secondo…, e al primo sottrarre il secondo…
36. **Serena: -** Ma deve essere…
37. **Sperimentatore: -** Forza! Senza paura! Che dice qua?
38. **Serena: -** Sicuramente queste cose le abbiamo fatte e le sappiamo fare, ma adesso… non riusciamo a ricordare… quindi
39. **Graziella: -** Trovare la soluzione di un problema, ma che significa trovare il problema?
40. **Sperimentatore: -** Di solito a scuola si risolvono problemi, non si inventano problemi.
41. **Graziella: -** Allora… la questione…
42. **Sperimentatore: -** Perché voi siete abituate a tradurre dal linguaggio naturale al linguaggio algebrico, a scrivere l'equazione…, qui vi si chiede la traduzione al contrario, dal linguaggio algebrico al linguaggio naturale… e non è facile questo…
43. **Graziella: -** Quindi ci sono due persone… con le caramelle… Allora ci sono due persone che…
44. **Serena: -** Vabbé come si fosse una…per esempio… bastava...
45. **Graziella: -** Ci sono due persone che vincono al lotto una cifra, che vincono al lotto una determinata cifra, la sottrazione tra queste due cifre è 18, la prima ha vinto tre volte, tre volte quello che ha scommesso e la seconda ha vinto zero volte quello che ha scommesso…
46. **Serena: -** Ma sempre complicato è…
47. **Graziella: -** Trovare quanto ogni persona ha giocato.
48. **Serena: -** Quindi è lontano dalla soluzione…
49. **Graziella: -** Quindi se due persone giocano una cifra, questo qua è uguale a 18. Ecco, la prima persona ha vinto il triplo di quanto ha scommesso, quindi 3 per x, l'altra ha vinto zero volte di quanto ha scommesso e viene 3 per un numero, quello che ha scommesso e viene 3 per 6 è 18 meno 0 per quello che ha scommesso è 0
50. **Serena: -** Ma deve essere 18
51. **Graziella: -** Mi sembra che dobbiamo fare all'incontrario, mettere questi due numeri…
52. **Sperimentatore: -** Questa equazione è quella che figura nel problema?
53. **Serena: -** No
54. **Graziella: -** No, è uscito…, no… non è quella che dice il problema. Dobbiamo mettere questi numeri, infatti… e abbiamo messo gli altri…
55. **Serena: -** Secondo me deve essere diversa questa cosa…
56. **Graziella: -** Allora due persone giocano una somma di denaro ciascuno. **A** *(nome della persona)***:** vince sei volte di quello che ha scommesso, l'altra persona vince tre volte quello che ha scommesso e la differenza… la differenza della loro vincita è uguale a 18. Quindi…
57. **Serena: -** Questo è uguale a quello che ci ha fatto fare… che abbiamo trovato la soluzione…
58. **Graziella: -** Pero qua la somma di denaro scommessa è diversa. Me lo dice che…
59. **Sperimentatore: -** Allora…
60. **Graziella: -** Ci sono due persone che giocano queste due somme di denaro diverse. La prima vince sei volte di quello che ha scommesso, la seconda vince tre volte di quello che ha scommesso, la differenza …
61. **Serena: -** … tra le vincite…
62. **Graziella: -** tra le vincite è uguale a 18. Trovare quanto hanno scommesso.
63. **Serena: -** Però è un problema che non ha soluzione…
64. **Sperimentatore: -** Non confondere, non ha soluzione o ne ha tante soluzioni?
65. **Serena: -** Ne ha tante soluzioni…, ha tante soluzioni, però in questo no ci ha chiesto di trovare le soluzioni…
66. **Sperimentatore: -** No, cosa chiede di trovare?
67. **Graziella: -** Trovare le somme di denaro che hanno scommesso.

THIRD QUERY

68. **Sperimentatore: -** Nel terzo quesito avete due equazioni, dovete dire la prima idea che vi viene in mente, quando vedete le equazioni… Per esempio, a me sembra che rappresenta questo, forse può essere anche altro…, ma qual è la prima cosa che vi viene in mente?
69. **Serena: -** Allora con la seconda …
70. **Graziella: -** La prima, a me viene in mente l'equazione della prima… Deve essere l'equazione di una retta…
71. **Serena: -** Sì
72. **Sperimentatore: -** Allora, scrivete…
73. **Serena: -** Allora… però con y viene l'equazione di una retta…
74. **Graziella: -** Questo è un fascio, un fascio proprio… (*per la seconda equazione*).
75. **Serena: -** E' un fascio proprio nelle due…
76. **Graziella: -** Sembra un fascio… con un solo parametro… Equazione della retta, però in forma generica, questa dovrebbe essere… Non quella, in forma generica *(per la prima equazione)*.
    Anche questa … Solo che noi di solito il coefficiente lo chiamiamo *n*
77. **Serena: -** Si, si.
78. **Sperimentatore: -** Va bene *n* o *q* è la stessa cosa. Voi scrivete $y = m\,x + n$, dipenda dal libro di testo, in alcuni libri si trova $m\,x + q$, in altri, invece, $m\,x + n$.
79. **Graziella: -** Equazione di una retta in forma generica.
80. **Serena: -** Equazione generica di una retta in forma implicita. Equazione generica di una retta in forma esplicita.
81. **Graziella: -** Equazione generica di una retta in forma implicita. Equazione generica di una retta in forma esplicita *(ripete mentre scrive le risposte).*

FOUR QUERY

82. **Graziella: -** Una persona paga mensilmente 5 € e 2 € per ogni ora di telefonate, quindi 5 € è il costo fisso, il minimo …
83. **Serena: -** … più 2 € …
84. **Graziella: -** 2 € per ogni ora di telefonata, che sarebbe x, il numero di ore mensili di telefonate effettuate.
85. **Serena: -** Allora x è … y …
86. **Graziella: -** Viene una retta
87. **Serena: -** x è il numero di ore mensili quindi x è uguale a…
88. **Graziella: -** y è la somma complessiva e x è uguale alle ore di telefonate. La y che è la somma complessiva è uguale a 5 €, che sarebbe il costo fisso in un mese, più 2 € per le ore di telefonate. Quindi 2, 2 € per le ore di telefonate, per x *(mentre scrive l'equazione).*
89. **Serena: -** Deve essere x…
90. **Graziella: -** La ci siamo
91. **Sperimentatore: -** Allora, non sei convinta?
92. **Serena: -** Allora pagare mensilmente 5 €, la somma complessiva pagata mensilmente è quindi 5 € …
93. **Graziella: -** No. Deve pagare 5 € mensile, che sarebbe il canone del telefono, più 2 € per ogni telefonata, 2 € per ogni ora di telefonate…
94. **Serena: -** Ah! Si.
95. **Graziella: -** Quindi ora dobbiamo disegnarla…Quindi il termine noto è 5, dobbiamo trovare due punti, almeno un punto, quindi se x è 2 più 5 uguale a 9 *(effettuano il grafico).*
96. **Serena: -** Dobbiamo scrivere qualche altro punto…, $x = 0$ e $y = 5$.
97. **Graziella: -** y è la somma complessiva pagata mensilmente, x è il numero di ore mensili di telefonate effettuate.
98. **Sperimentatore: -** Perché questa parte del grafico la avete fatta tratteggiata? Quindi va o non va?
99. **Graziella: -** Perché continua all'infinito quindi, si può prolungare dall'altra parte…

100. **Graziella: -** Ma l'equazione, dobbiamo trovare il numero di telefonate, di ore telefonate e la somma complessiva. Ma le equazioni di primo grado con due incognite non si possono risolvere, quindi…
101. **Sperimentatore: -** Che significa che non si possono risolvere?
102. **Graziella: -** Non si possono risolvere nel senso che ci sono infinite soluzioni.
103. **Sperimentatore: -** Perché?
104. **Graziella: -** In poche parole se il grado dell'equazione è inferiore al numero di incognite, non si possono risolvere… Al numero delle incognite l'equazione ha infinite soluzioni *(ripete mentre sta scrivendo).*
105. **Sperimentatore: -** Voi avete detto che la retta si può prolungare indefinitamente, nel contesto di questo problema si può prolungare indefinitamente?
106. **Graziella: -** Nel contesto? Credo di si, perché dipende da quante ore loro, da quante ore di telefonate… se ha avuto un'ora di telefonate, allora la retta… deve essere…
107. **Sperimentatore: -** Che cosa rappresentano gli assi?
108. **Graziella: -** x è le ore di telefonate, y è la somma complessiva, quindi a man mano che aumenta la x, il valore che diamo alla x, aumenterà anche il valore che diamo alla y quindi, il punto trovato sarà più in alto…
109. **Sperimentatore: -** E dall'altra parte?
110. **Graziella: -** A differenza di… diminuendo le ore di telefonate diminuirà anche il…
111. **Sperimentatore: -** Posso prolungare anche da questa parte? (*segnalando il secondo quadrante, per i valori negativi di x*)
112. **Graziella: -** Dipende se le ore di telefonate sono zero…
113. **Sperimentatore: -** Se sono zero mi trovo qua [*segnala il punto (0,5)*]
114. **Graziella: -** Quindi sarà il punto (0, 5). Da qui se utilizzano numeri negativi, quindi… non si possono avere ore di telefonate negative…
115. **Sperimentatore: -** Per questo ho formulato la domanda.
116. **Graziella: -** Eh…
117. **Sperimentatore: -** Da dove parte questa grafico?
118. **Graziella: -** Allora da questo punto…
119. **Sperimentatore: -** … che non è una retta…
120. **Graziella: -** Una semiretta
121. **Sperimentatore: -** Parte da questo punto…
122. **Graziella: -** Quindi dobbiamo cancellare questo che abbiamo fatto in più…

APPENDIX N° 3: PROTOCOL OF VITA AND ALESSANDRA

FIRST QUERY

1. **Alessandra: -** Somma totale, somma complessiva 300 €. Sappiamo che Carlo vince il triplo del denaro scommesso, quindi la somma scommessa totale …
2. **Vita: -** Pero quello che hanno scommesso, non sappiamo quant'è …
3. **Alessandra:** - Ah… vincono al lotto la somma complessiva di 300 €. Carlo vince il triplo del denaro.
4. **Vita:** - No, c'è il quadruplo…
5. **Alessandra: -** Allora Carlo e Lucia vincono al lotto la somma complessiva di 300 €. Carlo vince il triplo del denaro, quindi Carlo il triplo.
6. **Vita:** - Carlo e Lucia hanno vinto 300 € …
7. **Alessandra: -** Si, somma totale infatti, quella che hanno vinto, sono 300 € …
8. **Vita: -** Il triplo del denaro, pero noi dobbiamo sapere quanto hanno scommesso, quanto…, perché… quanto hanno scommesso?
9. **Alessandra: -** Determina la somma di denaro che Carlo e Lucia hanno giocato. Commenta il procedimento seguito. Quanto sono le possibili soluzioni? Motiva la tua risposta *(legge il testo)*.
10. **Vita: -** Ah… forse vedrò perché… Carlo vince il triplo di quello scommesso.
11. **Alessandra: -** Appunto dobbiamo vedere quanto hanno scommesso. Quadruplo di… Se la somma totale è di 300, giusto?
12. **Vita: -** Dobbiamo levare il triplo per sapere… che Carlo …
13. **Alessandra: -** Ma come facciamo… dobbiamo vedere… Allora mettiamo, mettiamo, no? che se Carlo ha scommesso, mettiamo tipo, non è un dato, che Carlo ammesso che tipo ha scommesso… 10 € e ne ha vinto il triplo…
14. **Vita: -** Ma non lo sappiamo…
15. **Alessandra: -** Perché di questo, giusto? Perché di questo qui… Ma dobbiamo sapere quanto hanno giocato alla…
16. **Vita:** - Si
17. **Alessandra:** - E siccome dobbiamo arrivare alla… a questo che è la somma totale e di questo dobbiamo dividerla una per quello che la scommessa di Carlo…., però dobbiamo vedere quanto hanno scommesso … Quindi la somma di questi due deve arrivare a 300 €.
18. **Vita:** - Si
19. **Alessandra: -** Pero dobbiamo vedere che questo con la somma che ha scommesso ne ha fatto il triplo, quindi ne ha vinto il triplo… E' compreso con questo perché la somma totale è questa, giusto?
20. **Vita: -** Si
21. **Alessandra: -** E' compresa con questo quello che ha scommesso, quello che ha vinto il triplo, mentre quella ha vinto il quadruplo… Quindi, si ammettiamo, no? E per questo si devono dare prima i dati…
22. **Vita: -** Non ci sono i dati delle scommesse … Dobbiamo dare una … Sono 300 €, giusto? Forse… se Carlo vince il …triplo e la somma complessiva di quello che hanno vinto è 300 e Lucia vince il quadruplo e dobbiamo dividere e dobbiamo dividere, no? 300 come il triplo di … vince il triplo di una scommessa…
23. **Alessandra: -** Si ma si deve vedere quanto è il triplo, infatti quello che hanno scommesso … quello che dobbiamo dare…, dobbiamo dare intanto… una… tipo…
24. **Vita: -** Si ma non sappiamo quanto vincono, supponiamo che vincono 300 € in tutto loro, e non sappiamo quanto hanno scommesso sia Carlo che Lucia pero sappiamo che Carlo vince il triplo.
25. **Alessandra: -** No, non sappiamo quanto hanno scommesso.
26. **Vita: -** Lo sappiamo, Carlo vince il triplo del denaro scommesso… e se lui vince il triplo e la somma è 300 …
27. **Alessandra: -** E quello vince il quadruplo…
28. **Vita:** - …il quadruplo di quello che ha scommesso…, tipo quello che …ha scommesso Carlo deve essere per forza 300 €, per forza … la somma complessiva
29. **Alessandra: -** Infatti, la somma totale è di 300 €,

30. **Vita: -** Mettiamo se Carlo scommette 50 € , Carlo, per esempio, Lucia, per esempio, scommette 100 €, 200 €…
31. **Alessandra: -** Tipo scommette 10 € il triplo, il triplo sono…
32. **Vita: -** 30 € , quindi… e Carlo mettiamo ha scommesso…
33. **Alessandra: -** Perché poi alla fine dobbiamo essere noi a dare delle soluzioni, perché non ci le dà. Quindi qua dice quante sono le possibili soluzioni… deve essere, però ci possono essere infinite soluzioni, infinite relativamente però ai numeri fino a quando possiamo arrivare, qua dice il quadruplo qua dice il triplo… Quindi se mettiamo… Se Carlo ha scommesso 10 €, giusto? Scommessa Carlo …
34. **Vita: -** Ma mettiamo 50 €.
35. **Alessandra: -** No poi è troppo… non so quanto è il quadruplo.
36. **Vita: -** Si va bene, non si sa quanto è veramente…
37. **Alessandra: -** Se è 30 €, il triplo… dovrebbe vincere 90. In totale… avrebbe dovuto vincere… ipotesi…
38. **Vita: -** Allora… se è 30… il triplo deve vincere 90
39. **Alessandra: -** Se è 30… il triplo deve vincere 90 e se…
40. **Vita: -** Stai parlando sempre del quadruplo, la somma totale deve essere di 90 deve arrivare a 300,
41. **Alessandra: -** Quindi da 90 per arrivare a 300, giusto? Ci sono 210…
42. **Vita: -** Si
43. **Alessandra: -** … quindi per vincere 210 Lucia …, giusto?
44. **Vita: -** Si
45. **Alessandra: -** … per vincere 210 € …
46. **Vita: -** Lucia è il quadruplo di quello… però è diviso 4, no?
47. **Alessandra: -** Non so…
48. **Vita: -** Vediamo se vengono più o meno giusto i numeri?
49. **Alessandra: -** Aspetta…
50. **Vita: -** Se vince 210 che è la somma totale…
51. **Alessandra: -** Quindi e questo deve essere il quadruplo…
52. **Vita: -** Diviso 4
53. **Alessandra: -** Contiene la somma scommessa.
54. **Vita: -** Questo è il triplo e forse l'ho capito… e questo dovrebbe essere il quadruplo… qua c'è la somma scommessa e poi ci sono 210 diviso 4.
55. **Alessandra: -** Si, ma non sappiamo se è veramente 30 €.
56. **Vita: -** Lo abbiamo visto come ipotesi
57. **Alessandra: -** E qua dice le possibili soluzioni
58. **Vita: -** Quante sono le possibili soluzioni?
59. **Alessandra: -** Però può essere… possono essere tante in base a quanto possano scommettere …
60. **Vita: -** Si… aspetta…, non è così, c'è… le soluzioni possono essere… possono essere appunto, ma così tante soluzioni vengono…
61. **Alessandra: -** Eh?
62. **Vita: -** …così tante soluzioni vengono… se per esempio questa signora fa il triplo… tante soluzioni vengono, invece qua dice quante sono le possibili soluzioni?
63. **Alessandra: -** Si, infatti, ti voglio dire, noi dobbiamo calcolare quante sono le possibili soluzioni, perché le somme di denaro che Lucia e Carlo hanno scommesso sono relative. Perché si quello vince il triplo e quello vince il quadruplo, però tipo non è che possiamo sapere quanto hanno scommesso, noi abbiamo fatto l'ipotesi che scommettano…, invece abbiamo il triplo, il quadruplo pero mi sembra che su questo…, per me c'è un limite, ci sono delle soluzioni che vanno… Mi sembra che non vanno però…
64. **Vita: -** No secondo me, se ci sono tante soluzioni alla fine…
65. **Alessandra: -** Questo è anche impossibile tipo… non impossibile, cioè inutile, perché si dice quante possono essere le possibili soluzioni, noi invece i valori che possiamo dare…, giusto? tipo una scommessa sia Carlo che Lucia possono quindi vincere il triplo; supponiamo che diamo 20 - 40 tipo vince il triplo 40 – 60 - 80 e sono 80 e se quello però per arrivare a 300, deve mettere sempre di meno, giusto?

66. **Vita: -** Certo...
67. **Alessandra: -** Quindi è sempre relativo le somme che noi mettiamo, secondo me, sono le possibili soluzioni possono andare tipo, tipo qua ci sono 300, tipo vanno da tot a tot, queste possibili soluzioni…, ma dobbiamo vedere…
68. **Vita: -** Intanto qua dobbiamo vedere quant'è la somma scommessa?
69. **Alessandra: -** Secondo me, non ha senso, perché si qua dice determina le somme… perché, infatti, questo non è qualche numero che dobbiamo mettere…
70. **Vita: -** Comunque…
71. **Alessandra: -** Allora dobbiamo vedere quante possono essere quindi queste soluzioni, da quando possano andare…
72. **Vita: -** Si
73. **Alessandra: -** …possono esserci, possono esserci 300 soluzioni possibili, può essere 150? Perché per…
74. **Sperimentatore: -** Che cosa ne pensate?
75. **Vita: -** Stiamo facendo qua… Come vengono … questo, poi dobbiamo vedere…
76. **Alessandra: -** Scriviamo dal più basso al più alto…
77. **Vita: -** Invece…
78. **Alessandra: -** Dal più piccolo al più grande…
79. **Vita: -** Perché non sai come sono…
80. **Alessandra: -** Ma possiamo scommettere qualsiasi somma?
81. **Vita: -** Per questo è impossibile…
82. **Alessandra: -** L'importante è che non superi questo qua, capisci quello che ti voglio dire io?
83. **Vita: -** Carlo vince il triplo del denaro scommesso, forse il denaro scommesso sarà uguale, quello che loro hanno scommesso…
84. **Alessandra: -** Eh, può essere…
85. **Vita: -** Sapendo che Carlo vince il triplo del denaro scommesso…
86. **Alessandra: -** Ma c'è il quadruplo e si sta parlando di scommesse per questo significa che non può essere un numero negativo.
87. **Vita: -** E se noi facciamo per esempio il triplo sempre di 10 €, sarebbe 30 e… invece è sempre 10 € sarebbe poi 40, il quadruplo che sarebbe 40, la somma dovrebbe venire poi 70, noi possiamo fare se la somma deve venire per esempio 300, noi possiamo fare…
88. **Alessandra: -** Come?
89. **Vita: -** Per esempio, del denaro scommesso è uguale sia per Carlo, mettiamo per esempio che è uguale sia per Carlo sia per Lucia, mettiamo per esempio 10 € in questo caso, se è 10 € per Carlo, il triplo è 30 €, per Lucia 40 e poi vincono al lotto la somma complessiva, quella somma deve essere per esempio, questo viene 70, quindi è sbagliato, invece deve mettere il denaro scommesso deve essere uguale e che poi con la somma deve fare 300
90. **Alessandra: -** Può essere…
91. **Vita: -** Se per esempio noi mettiamo 30 €, deve venire 90, giusto?
92. **Alessandra: -** Si …
93. **Vita: -** 30 € il quadruplo, 90 e 30, 120, 120 + 90
94. **Alessandra: -** Fanno 230
95. **Vita: -** No, 210, 210. Quindi non ci siamo. Per esempio, 40,
96. **Alessandra: -** 40 il triplo…
97. **Vita: -** 120, 40 il quadruplo, 120 + 40, fa 160, 160 + 120 fa
98. **Alessandra: -** 280
99. **Vita: -** 280 e ancora non ci siamo…
100. **Alessandra: -** No ci vuole poi tanto…
101. **Vita: -** Mettiamo per esempio 42, se facciamo…, per esempio 42
102. **Alessandra: -** …il triplo sarebbe…, 42 e 42 fanno…
103. **Vita: -** 84 più 42 fanno 126, 126
104. **Alessandra: -** il quadruplo…
105. **Vita: -** 126 + 42 fanno 168, giusto? 168 + 126,
106. **Alessandra: -** No, non è 300
107. **Vita: -** 126 + 168 è 294.

108. **Alessandra: -** No, non ci siamo
109. **Vita: -** 43, hai capito come risalire?
110. **Alessandra: -** Si, si
111. **Vita: -** Quindi, con 43…
112. **Alessandra: -** Anche così ci possono essere infinite soluzioni?
113. **Vita: -** No, dipende da quello che ha scommesso, ma così ti viene quanto hanno scommesso…
114. **Alessandra: -** Ma se hanno scommesso la stessa somma, non è che sappiamo se hanno scommesso la stessa somma…
115. **Vita: -** Secondo me, si, ma bisogna dirlo, che hanno scommesso la stessa somma…
116. **Alessandra: -** Tu prova a farlo…
117. **Vita: -** Non viene, viene 302. Il procedimento, come può essere? Non lo so
118. **Alessandra: -** Come possono essere…?
119. **Vita: -** Carlo e Lucia scommettono la stessa somma?
120. **Sperimentatore: -** Il problema lo specifica?
121. **Alessandra: -** No, appunto quello che dico io per questo…, noi possiamo supporre…, tutto è una supposizione, un'ipotesi, non è un dato certo…
122. **Sperimentatore: -** Si un'ipotesi, come voi avete supposto per esempio che…
123. **Vita: -** Avevamo scritto 30 €.
124. **Alessandra: -** Quindi, secondo me, no? Essendo che si deve arrivare ad un tot di 300 €, si devono calcolare da tanto a tanto per un massimo, per un totale, ossia che non si può superare quella…, giusto?
125. **Sperimentatore: -** Si
126. **Alessandra: -** E dobbiamo vedere quante sono le soluzioni, perché se noi dobbiamo determinare le somme di denaro che hanno scommesso, possono essere tante le somme di denaro che hanno scommesso per arrivare alla somma di 300 €.
127. **Sperimentatore: -** Si
128. **Alessandra: -** Pero, noi dobbiamo vedere quante sono le possibili soluzioni, cioè quante possono essere le soluzioni per arrivare…
129. **Sperimentatore: -** Fino a quello che voi siete arrivate qua, voi avete detto "questo problema è impossibile", attenzione che cosa significa che questo problema sia impossibile?
130. **Vita: -** Che non si può avere nessuna soluzione.
131. **Sperimentatore: -** In tutta la discussione, a quale conclusione siete arrivate, quante soluzioni si possono trovare?
132. **Vita: -** Ci possono essere fino ad arrivare ad un totale di 300 €.
133. **Alessandra: -** Quindi, sono 300 soluzioni.
134. **Vita: -** No
135. **Alessandra: -** 300 perché poi…
136. **Sperimentatore: -** Perché poi?
137. **Alessandra: -** Può anche…, però qua dice che ha scommesso…, può non avere scommesso niente e avere vinto il quadruplo e non avere scommesso niente.
138. **Vita: -** Non ha vinto nulla.
139. **Sperimentatore: -** Si, si potrebbe pensare questo. Voi volete vedere da dove a dove si va… La cosa importante di questo problema è determinare quante soluzioni esistono. Secondo voi, si può determinare un numero preciso di soluzioni?
140. **Vita: -** No, non so…
141. **Alessandra: -** Dipende…, pero essendo che…, possono essere anche tipo…
142. **Vita: -** Soluzioni … il doppio della somma scommessa perché può tipo …
143. **Alessandra: -** Si sono 300 €, giusto? ed è il totale devono partire…, partono sicuro da 0 perché possono anche aver scommesso 0
144. **Sperimentatore: -** Si
145. **Alessandra: -** Possono arrivare…
146. **Vita: -** Secondo me, va da 0 a 150 perché si l'altro può scommettere massimo 150, cioè per non avere quindi…
147. **Alessandra: -** No

148. **Vita: -** … che poi...
149. **Alessandra: -** … che poi... anche il doppio…, può non avere scommesso così…
150. **Sperimentatore: -** Allora facciamo il riassunto di tutto il problema…, che cosa dovresti scrivere?
151. **Alessandra: -** Che non sappiamo intanto quanto possono avere scommesso, quindi noi abbiamo…, dobbiamo essere noi ad imporre un dato.
152. **Sperimentatore: -** Allora scrivete questo.
153. **Vita: -** Si ma...
154. **Sperimentatore: -** …se siete d'accordo. Non sei d'accordo Vita?
155. **Alessandra: -** Scriviamo…
156. **Vita: -** Secondo me, si può vedere quanto hanno scommesso tutti e due, però… non so…
157. **Sperimentatore: -** Però dando qualche condizione, secondo te…
158. **Vita: -** Si delle ipotesi sempre…
159. **Alessandra: -** Ma è quello che dico io
160. **Vita: -** Però tu dici che non possiamo vedere le possibili…, le somme…, le possibili soluzioni…
161. **Alessandra: -** Sì perché non viene detto di valutare tutte le possibilità, ma possono essere tante, qua no poi alla fine determina le somme…
162. **Vita: -** Si, ma
163. **Alessandra: -** …noi possiamo imporre la somma che hanno scommesso e quindi, può essere, può variare, è una cosa che varia… in base a quanto hanno scommesso perché non abbiamo dati certi. Però le soluzioni per quanto possono essere, si possono sapere però…, cioè io penso…, però…, cioè…
164. **Vita: -** Non sono dati certi…, perché si possono…
165. **Alessandra: -** No, perché da un massimo di 300, proprio è il massimo, a 300 si può arrivare…
166. **Vita: -** Sì, per quello tipo… va a scommettere 300 € e quello mettiamo scommette, giusto? Il quadruplo poi già la somma viene sorpassata e quindi deve arrivare ad un massimo per quella…
167. **Alessandra: -** Allora…, aspetta…, secondo me è così, allora il triplo di 300…, cioè devo fare, 300 diviso 3…
168. **Vita: -** Si
169. **Alessandra: -** Perché quello può avere scommesso un numero che possa arrivare a 300, e che Lucia non ha scommesso niente, in questo caso, anche si ha vinto il quadruplo, mettiamo questa come possibile ipotesi, quindi diviso 3 e viene…
170. **Vita: -** 10
171. **Alessandra: -** Che stiamo facendo? Giusto 10
172. **Sperimentatore: -** Quant'è 300 diviso 3? 100, no?
173. **Vita: -** Appunto
174. **Alessandra: -** E' giusto, quindi il triplo…
175. **Vita: -** Non è che la…, il triplo non fa più di 100
176. **Alessandra: -** Quello che ha il triplo deve avere scommesso 100, giusto?
177. **Vita: -** Si
178. **Alessandra: -** Perché…
179. **Vita: -** …invece, il quadruplo è diviso 3
180. **Alessandra: -** …diviso 4…
181. **Vita: -**…diviso 4…
182. **Alessandra: -** 100 triplo… Facciamo per il quadruplo, 300, il 4 non spunta… Allora… (*Fanno 300 : 4*).
183. **Sperimentatore: -** 75, dai
184. **Alessandra: -** Allora… giusto?
185. **Vita: -** E' impossibile.
186. **Alessandra: -** No, non ho capito…
187. **Vita: -** Le soluzioni sono queste, 175, possono essere, perché poi alla fine si somma, quanto…
188. **Alessandra: -** No, non si può passare avanti…

189. **Vita: -** Se vediamo…
190. **Sperimentatore: -** Allora, una cosa importante, si può determinare un numero preciso di soluzioni in questo problema? Sì o no? Posso dire che le soluzioni sono 20, 30, 100 o che sono tantissime?
191. **Vita: -** Tantissime
192. **Alessandra: -** Sono variabili

SECOND QUERY

193. **Alessandra: -** Che vuole dire inventare una possibile situazione problema? Dobbiamo inventare un problema?
194. **Sperimentatore: -** Dovete inventare un problema. Al contrario di quello che si fa abitualmente, invece di avere un problema e risolverlo, è al contrario, data un'equazione, inventate un problema.
195. **Alessandra: -** E appunto…
196. **Vita: -** E' un problema, per esempio Carlo e…, oppure per esempio è un'eguaglianza?
197. **Sperimentatore: -** Come volete voi.
198. **Alessandra: -** Tipo… può essere per esempio…
199. **Vita: -** Se noi facciamo, per esempio, un sistema di due equazioni, ci viene questo? ci può venire questo?
200. **Alessandra: -** Come?
201. **Vita: -** Sai quando facciamo un sistema, un sistema di due equazioni…
202. **Alessandra: -** Si, tipo $6x - 3y = 0$ e $18 = 0$, questo dici tu?
203. **Vita: -** No, per esempio deve risultare questo, per esempio devono essere due equazioni che poi alla fine deve risultare questo…
204. **Alessandra: -** Quest'uguaglianza?
205. **Vita: -** Quest'uguaglianza
206. **Alessandra: -** Se per esempio noi consideriamo che devono venire due incognite… o che restano altre incognite…
207. **Vita: -** Si… Noi dobbiamo inventare un problema…, tipo di…
208. **Alessandra: -** Però come si fa di essere $6x - 3y = 18$, perché x e y, essendo due variabili, e sono diverse non si possono sommare…
209. **Sperimentatore: -** No
210. **Alessandra: -** E quindi non è possibile che si possa, giusto? Essendo due variabili, che sono diverse x e y, x e y non si possono sommare, e quindi non possiamo dare mai il risultato…, caso mai si potrebbero dare…, se ammettiamo tipo, se facciamo 6 – 3 viene 3xy
211. **Vita: -** Non può essere x e y non si possono mai sommare
212. **Sperimentatore: -** No. E' giusto quello che dici.
213. **Alessandra: -** E quindi non può essere, non può essere… Non si può risolvere, giusto?
214. **Sperimentatore: -** Tu vuoi risolvere l'equazione?
215. **Vita: -** Per esempio, se noi troviamo due equazioni, se per esempio troviamo due equazioni, le mettiamo in un sistema, giusto?
216. **Alessandra: -** Si, ma…
217. **Vita: -** Per esempio, mettere…, per esempio, …
218. **Alessandra: -** Non ti può venire questa del problema. Lo sai perché? Non si può risolvere perché abbiamo due incognite diverse, due incognite diverse non si possono mai sommare.
219. **Vita: -** Lo so
220. **Alessandra: -** Non si può quindi svolgere, non si può svolgere questo, noi dobbiamo fare un nuovo problema.
221. **Vita: -** Quando ci sono due incognite, perché non si può risolvere? Perché non si può risolvere? Si può trovare una soluzione. Se noi mettiamo due equazioni e le facciamo al sistema, possiamo ottenere…

222. **Alessandra: -** …quello della x, per esempio, si può imporre un numero, tipo x = 0 poi l'equazione verrà sicuro un'incognita soltanto, però noi dobbiamo fare il problema oppure dobbiamo svolgere il…, o entrambi.
223. **Sperimentatore: -** L'enunciato dice di inventare una possibile situazione problema… Inventare un problema. Di solito non fate questo, di solito si risolvono problemi a scuola, non si inventano… Questo quesito chiede il contrario.
224. **Vita: -** Eh!…
225. **Alessandra: -** Che cosa facciamo?
226. **Vita: -** Possiamo trovare per esempio la x…
227. **Alessandra: -** No, non dobbiamo…
228. **Vita: -** No va bene, si ti dico poi per esempio quello che ci troviamo, ci troviamo qua la y, se poi ci troviamo qua la y …
229. **Alessandra: -** Dobbiamo inventare il problema
230. **Vita: -** Il problema…, non si può fare per esempio un sistema, non il problema?
231. **Alessandra: -** Il problema, il problema dobbiamo inventare, dobbiamo dare i dati, un problema… come questo.
232. **Vita: -** Va bene, per esempio di risolvere due equazioni.
233. **Alessandra: -** No quale risolto, qua è un problema, altrimenti non era questo il problema.
234. **Sperimentatore: -** Non è un esercizio, un problema dovete fare.
235. **Alessandra: -** Secondo me, non possono dare mai un'uguaglianza perché non si possono sommare, quindi anche se noi poniamo un problema, non sarà mai un problema, giusto?
236. **Vita: -** Ma …, quindi…
237. **Alessandra: -** …un problema…
238. **Vita: -** …anche se abbiamo per esempio due equazioni e risolvere, non è un problema questo? Di trovare per esempio una relazione…
239. **Alessandra: -** Ma è falsa, è falsa… non possiamo trovare…, è un'ipotesi falsa, perché non può essere un'uguaglianza se $6x - 3y = 18$. Non si possono sommare due incognite e non si possono dare quindi un risultato uguale, un risultato giusto…
240. **Vita: -** Dobbiamo dare un numero alla x, un numero alla y che per risultato dia 18, non dovrebbe essere questo? Un valore alla x, un valore alla y che poi per risultato deve dare 18. Un'uguaglianza per essere vera deve essere così, no?
241. **Alessandra: -** Non ci può aiutare?
242. **Vita: -** Io, per esempio, volevo trovare un valore della y e poi, per esempio, mettendo qua trovare il valore della x, perché noi il problema è forse questo, dobbiamo trovare un valore della x, un valore della y e poi, per esempio, bisogna moltiplicare tipo…
243. **Alessandra: -** Se Marco, se Marco ammettiamo…
244. **Vita: -** Perché Marco…, perché un nome?
245. **Alessandra: -** Dai qualsiasi nome…
246. **Vita: -** Per forza dei nomi ci devono essere…, se..
247. **Alessandra: -** Allora nel mercato ci sono 6…, mettiamo e abbiamo 3 tipo… Ah! tipo Lucia…
248. **Vita: -** No nel supermercato ci sono 6, non so come dirlo, per questo…
249. **Alessandra: -** In totale erano 18 frutta, giusto?
250. **Vita: -** Non lo so…
251. **Alessandra: -** Vediamo se dà il risultato qua…
252. **Sperimentatore: -** Che cosa scrivi, lì?
253. **Alessandra: -** Sto facendo un'equazione…, sto facendo…
254. **Vita: -** Vediamo se è vero, se mettiamo per esempio…, x e y…
255. **Sperimentatore: -** Vorrei sapere che cosa scrivi? Spiegami
256. **Alessandra: -** Un attimo…
257. **Sperimentatore: -** Hai scritto x uguale?
258. **Alessandra: -** Ho fatto x, ho scritto prima x…
259. **Sperimentatore: -** Scrivetelo o non siete d'accordo con questo?
260. **Vita: -** Non lo so
261. **Sperimentatore: -** Allora…
262. **Alessandra: -** Sai perché, sai perché …

263. **Vita: -** Se questa è un'uguaglianza che deve risultare vera, perché: "Inventa una possibile situazione che possa risolversi utilizzando la seguente relazione"
264. **Alessandra: -** Situazione problema…
265. **Vita: -** Un problema è anche con due equazioni è, un problema, per esempio, il problema sarebbe che noi dobbiamo trovare la x e la y, il valore di x e il valore di y che messi in questa cosa da per risultato 18 perché è uguaglianza. Per esempio, qua per esempio diamo una delle due 2…
266. **Alessandra: -** Sì ma qua non dobbiamo dare un valore alla x e dare un valore alla y…
267. **Vita: -** …dobbiamo trovare tipo…, e qui tipo…
268. **Alessandra: -** Non è una cosa…, è che $6x - 3y = 18$... Capisci?
269. **Vita: -** e come fai, e come fai tu…?
270. **Alessandra: -** Non si può fare perché è già fatta l'uguaglianza…
271. **Vita: -** Perché è fatta se diamo un valore alla x e un valore alla y, l'uguaglianza vera è. Il problema questo è …
272. **Alessandra: -** Se noi per esempio diamo 3
273. **Vita: -** Sempre lì siamo, dipende di quanto siano i valori allora è, se diamo 3, se diamo 3…
274. **Alessandra: -** Aspetta, 6 per 3 è 18 meno…, mettiamo 0 qua alla y, meno 0 uguale a 18
275. **Vita: -** Se noi, per esempio non lo so, se mettiamo non si possono cambiare sempre i valori…
276. **Alessandra: -** Aspetta un minuto, 6 per 4 è 24, giusto? 24 meno… E allora cosa ho detto $6 \times 3 = 18$, $3 \times 0 = 0$ e viene 18. $6 \times 4 = 24$, e viene $3 \times 1 = 3$, $24 - 3 = 21$ e questo viene… $6 \times 5 = 30$, $3 \times 2 = 6$ … *(Sostituisce x e y con diversi valori).*
277. **Vita: -** E' sbagliato questo …
278. **Alessandra: -** Perché deve venire uguale a questo numero, capisci… Deve essere un problema io non lo so come si fa un problema, può essere che sia sbagliata la consegna…
279. **Vita: -** E' sbagliata… Ci vuole un'incognita e solo se è la stessa potevamo…
280. **Alessandra: -** Può essere…
281. **Vita: -** Una possibile soluzione, una perché ce ne sono tante soluzioni, giusto? Se noi inventiamo una ed è giusta e fa venire questa uguaglianza vera…
282. **Alessandra: -** E allora …, $6 \times 3 = 18$…
283. **Vita: -** Inventa una possibile soluzione…
284. **Alessandra: -** Problema, problema…, non centrano le soluzioni…
285. **Vita: -** Il problema e se tu per esempio, il problema no…, noi non abbiamo né la x né la y…
286. **Alessandra: -** Sì ma dobbiamo inventare un problema, un problema proprio.
287. **Vita: -** Ma perché un problema proprio?
288. **Sperimentatore: -** Dovete inventare un problema.
289. **Alessandra: -** Dobbiamo inventare un problema, non è che lo dobbiamo risolvere.
290. **Vita: -** Io metto per esempio…, Marco…
291. **Alessandra: -** Un problema…
292. **Vita: -** Eh!
293. **Alessandra: -** Allora "è andato", cominciamo a scrivere, giusto? Allora, "al mercato…" (*Frase scritta: Marco è andato al mercato*...).
294. **Vita: -** Se mettiamo i valori alla x e alla y?
295. **Alessandra: -** Ma risolvere no…, ma qua diceva il problema, capito?
296. **Sperimentatore: -** Non chiede di risolvere
297. **Vita: -** Forse è così, per esempio, la somma di tot complessiva, giusto? Per esempio…, forse ti immagini per esempio a questo, per esempio 6, vince per esempio quello…
298. **Alessandra: -** Carlo e Lucia vincono al lotto…
299. **Vita: -** Carlo e Lucia…
300. **Alessandra: -** Ah! Si! Carlo e Lucia vincono al lotto la somma complessiva di 18 €. Sappiamo che Carlo vince il triplo del denaro scommesso, in questo caso vince il doppio, il triplo del denaro scommesso e dovrebbe avere scommesso 2…
301. **Vita: -** Meno 3, c'entra?
302. **Alessandra: -** E Lucia…
303. **Vita: -** Secondo me, non c'entra perché 6 per…, aspetta…

304. **Alessandra: -** Se noi mettiamo, no? Che… mettiamo…
305. **Vita: -** Per esempio, il tot complessivo è 18…
306. **Alessandra: -** Non ha senso, non ha senso… Perché no…
307. **Vita: -** Lo sai perché, perché noi non siamo riuscite a fare quello di prima…
308. **Alessandra: -** Non c'entra, per me è stato facile perché possono essere tante soluzioni però… Sai che cosa è? Tu no? vuoi fare tipo… ammettiamo no? tipo *al mercato ci sono 18 mele* che erano già di prima poi ne ha prese, tipo *Marco ne ha prese 6* e sono rimaste qua. *Quante*, poi tipo devi mettere quindi un altro dato, tipo poi li devi mettere eh! C'è n'erano 3 nell'armadio quante sono rimaste complessivamente? Giusto? Tipo questo, no? Così.
309. **Vita: -** Si, si
310. **Alessandra: -** Allora, allora, *Marco è andato…*, no Marco…
311. **Vita: -** *…ha*, per esempio, *18 mele*, no Marco…
312. **Alessandra: -** …al mercato…
313. **Vita: -** Queste 3 sono perché…
314. **Alessandra: -** Guarda…, *al mercato…*
315. **Vita: -** Non sarebbe, non è mercato… Marco, per esempio, e Lucia hanno 18 mele…
316. **Alessandra: -** Sai che cosa ti voglio dire io che un negoziante va a comprare 18 mele, da cui 6 le mette fuori così, le altre le dà…, le porta…, insomma le mette dentro qualche parte, tipo… le vuole mettere…, non lo so, le vuole mettere…, e poi il negoziante è andato a prendere tot mele. Quante ci sono complessivamente tra quelle rimaste e quelle…, giusto?
317. **Vita: -** Per me però…, non ci possiamo aiutare a fare questo problema dando, per esempio, le soluzioni?
318. **Alessandra: -** Secondo me, tu dimmi quale…
319. **Vita: -** Una possibile soluzione qua c'è scritta, no? Se noi, per esempio, diamo un valore per esempio…
320. **Alessandra: -** Non ci può essere perché…
321. **Vita: -** Sono tante le soluzioni…
322. **Alessandra: -** Nooo
323. **Vita: -** Sono tante le soluzioni, come no? Se noi, per esempio, aspetta…
324. **Alessandra: -** Un problema secondo me…
325. **Vita: -** Con due… due incognite
326. **Alessandra: -** E per questo dico io che secondo me non si può svolgere, giusto?
327. **Sperimentatore: -** Allora che cosa vi traumatizza tanto, che ci siano due incognite diverse?
328. **Alessandra: -** Cioè, dobbiamo fare un problema, noi non è che abbiamo fatto tipo… non abbiamo risposto mai ad un problema, noi abbiamo avuto sempre l'equazione senza risolvere…
329. **Vita: -** Cioè, l'abbiamo soltanto…
330. **Alessandra: -** Però no, no…
331. **Vita: -** Ma poi Lei ci dice com'è?
332. **Sperimentatore: -** Si, poi vi lo dico… E allora concludendo possiamo scrivere questo problema o no?
333. **Alessandra: -** Vicine c'eravamo?
334. **Sperimentatore: -** Che?
335. **Alessandra: -** Vicine c'eravamo?
336. **Sperimentatore: -** Lo possiamo scrivere o no? Succede che non vi mettete d'accordo, una dice una cosa e l'altra risponde sempre di no…
337. **Alessandra: -** Secondo me…
338. **Sperimentatore: -** In parte ha ragione ciascuna, ma non mi mettete d'accordo tra voi… Vediamo Vita come faresti tu il problema?
339. **Vita: -** Non lo so…, non lo so, perché mi sa che sarà qualcosa …come il primo, sarà qualcosa di simile.
340. **Sperimentatore: -** Hai trovato un collegamento tra il secondo ed il primo quesito? Tu cosa ne pensi?

341. **Alessandra: -** Sì perché c'è solamente un'equazione del sistema come già ha detto la mia compagna, per cui per trovare la x e la y possiamo mettere anche infinite soluzioni per avere…
342. **Vita: -** Pero qua abbiamo il triplo e il quadruplo e qua abbiamo 6 e 3, 6 meno 3, quindi secondo me queste cose verranno simile da svolgere, poi non lo so… Perché c'è 18 e qua c'è 300 la somma complessiva, Carlo vince, per esempio, il triplo del denaro invece qua l'incognita x ha 6… La somma complessiva…
343. **Sperimentatore: -** E allora come possiamo fare?

Le alunne riprendono il PRIMO QUESITO

344. **Alessandra: -** 300 che è uguale a … Ah! perfetto, possiamo dare una cosa qualsiasi, se noi mettiamo, mettiamo il triplo del denaro, triplo 3…
345. **Sperimentatore: -** State parlando del primo quesito o del secondo?
346. **Alessandra: -** Del primo…
347. **Vita: -** Nella domanda ci mettiamo per esempio infinite soluzioni, però qua mettiamo per esempio infinite soluzioni, però mettiamo per esempio una, solo un esempio, non era così…
348. **Sperimentatore: -** Allora siete tornate al primo? Nel primo che cosa hai scritto?
349. **Alessandra: -** Un'equazione, un'uguaglianza, cioè la somma totale è di 300, però noi non sappiamo quant'è la x e quant'è la …
350. **Vita: -** Tu sei sicura di…
351. **Alessandra: -** Io non lo posso sapere, io sto facendo altre ipotesi, cioè anzi che avere una soluzione possibile ho un'equazione generale come viene data qua, in quanto 300 è la somma totale e mettiamo che 3 per non sapendo quanto ha scommesso e 4 per non sapendo quanto ha scommesso…
352. **Vita: -** Perché x e y forse erano diverse.
353. **Alessandra: -** Però…
354. **Vita: -** Infatti, perché x e y rappresentano forse il denaro scommesso, giusto? Il denaro scommesso…
355. **Alessandra: -** Se noi per esempio….
356. **Vita: -** Ma noi non lo possiamo sapere mai…
357. **Alessandra: -** Infatti, secondo me sono delle variabili, è tutto un'ipotesi anche questa, noi dobbiamo vedere quante sono, però io penso che se mettiamo, per esempio, che sono le possibili soluzioni…
358. **Sperimentatore: -** Quante sono le possibili soluzioni?
359. **Vita: -** Non sono certa, tranne 0 forse…
360. **Alessandra: -** No, perché?
361. **Vita: -** Perché può essere che non ha scommesso niente.
362. **Alessandra: -** Aspetta però… No però poi dice che quello ha scommesso il triplo, quindi dà per certo che hanno scommesso qualcosa…
363. **Vita: -** C'è una relazione tra questi due? Sono simili? *(il primo e il secondo quesito)*
364. **Sperimentatore: -** Si tratta di fare un problema per volta. Con quale quesito dobbiamo finire, con il primo?
365. **Vita: -** Con il primo
366. **Sperimentatore: -** Che cosa dovresti scrivere? Quante sono le possibili soluzioni? Scrivete quello che ne pensate.
367. **Alessandra: -** Dobbiamo calcolare un numero preciso? Sempre…
368. **Vita: -** Un numero preciso non c'è, ma possono essere tante soluzioni…
369. **Sperimentatore: -** Perché avete tanta paura di scrivere?
370. **Vita: -** Si sbagliamo…
371. **Sperimentatore: -** Non si tratta di giusto o sbagliato, io analizzo forme di pensiero non correggo per giusto o sbagliato…
372. **Vita: -** Infinite tranne 0 questo è sicuro, perché si hanno vinto qualcosa hanno scommesso… Per esempio, qui x e y sempre dobbiamo trovare, per esempio, il denaro scommesso…

373. **Alessandra: -** Si, però…
374. **Vita: -** Infinite soluzioni, secondo me sono, perché dipende per esempio, possiamo dare, per esempio, che Marco, per esempio…
375. **Alessandra: -** Ma infinite soluzioni… può essere anche che tu ci metti 6 per 300 e 6 per 300 è maggiore di 300, della somma totale…
376. **Vita: -** Deve essere maggiore di 0 e minore di 300?
377. **Alessandra: -** Qui invece dobbiamo mettere…, però se noi mettiamo la x poi vengono uguali questi due, giusto? Se noi mettiamo tipo, perché voglio fare tipo, com'è che stiamo facendo adesso, tipo quelli lì che vanno…, qui la variabile può andare da… sia maggiore di 0 sia minore di 300, non compreso 300, compreso 300…
378. **Vita: -** No aspetta, maggiore di 0 giusto? Ma deve essere minore di 300 non uguale, perché tu non puoi, per esempio, se qua dice…
379. **Alessandra: -** No, può essere che uno è 0 e l'altro è 300.
380. **Vita: -** Allora è compreso…
381. **Alessandra: -** No, no, no, 300 no, perché se poi noi mettiamo 300, il triplo di 300… 150
382. **Vita: -** Che 150? 150, la metà tu dici? Forse 150 per x…
383. **Alessandra: -** Forse…, qual è il quadruplo di 300? Qual è il quadruplo di 300?
384. **Vita: -** Sarebbe 300 diviso 4
385. **Alessandra: -** Il quadruplo…, 300 diviso 4 …
386. **Sperimentatore: -** 75
387. **Alessandra: -** 75
388. **Vita: -** Viene 75? Perché se è il triplo, il triplo di 300…
389. **Alessandra: -** Capisco questo
390. **Vita: -** Capito?
391. **Alessandra: -** Il triplo, ci sono tutte queste soluzioni.
392. **Sperimentatore: -** Allora scriviamo…
393. **Alessandra: -** No, però il triplo di 300?
394. **Sperimentatore: -** 900
395. **Vita:** Il triplo?
396. **Sperimentatore: -** Si
397. **Vita:** No, voglio dire…
398. **Sperimentatore: -** La terza parte, 100
399. **Alessandra: -** 100, quindi vanno da questo a 100.
400. **Sperimentatore: -** x va da 0 a?
401. **Alessandra: -** 100
402. **Vita: -** Perché è triplo…
403. **Alessandra: -** Volevo fare tutte le possibili soluzioni. Perché se non ci fosse qua può andare al massimo 75 e 75 già è compreso. Il massimo che si può scommettere è 100 per arrivare ad un totale del triplo di 300
404. **Vita: -** Si
405. **Alessandra: -** Quindi…
406. **Sperimentatore: -** Allora scrivi, le possibili soluzioni sono…
(*Scrivono* $0 \leq x \leq 100$).
407. **Sperimentatore: -** Allora tornando al secondo quesito potete scrivere il problema o no?
408. **Vita: -** Se non è una specie così …
409. **Alessandra: -** Perché è compreso 0 ed è compreso 100.

Le alunne riprendono il SECONDO QUESITO

410. **Vita: -** Se facciamo, per esempio, il secondo sarà pure così… Dobbiamo trovare, però qui ci sarà sicuramente una soluzione, queste forse vogliono tutte le soluzioni possibili invece qua ci dice una sola soluzione…
411. **Alessandra: -** Soluzione problema, non soluzione della…, soluzione problema. (*Risponde: "soluzione" per "situazione"*).
412. **Vita: -** Si.

413. **Alessandra: -** Problema, quindi…
414. **Vita: -** Non possiamo fare, per esempio, scriviamo il testo del secondo…
415. **Alessandra: -** Non va bene, mettiamo il numero due.
416. **Vita: -** No il testo, il testo lo dobbiamo dare noi.
417. **Alessandra: -** Sì ma soltanto questo dobbiamo scrivere, questo è il problema.
418. **Vita: -** Infatti.
419. **Alessandra: -** Questa è la cosa…
420. **Vita: -** Se mettiamo, per esempio, un'altra cosa… Le possibili soluzioni di questo funziona.
421. **Sperimentatore: -** Di quale quesito state parlando, del primo o del secondo?
422. **Alessandra: -** Del secondo, sì ma qua è il problema…
423. **Sperimentatore: -** Il secondo non è necessario risolverlo…
424. **Alessandra: -** Dobbiamo fare un problema, un problema…
425. **Vita: -** Simile a quello di prima facciamo. Per esempio, Carlo e Lucia hanno 18 mele, per esempio, perché quello è complessivo…
426. **Alessandra: -** Carlo e …
427. **Vita: -** No, prima lo possiamo vedere, prima di scrivere
428. **Alessandra: -** Carlo e Lucia…
429. **Vita: -** No
430. **Alessandra: -** Allora al mercato ci sono 18 mele, il negoziante le mette 6, quelle le mette un tot di mele di queste 18, le mette, cioè le vende, l'altro decide di metterle di riserva, che ne so… Il negoziante arriva e ne prende 3. Quante devono essere le mele totale che ci sono tra quelle che si è preso il negoziante e …
431. **Vita: -** Va bene, mancano di 3 e quelle che ne so, che ha preso suo cugino…
432. **Alessandra: -** Sì ma no, 6 quelle di là…
433. **Vita: -** Sì ma se uno ne prende 13, ne restano 18? *(Silenzio)*
434. **Sperimentatore: -** Vita, secondo te, come lo faresti?
435. **Vita: -** Simile a questo lo farei…
436. **Sperimentatore: -** E come lo faresti?
437. **Vita: -** Per esempio, vincono 18 € e mettiamo che Carlo ha vinto, per esempio, il triplo…
438. **Alessandra: -** No, vince 6 € del denaro scommesso, perché qui noi sappiamo che il triplo e il quadruplo, pero qua noi abbiamo 6 e 3.
439. **Vita: -** Pero questo meno 3 sarà qualche cosa? C'è qualche cosa con questo meno 3?
440. **Alessandra: -** Certo questi non ci sono più, qua invece sono…, quant'è il denaro totale quindi quello più quello.
441. **Vita: -** Può essere che si sottraggono, quelli lì hanno in tutto, hanno…
442. **Alessandra: -** Ah!…, scommesso…, allora…
443. **Vita: -** …qua vincono…
444. **Alessandra: -** …qua perdono…
445. **Vita: -** Forse…, può essere?
446. **Alessandra: -** Allora, Carlo e Lucia perdono la somma complessiva di 18 €, sappiamo che Carlo ne perde, ne…
447. **Vita: -** …ne perde, per esempio, uno ne perde il triplo, l'altro ne perde 6, come si dice per esempio 6?
448. **Sperimentatore: -** Sestuplo
449. **Vita: -** Il sestuplo
450. **Alessandra: -** E 18 €… Allora, Carlo perde 6 € del denaro scommesso, mentre Lucia…
451. **Vita: -** Il denaro scommesso è questo x. x e y sono le incognite di quanto hanno scommesso. Noi abbiamo 3 che sarebbe forse il triplo, uno ne perde il triplo, uno ne vince…
452. **Alessandra: -** Sì ma alla fine deve essere 18 e non ho capito perché… Se Carlo…
453. **Vita: -** Forse perché… Se noi, per esempio, alla y forse…, ci sarebbe per esempio che Lucia metta soldi forse, poi ne toglie il triplo…
454. **Alessandra: -** Allora Carlo e Lucia vincono la somma complessiva di 18 €. Sappiamo che Carlo ne vince 6 € del denaro scommesso, mentre Lucia ne perde il triplo…
455. **Vita: -** Se giocano insieme com'è che uno vince e uno perde?

456. **Sperimentatore: -** Allora, come finisce?
457. **Vita: -** Non si può fare…
458. **Sperimentatore: -** Non siete molto lontane…
459. **Alessandra: -** C'era l'ipotesi di poco fa, quella quando dicevo "vince 6 €..."
460. **Vita: -** Non è 6 €, perché il 6 è sestuplo …
461. **Alessandra: -** Ah! Si…
462. **Vita: -** …perché x e y sono le incognite che rappresentano il denaro scommesso. Il sestuplo, per esempio, ne vince il sestuplo…
463. **Alessandra: -** 6 €…
464. **Vita: -** Se fosse meno ne perde…
465. **Alessandra: -** Appunto e quindi non possono giocare assieme, ma però la somma complessiva…
466. **Sperimentatore: -** Possono giocare in tavoli diversi, o no?
467. **Vita: -** Fare un problema è più difficile di svolgerlo, perché poi deve venire sempre questa equazione…
468. **Alessandra: -** E' lo stesso, è lo stesso…
469. **Sperimentatore: -** E allora?
470. **Alessandra: -** Carlo e Lucia perdono al lotto la somma complessiva di 300 €. Sappiamo che Carlo ne perde il triplo, ne perde 6 € del denaro scommesso mentre Lucia ne perde 3 € del denaro scommesso…
471. **Vita: -** Sì perché forse, per esempio, Lucia ne ha messo forse di più soldi anche se hanno vinto forse ci verrebbe una perdita
472. **Alessandra: -** Carlo e Lucia perdono al lotto (*ripete mentre scrive*).
473. **Vita: -** No, mettiamo, per esempio, hanno 18 € Carlo e Lucia giusto? Uno…
474. **Alessandra: -** Com'è che nessuno vince? Aspetta, la somma complessiva di 18 €, giusto? (*ripete mentre scrive*).
475. **Vita: -** Carlo perde 6 € del denaro scommesso, quindi loro 6 € del denaro che hanno scommesso. Può essere che hanno scommesso di più? Tipo che hanno perso ancora di meno, cioè non hanno perso tutti i soldi, può essere che di quello che hanno scommesso ne hanno perso una parte.
476. **Sperimentatore: -** E allora? Siete d'accordo o no? *(Silenzio).*
Allora, qua c'è una differenza, che cosa indica questa differenza?
477. **Alessandra: -** Una sottrazione abbiamo…
478. **Sperimentatore: -** Si
479. **Vita: -** Forse può essere che uno vince e l'altro perde…
480. **Alessandra: -** Infatti. Sì ma non può essere che uno vince sempre
481. **Vita: -** Se noi mettiamo, per esempio, o che perdono o che hanno 18 € complessivamente. Come possiamo dire, per esempio, che uno perde e uno vince?
482. **Sperimentatore: -** Forse uno ha fato una giocata e l'altro ha fatto un'altra…
483. **Vita: -** E quindi forse Carlo ha 18 €, giusto? e Lucia…
484. **Alessandra: -** Carlo ne ha perso 6 e…
485. **Vita: -** No, per esempio, Carlo ne ha vinto un sesto della somma che abbia messo e, invece, Lucia forse ne ha perso il triplo…
486. **Sperimentatore: -** E allora?
487. **Alessandra: -** Mettiamo che siano questioni…
488. **Vita: -** Ma sono questioni diverse
489. **Alessandra: -** Ma si questi due sono in comunità, hanno dei soldi in comune però giocano…
490. **Vita: -** Non può essere che hanno soldi in comune, non può essere che uno vince e uno perde si sono in comunità
491. **Alessandra: -** Sono questioni diverse, hanno soldi in comune, però fanno due giocate diverse, uno vince e l'altro perde…
492. **Vita: -** Se loro mettono i soldi in comune giocano assieme, perché devono giocare in tavoli diversi? Forse sono due soli, per esempio, la somma complessiva forse è di 18 €, uno vince il sesto, invece quello ne perde il triplo. Sono diversi?
493. **Alessandra: -** Secondo me, si

494. **Sperimentatore: -** E allora, concludendo che cosa dovete scrivere?
495. **Alessandra: -** Carlo e Lucia giocano al lotto la somma complessiva di 18 €.
496. **Vita: -** No la somma complessiva, non sono assieme, secondo me, questi, perché non può essere che uno perde e uno vince la somma complessiva di 18 €.
497. **Alessandra: -** Carlo e Lucia hanno, cioè ognuno ha 18 €, giusto?
498. **Vita:** - Si
499. **Alessandra: -** Deve fare in modo che scommettendo 6 €, Lucia… 6 per 3 è 18…
500. **Vita:** - Non possiamo, per esempio, mettere invece somme diverse. Carlo e Lucia, la somma complessiva di 18 €
501. **Alessandra: -** Non può essere…
502. **Vita: -** Se c'era più se poteva fare che vincevano assieme 18 €, ma come c'è meno, c'è un meno, forse si dovrebbe mettere…
503. **Sperimentatore: -** Andate troppo nei dettagli delle cose, non interessa se si sono implicati gli stessi numeri o diversi, state inventando un problema…
504. **Vita:** - Facciamo quello che ci ho detto io, scrivi…
505. **Alessandra: -** La somma complessiva di 18 €. Sappiamo che Carlo perde 6 € del denaro scommesso perché poi c'è la sottrazione, vince l'altro perde…
506. **Vita: -** Come li vince questi soldi? Perché è la somma complessiva e perde?
507. **Alessandra: -** In un primo momento vincono 6 € ognuno
508. **Vita: -** Non c'è primo momento, complessivamente devi fare, non in un primo momento
509. **Alessandra:** - Sì ma nella serata è, nell'arco della serata, e poi alla fine ti ritrovi quello, giusto? In un primo momento vincono 6 € del denaro scommesso, poi perdono 3 €
510. **Vita:** - Forse sarebbe che, per esempio, Carlo non mette, facciamo quello, per esempio, che non mette proprio nessun euro, capisci? Però come si scrive?
511. **Alessandra:** - Determina le somme di denaro…
512. **Vita:** - Vincono complessivamente, hanno vinto complessivamente 18 €, Carlo, per esempio, ha vinto, per esempio, il sesto della somma che aveva messo, invece, Luigi, per esempio, meno 3, per esempio, y dovrebbe essere meno 3, che hanno vinto…
513. **Alessandra: -** Allora guarda com'è: Carlo e Lucia vincono al lotto la somma complessiva di 18 € . In un primo momento vincono 6 € del denaro scommesso, poi ne perdono 3 € sempre del denaro scommesso tra parentesi, giusto? In un secondo momento ne perdono 3 €. Determina le somme di denaro che Carlo e Lucia hanno giocato. E poi: Quante sono le possibili soluzioni? (*Riscrivono il testo*).
514. **Vita: -** Non sono 3 €, è il sesto ormai e quello è il triplo
515. **Sperimentatore:** - Allora scrivete
516. **Alessandra:** - *(Si correggono)* Sei volte del denaro scommesso e tre volte
517. **Sperimentatore: -** Va bene
518. **Vita: -** Ma non è così, per tre volte, aspetta …
519. **Sperimentatore: -** E poi…
520. **Vita: -** No, niente…
521. **Alessandra: -** Dobbiamo mettere "determina le somme di denaro", perché poi fa sempre lì, non è che posso fare, queste vanno da questo a questo, perché questi qua poi come fanno?
522. **Sperimentatore: -** E allora…
523. **Alessandra: -** Ah! Forse, per esempio, e va bene, può andare determina: "Determina…
524. **Vita:** - No, mettiamo: "Quante possibili soluzioni?"
525. **Alessandra: -** …le somme di denaro che Carlo e Lucia hanno giocato"
526. **Vita:** - E noi non le sappiamo determinare…, ma le possibili soluzioni… qualunque sono, quante sono le possibili soluzioni…
527. **Alessandra: -** Può essere che sono infinite, che non c'è una somma di denaro ben precisa
528. **Vita: -** E va bene, scriviamo…
529. **Sperimentatore:** - Allora, andiamo al numero tre.

THIRD QUERY

530. **Alessandra: -** Equazione di primo grado con due incognite e quella, invece, un'equazione della retta oppure del fascio…

531. **Sperimentatore: -** Dovete dire la prima idea che vi viene in mente quando vedete le equazioni…
532. **Alessandra: -** Equazione di primo grado con due incognite *(per la prima)*.
533. **Vita: -** Oppure può essere, aspetta… No così, potrebbe essere una parabola, una parabola no, non c'entra…
534. **Alessandra:** - … è al quadrato…
535. **Vita:** - Poi, circonferenza non è...
536. **Alessandra: -** Dobbiamo, per esempio, spiegare è una curva…?
537. **Vita: -** Intanto scriviamo equazione di primo grado con due incognite.
538. **Sperimentatore: -** Quando vedete le equazioni dovete dire la prima idea che vi viene in mente …
539. **Alessandra:** - Questa è un fascio di rette proprio *(per la seconda)*.
540. **Vita:** - Si. Aspetta…
541. **Sperimentatore:** - Sì o no?
542. **Alessandra:** - Sta rappresentando la...
543. **Vita:** - Sì ma anche qua…
544. **Alessandra: -** Va bene tu hai detto che ha i numeri, come mx + n, questo qua, questo è…
545. **Vita:** - Proprio
546. **Sperimentatore:** - Allora andiamo al numero 4
547. **Vita: -** Ancora un problema?
548. **Sperimentatore:** - Si.

FOUR QUERY

549. **Alessandra: -** Una persona per poter usufruire di un telefono fisso installato presso di un'altra, pattuisce con questa ultima di pagare mensilmente 5 € più 2 € per ora di telefonate effettuate. Detto: x il numero di ore mensili di telefonate effettuate e y la somma complessiva pagata mensilmente. Una persona per poter usufruire di un telefono fisso installato presso di un'altra, pattuisce con questa ultima di pagare mensilmente 5 € più 2 € per ora di telefonate effettuate. (*Legge e rilegge il testo del problema*).
550. **Vita:** - 5 € ce li dà e 2 € all'ora.
551. **Alessandra: -** Una persona per poter usufruire di un telefono fisso installato presso di un'altra, pattuisce con questa ultima di pagare mensilmente 5 €. Quindi... *(Rilegge il testo del problema*).
552. **Vita: -** Si, spiega quale tipo di relazione intercorre tra x e y. Sia per esempio che x… Allora, x…
553. **Alessandra:** - x è il numero delle ore da fare. Prima dobbiamo fare x
554. **Vita: -** Mentre y
555. **Alessandra: -** Allora, x è il numero delle ore mensili telefonate effettuate
556. **Vita:** - Forse questa cosa non si può fare
557. **Alessandra: -** y è la somma complessiva pagata mensilmente. Intanto, c'è una riga per disegnare?
558. **Vita:** - Va bene, niente fa.
559. **Sperimentatore:** - A mano, a mano. (*Disegnano il diagramma cartesiano*)
560. **Alessandra:** - x, che rappresenta x?
561. **Vita:** - Ore mensili di telefonate, giusto?
562. **Alessandra:** - Si.
563. **Vita:** - E y…
564. **Alessandra: -** Quindi, y la somma complessiva…
565. **Vita: -** La somma complessiva, metti ore più 5 €, che sarebbe la somma complessiva ore più…
566. **Alessandra:** - Si ore più 5 €
567. **Vita: -** La somma complessiva sarebbe 5 € e 0
568. **Alessandra: -** No, 0 delle ore di telefonate, 1 e 2
569. **Vita:** - Aspetta, allora… E x…, va avanti… *(Rilegge il testo del problema)*

570. **Alessandra: -** Più 5 €…
571. **Vita: -** Più 2 € per le ore di telefonate
572. **Alessandra: -** 1, 2, 3, 4 e 5
573. **Vita: -** Perché 5, tu devi sommare 5 più l'euro
574. **Alessandra: -** y è la somma complessiva pagata mensilmente
575. **Vita: -** E y è la somma complessiva, giusto?
576. **Alessandra: -** E' 5 €
577. **Vita: -** No, non è 5 € la somma complessiva, la somma complessiva sarebbe 5 € più 2 €, 2 €per ora. Se un'ora sarebbe 7 € sarebbe…
578. **Alessandra: -** Per 2 ore
579. **Vita: -** Per esempio se…, allora pattuisce con questo e paga 5 €, 5 €più 2 €sarebbe 7 €, 7 € per esempio, per un'ora. Giusto? Hai capito?
580. **Alessandra: -** Si. Quindi, ho sbagliato il grafico, lo faccio di nuovo qua
581. **Sperimentatore: -** Va bene, non ti preoccupare
582. **Vita: -** Per esempio, 7 €…
583. **Alessandra: -** … e qua c'è uno… Qua è il primo punto, intanto…
584. **Vita: -** Hai fatto il 7?
585. **Alessandra: -** Si
586. **Vita: -** Il 7 è un'ora
587. **Alessandra: -** Si
588. **Vita: -** Forse con il grafico, forse con il grafico…
589. **Alessandra: -** Si
590. **Vita: -** Non si potrebbe fare in altro modo, senza grafico? (*Sorridono*).
591. **Alessandra: -** Ricominciamo?
592. **Vita: -** Secondo me, per esempio a 7 ⟶ 14
593. **Alessandra: -** Al 7 ⟶ 1
594. **Vita: -** Ah! si, al 7 ⟶ 1
595. **Alessandra: -** E questo è il primo punto.
596. **Vita: -** Mettici a 7 ⟶ 1, però.
597. **Alessandra: -** Aspetta, aspetta 1, 2, 3, 4, 5, 6, 7. A 7 ⟶ 1, è qui.
598. **Vita: -** Poi metti 2
599. **Alessandra: -** 8, 9, …., 14.
600. **Vita: -** Scrivi 14
601. **Alessandra: -** Allora dobbiamo fare mensilmente, però…
602. **Sperimentatore: -** A 1 ⟶ 7, a 2?
603. **Alessandra: -** A 1 ⟶ 7, a 2 ⟶ 14
604. **Sperimentatore: -** 14. E come avete trovato il 14?
605. **Alessandra: -** Perché, allora se x è uguale a 1, se noi mettiamo quindi un'ora quindi possiamo vedere appunto che si paga…
606. **Vita: -** Un'ora è 7, no 2 €, se poi dobbiamo sommare 5 €che ci paga, sono 7 €.
607. **Sperimentatore: -** Sono 7 e poi per due ore?
608. **Vita: -** Per due ore, sarebbe il doppio, 14.
609. **Alessandra: -** Aspetta non ho capito. Ah! forse proprio così è.
610. **Vita: -** Mensilmente 5 €, non è ad esempio a ora. Mensilmente 5 € più 2 ore.
611. **Alessandra: -** Mensilmente quindi sono 7 €in tutto.
612. **Sperimentatore: -** Per un'ora.
613. **Alessandra: -** Per un'ora….
614. **Sperimentatore: -** Per due ore?
615. **Vita: -** Se un'ora è di 7, giusto? Per un'ora paghi 7 €, per due ore ne paghi 14, no?
616. **Alessandra: -** 5 più due…
617. **Vita: -** Sai come forse è? Tu paghi, per esempio, 2 €all'ora, in due ore paghi, per esempio, 4 €, poi mensilmente sarebbe, per esempio, per 30
618. **Alessandra: -** 30…
619. **Vita: -** La somma complessiva è, per esempio 30, per 30 più 5 € perché è mensilmente. Allora…
620. **Alessandra: -** Aspetta, facciamo così, intanto svolgiamolo

621. **Vita: -** Questo è sbagliato…, perché non è detto mensilmente.
622. **Alessandra: -** Facciamo di nuovo il grafico?
623. **Sperimentatore: -** Se lo volete fare. Comunque, potresti lasciarlo così…
624. **Alessandra: -** Allora…
625. **Vita: -** Prima si sbaglia e poi
626. **Alessandra: -** Sì ma dobbiamo fare…
627. **Vita: -** Entro un'ora
628. **Alessandra: -** …ma non ci va qui…, se noi facciamo 30, per 30
629. **Vita: -** Va bene lo mettiamo, per esempio, ci mettiamo 30 non è che deve andare per forza 30, ci lo metti tu 30, lo devi scrivere, niente fa.
630. **Alessandra: -** Mettiamo, ore…
631. **Vita: -** 30
632. **Alessandra: -** Ore, ore di telefonate, giusto? y è la somma complessiva, giusto?
633. **Vita: -** Metti per esempio un'ora paghi 2 €, 2 ore paghi 4 €
634. **Alessandra: -** 1 uguale a 2
635. **Vita:** Qua ci devi mettere uno
636. **Alessandra: -** Uno, vedi un euro
637. **Vita: -** No, 2 €. In un'ora tu paghi 2 €qua è scritto… Allora, più 2 €per ora
638. **Alessandra: -** 2 €per ora, giusto è 1 €e 2
639. **Vita: -** Giusto, giusto. Un'ora 2 €, per 2 ore
640. **Alessandra: -** Paghi 4 €
641. **Vita: -** Si. *(Silenzio).* In due ore paghi 4 €. Poi mensilmente che sarebbe 30, la somma complessiva pagata mensilmente e il numero di ore mensili
642. **Alessandra: -** Allora…
643. **Vita: -** Il numero di ore mensili.
644. **Alessandra: -** …che paghiamo per mese.
645. **Vita: -** 30, 30 giorni, per esempio
646. **Alessandra: -** Per ore, 30 ore
647. **Vita: -** Si, 30 ore che, per esempio, sarebbero 60, 60 più 5 fa 65 questo. Giusto?
648. **Alessandra: -** Non mi convince
649. **Vita: -** Questo giusto è
650. **Alessandra: -** Se dice: "Determina la somma complessiva che viene pagata mensilmente", un mese è fatto da 30 giorni come potrebbe essere fatto di 31 oppure di 28.
651. **Sperimentatore: -** Si
652. **Vita: -** E noi dobbiamo prendere ad esempio, prendiamo ad esempio come riferimento…
653. **Alessandra: -** Facciamo 28, facciamo 31 o 28 di Febbraio.
654. **Sperimentatore: -** E quante ore potresti parlare al mese?
655. **Vita: -** E' variabile
656. **Alessandra: -** Dipende dai giorni la cosa
657. **Vita: -** No, aspetta se il giorno ha 24 ore, giusto?
658. **Alessandra: -** Sì ma tu che fai 24 ore giusto? 12 ore sto a … In un giorno di 24 ore parli 12 ore al telefono, come fai?
659. **Vita: -** No, se tu parli per esempio, aspetta…
660. **Alessandra: -** E' variabile.
661. **Vita: -** Ma con questo "variabile"…
662. **Alessandra: -** E' tutto, è tutto variabile…
663. **Vita: -** Problemi variabili fecero.
664. **Alessandra: -** Proprio qua è il problema. *(Ridono).* Perché non possiamo sapere quanto possiamo parlare in un'ora c'è quando non telefoniamo proprio, c'è quando telefoniamo per…, per 5 ore, giusto?
665. **Sperimentatore: -** Si, e allora cosa dite?
666. **Vita: -** Determina la somma complessiva pagata mensilmente. *(Legge il testo).*
667. **Alessandra: -** Quante sono le possibili soluzioni? *(Ridono)*
668. **Vita: -** E' quindi si deve parlare…
669. **Alessandra: -** Sono della probabilità questi problemi?
670. **Sperimentatore: -** No, poi vi spiego cosa sono…

671. **Vita: -** Questo è giusto, però…
672. **Alessandra: -** Vicine ci siamo? Vicine ci siamo?
673. **Sperimentatore: -** Allora…
674. **Vita: -** Stabilire quale tipo di relazione intercorre tra x e y. *(Legge il testo)*
675. **Alessandra: -** Appunto questo qua di nuovo, questo qua…
676. **Vita: -** Non è valido questo.
677. **Alessandra: -** …relazione tra x e y…
678. **Vita: -** …determina la somma complessiva pagata mensilmente…, la somma complessiva…*(Legge il testo)*
679. **Alessandra: -** Però non sappiamo quante ore al telefono ci sta…
680. **Vita: -** …e il numero delle ore mensili… di telefonate effettuate.
681. **Alessandra: -** Quante sono le possibili soluzioni? *(Continuano a leggere il testo).*
682. **Sperimentatore: -** E allora, che cosa chiede all'inizio?
683. **Alessandra: -** Rappresentarla graficamente nel piano cartesiano
684. **Vita: -** Stabilire il tipo di relazione.
685. **Sperimentatore: -** Graficamente che cosa verrebbe?
686. **Alessandra: -** Sarebbe il numero delle ore.
687. **Sperimentatore: -** Provate a fare un grafico di questo…
688. **Alessandra: -** … e quindi , ci sono 3, giusto? Qui è 9. *(Silenzio).* Qua continuerà il grafico sarà sempre elevato, continuerà sempre durante il mese a diventare sempre più…
689. **Vita: -** Quindi, va bene aumenterà la somma…
690. **Alessandra: -** Se aumenta il mese aumenta la somma…
691. **Vita: -** Aumenta la somma e qua mettici 5 € mensilmente.
692. **Sperimentatore: -** E che cosa è il grafico?
693. **Alessandra: -** Una retta
694. **Sperimentatore: -** Fate il grafico
695. **Vita: -** Forse sarebbe che…, la somma complessiva pagata mensilmente…, 30 giorni .
696. **Alessandra: -** Dobbiamo fare adesso per 30
697. **Vita: -** Un giorno è formato da 24 ore
698. **Alessandra: -** Si, ma non sappiamo quanto può stare una persona al telefono.
699. **Vita: -** Se tu per esempio 2 ore, 3 ore… se tu, per esempio, metti 24 ore che è un giorno… Per esempio, mettiamo…, però ma non sappiamo se è giusto 30 € al mese, poi alla somma tu aggiungi più 5 € perché mensilmente si deve aggiungere 5 €, perché deve essere la somma complessiva… Però da quanto è questo mese?
700. **Sperimentatore: -** Scegliete voi
701. **Vita: -** Ah! possiamo dare noi i giorni al mese?
702. **Sperimentatore: -** Si
703. **Vita: -** Mettiamo a caso che il mese è di 28 giorni, giusto? Se un giorno è, per esempio, di 24
704. **Alessandra: -** Allora, 30 è di più, mettiamo uno di 30
705. **Vita: -** 30 ad esempio, 30 giusto? 30 giorni, un giorno è di 24 ore
706. **Alessandra: -** Ogni giorno 24 ore. Stai facendo una cosa, tu…
707. **Vita: -** Si ma ti dico in un giorno 24 ore, giusto? E sarebbe che noi, per esempio, mettiamo nel grafico qua 24 ore, mettiamo a caso che sia 24 ore… 24 ore…
708. **Alessandra: -** E' sbagliato. Tu che fai per 30 giorni e vai a mettere 24 ore…
709. **Vita: -** Per ogni ora è forse, no?
710. **Sperimentatore: -** Il problema considera ogni ora, non è specificato quante sono le ore
711. **Alessandra: -** Infatti, non è specificato, capito?
712. **Sperimentatore: -** Dovresti trovare il valore massimo
713. **Vita: -** Si, tutte le ore vorrei calcolare, giusto? Per esempio 24 ore se calcoliamo quanto è la somma complessiva.
714. **Alessandra: -** E' 30 giorni
715. **Vita: -** E' sarebbe la somma complessiva è, per esempio, ogni giorno, poi la facciamo per 24, la facciamo per 30 giorni e aggiungiamo più 5 €
716. **Sperimentatore: -** E questo sarebbe la somma…
717. **Vita: -** Per esempio, per esempio…

718. **Alessandra: -** Per ogni mese
719. **Vita: -** Si, per esempio, prima lo facciamo per 24 ore che sarebbe un giorno.
720. **Alessandra: -** Quindi è sempre come quella che…
721. **Vita: -** Poi dobbiamo fare, per esempio sì come, per esempio, questo… sappiamo che…
722. **Alessandra: -** Allora, dobbiamo dire che si uno è fatto da 2 ore 2 allora 24, no?
723. **Vita: -** Ci vuole un grafico?
724. **Sperimentatore: -** Il grafico è orientativo, per poi rispondere alle altre domande
725. **Alessandra: -** Allora, intanto dobbiamo vedere per 48. *(Fanno i calcoli).*
726. **Vita: -** 48 € per 24, 48 € e questo sarebbe un giorno.
727. **Alessandra: -** Si
728. **Vita: -** Un giorno, per esempio se tu metti…
729. **Alessandra: -** 30
730. **Vita: -** Se tu ora fai, metti per esempio, questo sono le ore, 24 ore questo è messo. Se tu, per esempio, fai un altro metti in un grafico poi, metti 30 giorni, giusto? Se tu, per esempio, metti qua… questo è per ore il grafico. Se tu per esempio prendi 30 giorni devi mettere, per esempio, 48 € che sarebbe per un giorno, 48 per 30 più 5 €
731. **Alessandra: -** Ma, secondo me… il massimo è di 24 ore.
732. **Vita: -** Ma, così non è giusto per esempio perché…
733. **Alessandra: -** 24 per 30. Dobbiamo calcolare per 30 giorni quante ore ci sono?
734. **Sperimentatore: -** 24 per 30.
735. **Alessandra: -** No, aspetta se 24 ore è in giorno di quante ore?
736. **Vita: -** Di quante ore è formato il mese?
737. **Sperimentatore: -** 720
738. **Alessandra: -** 320
739. **Sperimentatore: -** 720
740. **Vita: -** 720, poi sarebbe
741. **Alessandra: -** 720 l'ora
742. **Vita: -** No, 720 ore ha detto che è un mese.
743. **Alessandra: -** Un mese è 720 ore
744. **Sperimentatore: -** 30 per 24 è uguale a 720.
745. **Vita: -** 720, metti un attimo 720
746. **Alessandra: -** Facciamo che è qui, giusto?
747. **Vita: -** 720, giusto? Se 24 era 48 €…
748. **Alessandra: -** 720, dobbiamo vedere a quanto…?
749. **Vita: -** …per 2 più 5 €…
750. **Alessandra: -** Perché per 2?
751. **Vita: -** Perché era 2 €, la somma…
752. **Alessandra: -** All'ora, all'ora…
753. **Vita: -** No, Alessandra perché…
754. **Alessandra: -** 720
755. **Vita: -** Come abbiamo fatto, sempre così abbiamo fatto, qua veniva due, due, 24 per 2 e 720 per 2 più 5 €, no?
756. **Alessandra: -** Più 5 €
757. **Vita: -** Più 5 €. Sarebbe 1440 più 5 €. 1445 €
758. **Alessandra: -** Può essere?
759. **Sperimentatore: -** E questo, cosa sarebbe?
760. **Vita: -** Questo sarebbe la somma complessiva di un mese.
761. **Sperimentatore: -** Se parla al telefono…
762. **Alessandra: -** 24 ore, ma non è sempre così.
763. **Vita: -** Se parla 24 ore su 24 è questa la somma complessiva.
764. **Sperimentatore: -** Sarebbe la somma…?
765. **Vita: -** La somma di un mese
766. **Alessandra: -** Quindi sempre con i limiti si può fare non essendo…
767. **Vita: -** Intanto questo lo metti nel grafico, questo dato e poi se, per esempio, parla 24 ore su.
768. **Alessandra: -** Mettiamo qua sicuro dovrebbe venire qua sopra, non possiamo disegnarlo.

769. **Sperimentatore: -** Il grafico è orientativo per poi rispondere alle domande.
770. **Vita:** - Poi sarebbe… determina la somma complessiva pagata ed il numero delle ore mensili…
771. **Sperimentatore: -** E allora?
772. **Vita:** - Le possibili soluzioni sono, dipende sempre dipende da quante ore parla al telefono.
773. **Alessandra: -** E non si sa questo
774. **Vita: -** Qua abbiamo messo 24 ore su 24, Alessandra, questa somma…
775. **Alessandra: -** E quindi dobbiamo mettere sempre là, quella che ne parlano, 24 e dobbiamo mettere quindi tutte le altre…
776. **Sperimentatore: -** Scrivete allora quello che avete detto.
777. **Vita: -** Queste soluzioni prima…
778. **Alessandra: -** Che possono andare per…
779. **Vita: -** Per x sarebbe il numero delle ore, giusto? *(Silenzio)*. Se sono, per esempio, 24 ore che sarebbe un giorno…
780. **Sperimentatore: -** Scrivete quello che avete detto.
781. **Alessandra: -** Non sappiamo se è giusto.
782. **Vita: -** Ma è giusto questa cosa così?
783. **Sperimentatore: -** Allora, scrivete la vostra conclusione.
784. **Alessandra: -** Sempre come quella che abbiamo fatto?
785. **Sperimentatore: -** Sì quello che avete detto alla fine, scrivetelo così resta qualcosa scritta.
786. **Alessandra: -** Scriviamo…
787. **Vita: -** Scriviamo questo così? Quest’ultima cosa?
788. **Sperimentatore: -** Si
789. **Alessandra: -** Allora, per quindi, per un mese…
790. **Vita: -** …la somma complessiva…
791. **Alessandra: -** No totale 720 ore le pagano 1445 €
(Scrivono: *“Per un mese (totale 720 ore) si pagano 1445* €”).
792. **Vita: -** Per un mese…
793. **Alessandra: -** Per un’ora…
794. **Vita: -** Scrivi questa cosa…
795. **Alessandra: -** Va bene si capisci… Il totale di ore per un’ora si paga…
796. **Vita: -** Si paga….
797. **Alessandra: -** 7 €
798. **Vita: -** No, non è 7 €, aspetta… è sbagliato 7 €.
799. **Alessandra: -** Perché?
800. **Vita: -** Per un’ora è 2 €, perché 7 €?
801. **Alessandra: -** 2 €più però 5
802. **Vita: -** No, no perché 5 € vengono messe al mese, no? Mensilmente 5 €paga Alessandra, in un’ora paga 2 € poi arrivando ad un mese
803. **Alessandra: -** Quindi non sono 7 €
804. **Vita: -** Paga 5 €
805. **Alessandra: -** Si. Sempre l’incognita c’è
806. **Vita: -** Si è normale… *(Silenzio)*.
807. **Alessandra: -** No, bisogna vedere…
808. **Vita: -** No per x, per x per esempio un’ora y … così è. Perché x rappresenta le ore e y rappresenta la somma.
809. **Sperimentatore: -** Si
810. **Vita: -** Se tu metti, per esempio, per x 1 ora metti y è uguale…
811. **Alessandra: -** Per x questa cosa…, uguale… Non è che c’è per x
812. **Vita: -** Per, per, scrivi, per x un’ora, mettici y uguale a 2 ore
813. **Alessandra: -** 2 €, appunto
814. **Vita: -** Per y, per x giusto, uguale a 2 ore y uguale a 4 €
815. **Alessandra: -** Per x
816. **Vita: -** Per x, giusto, 2 ore, uguale a 2 ore, y uguale a 4 €.
817. **Alessandra: -** Per x uguale a…

818. **Vita: -** 720 ore y è 1445 €.
819. **Alessandra: -** E che relazione c'è ora tra…?
820. **Vita: -** E' il doppio sempre
821. **Alessandra: -** Compreso
822. **Vita: -** Aspetta un attimo…
823. **Sperimentatore: -** Allora, cosa stai calcolando?
824. **Alessandra: -** Se sbaglio
825. **Sperimentatore: -** Allora, che cosa calcoli?
826. **Vita: -** Di sicuro che questo è così?
827. **Alessandra: -** Non lo so
828. **Vita: -** Facciamo di nuovo questo calcolo
829. **Sperimentatore: -** E allora abbiamo finito?
830. **Alessandra: -** Per y mentre x *(Scrivono 2 € ≤ y ≤ 1445 €, 1 ora ≤ x ≤ 720 ore).*
831. **Sperimentatore: -** Va bene…

CHAPTER FIVE: FINAL CONCLUSIONS

The historical analysis carried out in the first chapter on the construction of the algebraic language allowed us to highlight the principal conceptions, the preceding procedures, the passages from one concept to the other and, particularly, the passages through the linguistic levels of the different phases: rhetoric, syncopated and symbolic. Beginning from this study we drew some conclusions applicable to the study of the epistemological obstacles that the pupils meet in the situations of learning the algebraic language.

The conclusions that we think are important for the realization of the experimental work are the following: the passages from rhetorical algebra to symbolic algebra are very slow: from certain *names* denoting the unknown and certain relations, to the *abbreviations of these words*, to the *intermediary codes* between rhetorical language and syncopated one and finally to the *symbols*. In the process of elaboration of a syntactically correct and operationally efficient algebraic symbolism, the progressive abandonment of the natural language as mediator of expression is observed. In the syncopated phase the natural, arithmetic and geometric languages are used as support to the algebraic language in the process of elaboration. These languages –semantically richer than the algebraic one– allow to interpret adequately the problems to solve, to obtain its solution, to formulate the rules and to justify the passages effected algebraically. In fact, precisely in the construction of the algebraic language two levels of conceiving the generality of a method exist: one regarding the feasibility of applying it to a plurality of specific cases and the other one concerning the possibility of expressing it through the language of the symbolic algebra.

It is interesting to observe that arithmetic plays a role of support/obstacle to the evolution of algebra. In the phase of transition, indeed, from the arithmetical thought to the algebraic thought, certain obstacles at arithmetical level can delay the development of the algebraic language. But it can also happen that the introduction of new strategies and of the new algebraic subject can eclipse the preceding arithmetical knowledge (Cfr. Malisani, 1990 and 1993).

From the analysis of the different resolutive procedures we can deduce that the visual representative registers are present in those that use the geometric language, but they are also recalled in some arithmetical or analytical methods. In the rhetoric and syncopated phase the concept of equality varies according to the adopted procedures; it can

represent: the result of an arithmetical operation, the equivalence of plain figures, the equivalence between two ways of expressing the same quantity or “the conditioned equality” between two members of an equation.
The notions of unknown and of variable as “thing that varies” have a totally different origin and evolution. The concept of variable is developed slowly passing from the initial relation among the numbers included in the tables, to the dynamic quantities reported through a formula, to the variable connected to continuous quantity in the study of physics, to the curves described in kinematical terms, to the relation among variables that finally leads to the concept of function. The unknown, instead, has its origin in the resolution of problems that ask the calculation of one or more quantities. The preponderance of this notion in the resolutive procedures is notable up to 1600.
Nowadays the idea of variable as any number seems so obvious and simple to us that it is difficult to understand why it took so long to consolidate. But if we reflect a moment on the mental process that activates, we discover that it is a thought in functional terms and therefore it requires the ability to think simultaneously on whole families of numbers rather than on a specific quantity, as well as on the reciprocal relations between families of numbers (Arzarello *et alii*, 1994).
Different studies consider that the concept of variable represents a point of critical transition (Matz, 1982; Wagner, 1981, 1983). This is a complex concept because it is used with different meanings in different situations. Its management depends on the particular way of using it in the resolution of problems. The notion of variable could take a multiplicity of aspects: generalized number, unknown, “thing that varies”, entirely arbitrary sign, register of memory, etc. Usiskin (1988) thinks that the plurality of conceptions is exactly the reason why this notion becomes difficult to define and it is possibly the cause of most of the difficulties that pupils meet in studying algebra.
The experimental research of the second chapter has the purpose of studying some characteristics of the period of transition between the arithmetical language and the algebraic language. We want to analyze if the different conceptions of variable are evoked by the pupils in the resolution of problems and if the procedures in natural language and/or in arithmetical language prevail as resolutive strategies, in absence of adequate mastery of the algebraic language.
We set up some a-didactical situations on the resolution of the magic square: “complete the square inserting the lacking numbers, so that the sum of the numbers of every line, column or diagonal is always the same”. Twenty seven pupils attending the first year of

middle school and thirty nine pupils belonging to two first classes of the Psycho-Pedagogical High school participated in this experimentation.

From the qualitative and quantitative analysis of the data we deduce that the symbol *"a"* of the magic square can take very different aspects for the pupils of 11 and 14 years old: constant, constant not negative, numerical value, variable, unknown and symbol no value. The variety of evoked conceptions depend on the individual development of the algebraic thought (more numerous in the pupils of 14 year old) and on the particular way of using *"a"* within the activity of problem solving. Therefore the pupils recall the different conceptions of variable even in absence of a suitable mastery of the algebraic language (hypothesis 1).

It is interesting to observe that the pupils of the middle school use resolutive strategies in natural language or in arithmetical language. The algebraic language is almost absent because they have not succeeded in operating with the literal value *"a"* . The high school students applied the arithmetical procedure or the arithmetical-algebraic method and they effected algebraic calculations, but in some cases they made some errors. The pupils did not use any algebraic strategies anticipated in the a-priori analysis. Therefore the procedures in natural language and/or in arithmetical language prevail as resolutive strategies, in absence of an appropriate mastery of the algebraic language (hypothesis 2). From this point of view, a clear correspondence is recorded between individual development and historical development.

The experimental works introduced in the third and in the fourth chapter aim to study the relational-functional aspect of the variable in the problem-solving, considering the semiotic contexts of algebra and analytical geometry. We want to analyze if the notion of unknown interferes with the interpretation of the functional aspect and if the natural language and/or the arithmetical language prevail as symbolic systems in absence of adequate mastery of the algebraic language. We also want to investigate the difficulties that the students meet to interpret the concept of variable in the process of translation from the algebraic language into the natural one.

To effect this research we made up a questionnaire composed by four questions on the linear equation in two variables. In the first and the fourth of them the variable takes on the relational-functional aspect in the context of a concrete problematic situation. In the first problem the pupil was free to choose the resolutive context; in the fourth one, instead, we force him to operate within the analytical geometry. The second question asks for the formulation of a problem that can be resolved by means of a given equation and the third one inquires the interpretation of two relations of equality: $ax + by + c = 0$ and $y = mx + q$.

The experimentation was carried out in two phases. One hundred eleven students of 16-18 years old of the Experimental High School participated in the first phase and resolved the questionnaire individually. The second experimentation was carried out with four pupils of 16-17 years old of the Scientific Experimental High School. They worked in pairs.

From the analysis of the data we observe that the strategies to solve the first problem are the following:

- *Procedure in natural language*: the student adds a datum and finds a particular solution that verifies the equation. This procedure leads to a single solution and it turned out to be the most used by the pupils. The predominant conception of variable is that of unknown.

- *Methods by trials and mistakes in natural language and/or in half-formalized language*: generally arithmetical, it conducts to several solutions. The dependence of the variables is evoked, but a strong conception of the relational-functional aspect does not appear yet.

- *Pseudo-algebraic strategy*: it is little used by the pupils and it leads to the correct solution of the problem only in some cases.

From the study effected we notice that to solve the first problem the procedures are based on the natural language and they follow the pace of the spoken thought in which the semantic control of the situation is developed and takes place. The pupils also use, in an explicit or implicit way, the arithmetical language in a not purely algebraic context. From the analysis of the protocols of the interviews we observe that the symbolic language is practically absent in one of them. In the other one, instead, it is used in a superficial way, only to communicate, not to solve the problem. Therefore the check that the formula could operate on the flow of the verbal reasoning is missing. So it is possible to falsify the second hypothesis: "the natural language and/or the arithmetical language prevail as symbolic systems, in absence of adequate mastery of the algebraic language". Therefore, a clean correspondence between historical development and individual development is recorded.

It is interesting to observe that no pupil uses visual representative registers to solve the first problem, and that many students answer that the question has only one solution (variable as unknown). In the fourth problem, instead, with a concrete situation similar to the preceding one but formulated in the context of the analytical geometry, the students consider that it is verified by a plurality of solutions (variable in functional relation).

These results show that the students, with insufficient mastery of the algebraic language, can consider more easily the plurality of solutions in the presence of visual representative registers, by evoking the mental model of straight line.

For the third question almost all of the pupils have interpreted the expressions $ax + by + c = 0$ and $y = mx + q$ within the analytical geometry, but the model of straight line has not been resumed with the equation of the first problem. Thus the graphic representation is totally absent from the resolutive process. This behaviour called "avoidance of visualization" was already found in the didactical research (Cfr. Eisenberg & Dreyfus, 1991; Vinner, 1989; Furinghetti & Somaglia, 1994; Chiarugi, I. *et alii*, 1995).

In this situation we think that the "avoidance of visualization" is linked to a matter of didactical contract. Usually, the problems with equations given at school are solved in an algebraic context where the variable engages the unknown aspect. The concrete problematic situations generally are never solved within the analytical geometry, recalling visual representative registers. The problems of analytical geometry given at school are different. In the fourth problem, the pupil is forced to use the model of straight line with its Cartesian representation. Therefore the equation becomes "perceivable" through the graph and the student can "visualize" more easily the plurality of solutions. So it is possible to falsify the fourth hypothesis: "The student understands more easily the relational-functional aspect of the variable in the presence of visual representative registers".

It is interesting to underline that a clear convergence with the historical point of view is manifested. The notion of unknown appears in the resolution of problems that require the calculation of one or more quantities and exert a strong predominance in the resolutive procedures up to 1600. Historically the relational-functional conception often appears in the presence of visual registers: tables, curves, descriptions of a motion, etc.

From the statistical survey and qualitative analysis of the protocols we can observe that the student is more inclined to consider the variable under the unknown aspect (searching the oneness of the solution of the linear equation) in the context of a concrete situation and in absence of representative graphic registers. In few cases, although the conception of unknown prevails, we verified the passage from the single solution to a plurality of solutions. This takes place through the systems of equations. In others words, for this students the infinite solutions constitute a set of single solutions coming from the resolution of different linear systems that contain the given equation. Accordingly they do not state the problem of the bonds imposed by the context in which the expression is considered.

In the cases in which the functional-relation conception of the variable prevails, the infinite solutions of the linear equation constitute a set of pairs of values that are obtained by varying one of them and calculating the other one, beginning from the linear dependence of a variable on the other one. In this way the pupils also succeed in considering the bonds of the numerical universe that the contextual sense of the equation imposes. As a consequence, we can affirm that there is an interference of the conception of unknown on the functional aspect, in the context of a problematic situation and in absence of visual representative registers (Hypothesis 1). However, we believe that the matter must still be deepened analyzing how the passage from a conception to the other could occur without interference in the process of resolution of a concrete problematic situation.

The translation from the algebraic language ($6x - 3y = 18$) into the natural language results a difficult exercise for the pupils, therefore it is possible to falsify the third hypothesis.

Some students are able to produce the text of a problem that does not result meaningful for the given relation. Others, instead, carry out only a purely syntactic manipulation of the formula, because they confuse the activity of solving an equation with that of inventing a problem which originates from an equation. We think that this difficulty is due to a matter of didactic contract: at school the students usually resolve problems, they do not invent problems.

The formulation of a problem from an equation implicates fundamentally three activities:

- Choosing an adequate context to attach meaning to the equation;
- Identifying the objects of the context that represent the variables;
- Individualising the properties of the objects that are pointed out by the relation expressed in the equation.

From the analysis of the protocols of the interviews we observe that the critical stage is precisely: “to individualize the elements of the context to be associated with the variables". In a protocol we observe the attempt to formulate the problem in a context of "market and apples", but the students do not succeed in identifying *x* and *y* with the quantities of apples of two different subject-objects: two shopkeepers, two different cassettes, two different varieties, etc. Thus they formulate the text of a classical arithmetical problem with specific numerical values (the coefficients of the equation); in the attempt of bettering the statement, they succeed only in inserting a variable and therefore they abandon this context.

The two couples of students resolve the query producing a text similar to the first problem. This means to deal with the context "money and bets" and the elements "two persons that play". They must only adapt the properties of the objects to the new relation that the equation expresses. We thought that this activity would have brought about the paraphrasing of the text of the first problem, but it was not so obvious. One couple felt the need to make the variable emerge in the text of the problem and to interpret the minus sign; their final formulation is the consequence of a gradual elaboration.

In the two protocols we clearly observe an important loosening between the symbolic language and the possibility of finding a different context from "money and bets" to give meaning to the equation. We think that this is not the consequence of the lack of creativeness, but the result of an insufficient control on the symbols. This is revealed in the impossibility of associating the variables to some elements of the context.

Therefore, we notice that a formula alone does not activate forms of productive thought, it is not at all considered like the interpretative model of a problem or even better as a class of problems.

In the construction of the algebraic language the expression of a class of problems through a formula was the result of a long conquest, because there are two levels of conceiving the generality of a method: one regarding the possibility of applying it to a plurality of particular cases and the other one concerning the feasibility of expressing it through the language of the symbolic algebra. To achieve the second level it is necessary to introduce the parameters, so that whole families of problems can be treated by concise procedures, that is by a formula.

To study these conclusions in depth it would be interesting to examine the first level of generalization analyzing the existing relation between the variables of an equation and the objects of the context that represent them, from a semiotic point of view. It would be important to analyse how the construction of the sense of a symbolic expression takes place in the space in which the dominion of the symbolic narrative still has not been achieved completely and the story of the problem is just outlined (Cfr. Radford, 2002).

Other central matters that emerge from this research and that would be interesting to study in depth are the following:

- how could the passage from the conception of unknown to the relational-functional one (or vice versa) occur without interference in the process of resolution of a problematic situation?
- how does the semiotic context influences the conceptions of the variable from the pupil's point of view? That is, it would be interesting to study the interaction of other contexts: natural language, geometric language, perceptive schemes, etc. with the operating of the pupils in a strictly algebraic context.
- what influence do the visualization and the coordination of different representative registers in the problem-solving on the conceptions of the variable exert from the pupil's point of view?

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