Fractions: conceptual and didactic aspects

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In this paper we present the findings of a principally bibliographical long-term research project, concerning "fractions".

This is one of the most studied questions in Mathematics Education, since the learning of fractions is one of the major areas of failure.

Here we present a way of understanding lack of success based on Mathematics Education studies, rather than on mathematical motivation.

Mathematical aspects

It must be said that a number of teachers are unaware of the fact that there is a considerable difference between a fraction and a rational number (this study will deal only with absolute rational numbers Q^a).

Few are aware of the purpose of constructing Q^a starting from ordered pairs of $N \times N+$.

The fact that an absolute rational number is a class which contains infinite ordered equivalent pairs of natural numbers (the second of which is not zero) is by no means clear to all.

Let us examine a mathematically acceptable definition of Q^a, starting from N, by considering the pairs $(a; b)$, $(c; d)$ of the set $N\times(N-\{0\})$, where a, b, c, d are any natural number, with the sole restrictions $b\neq0$, $d\neq0$, and taking the following relation (indicated by **eq**):

$$[(a; b)\ \underline{\mathbf{eq}}\ (c; d)] \text{ if and only if } [a\times d = c\times b].$$

This relation belongs to a special category, that of the relations of equivalence, in that is [leaving aside the simple demonstration]:

- ***reflexive***: for each pair (a; b) of N×(N-{0}), the following statement is true: (a; b) **eq** (a; b);
- ***symmetrical***: for each pair of pairs (a; b), (c; d) of the set N×(N-{0}), the following statement is true: **if** [(a; b) **eq** (c; d)] **then** [(c; d) **eq** (a; b)];
- ***transitive***: for each set of three pairs (a; b), (c; d), (e; f) of the set N×(N-{0}), the following statement is true: **if** {[(a; b) **eq** (c; d)] **and** [(c; d) **eq** (e; f)]} **then** [(a; b) **eq** (e; f)].

In this way the initial set $N\times(N-\{0\})$ can be distributed by subdividing it in equivalence classes via the operation known as “passage to the quotient”, thus indicated: $[N\times(N-\{0\})]$ / **eq**.

$[N\times(N-\{0\})]$ / **eq** contains infinite classes, which are the elements that constitute it; in each class there are infinite pairs of natural numbers.

$[N\times(N-\{0\})]$ / **eq** is the set Q^a. Each infinite class of equivalent pairs is called an absolute rational number. Normally a representative for each class is chosen and can be expressed through different written forms.

In Q[a] the operation of division can be defined, whereas in N it was not: to divide the pair (a; b) (with $b \neq 0$) by the pair (c;d) (with $d \neq 0$), we need only multiply (a; b) by (d; c) (with the same necessary condition $c \neq 0$).

Fractions as an object of scholastic knowledge

The passage from "Knowledge" (academic) to "learned knowledge" (of the student) is the result of a long and delicate path leading first to the knowledge to be taught, then to the knowledge actually taught and finally to the knowledge learnt.
In this sequence the first step of transforming "Knowledge" into "knowledge to teach" is called *didactic transposition* and constitutes a moment of great importance in which the professionalism and creativity of the teacher are of utmost importance.

As we have already seen, the object of Knowledge Q^a cannot simply be transferred to the pupil, neither at primary nor at secondary level. The pupil simply does not possess the critical maturity or cognitive ability to construct such Knowledge.

Nonetheless, among the “learned knowledge”, it is necessary to include Q^a, together with the use of the point, of decimal numbers including those between 0 and 1, and so on. Moreover, the monetary system of almost all countries presupposes that citizens should possess a basic ability to handle absolute rational numbers; the international measurement system adopted it since the end of the eighteenth century, making it necessary; and in practically all jobs it is at least necessary to grasp the intuitive meaning of 0.5 or 2.5. Thus rational numbers have a social statute that makes them an ability that all should develop.

As mentioned above, an act of didactic transposition is clearly necessary in order to transpose Q^a into something accessible to primary and then secondary pupils.

The history of Mathematics teaching clearly places the path of this transposition within the following line of development: fractions (primary and secondary school), decimal numbers (primary and secondary school), Q^a (upper secondary school or, at times, university).

It would be wrong to suppose that “didactic transposition” is the same as “simplification”. Often the concepts that our student must go through are bristled with complications, compared to the ones relative to the Knowledge.

For example, with fractions numerous conceptual problems arise concerning objects of knowledge which do not exist in Q^a: apparent fractions (m/n with n divisor of m) or improper fractions (m/n with $m>n$), are cumbersome, while in Q^a these options simply do not exist.

Indeed, if it were possible to avoid passing via fractions and go straight to absolute rational numbers things might well be more simple and natural.

But this is impossible. It still seems natural to pass via fractions, even if it is not at all clear that this is the most effective path. What is clear is that it poses many difficulties.

Thus fractions, while not a part of academic Knowledge, are nonetheless an issue of Mathematics Education as an object of knowledge, a knowledge that we could call "scholastic".

Theoretical framework of didactic researches into fractions

Introducing the concept of fractions has a common basis the world over. A given concrete unit is divided into *equal* parts and some of these parts are then taken.

This intuitive idea of fraction of the unit is clear and easily grasped, as well as being simple to modelize in everyday life. It is, however, theoretically inadequate for subsequent explanation of the different and multiform interpretations given to the idea of fraction.

As we shall see, one single "definition" is not sufficient.

At times it seems that many teachers are unaware of the conceptual and cognitive complexity involved. I believe that it is necessary to dedicate a whole section to different ways of understanding the concept of fraction, that we would like the pupil to acquire.

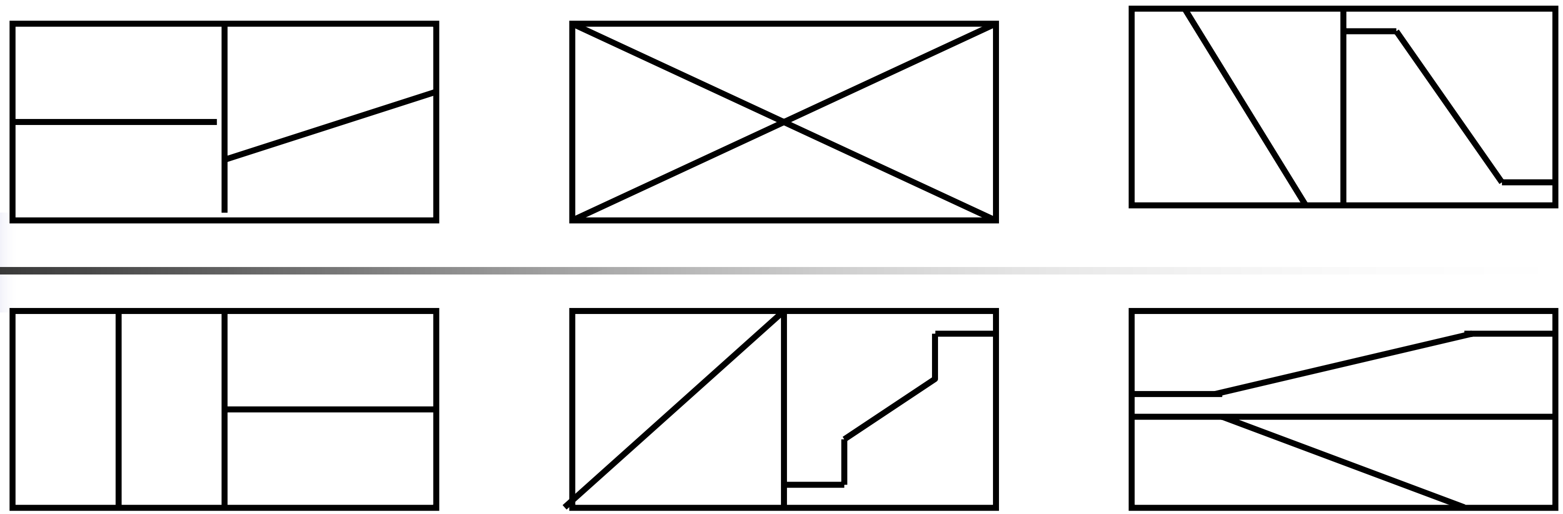
To give reliability to my work I am obliged to propose an overview of international research into this delicate field, certainly one of the most cultivated the world over. It is impossible to quote the whole of these researches, since its vastness goes beyond our imagination.

In this occasion, for the sake of brevity, I will drop this point; but in the text I gave you, 11 pages are dedicated to an analysis of the principle researches into this field and many others to the bibliography.

Different ways of understanding the concept of fraction

Something which often strikes teachers on training courses is how an apparently intuitive definition of fraction can give rise to at least a dozen different interpretations of the term

1) A fraction as part of a one-whole, at times continuous (cake, pizza, the surface of a figure) and at times discrete (a set of balls or people). This unit is divided into “equal” parts, an adjective often not well defined in school, with often embarrassing results such as the following, concerning continuous situations:



or discrete ones: how to calculate ⅝ of 12 people.

Providing students with concrete models and then requesting abstract reasoning, independently of the proposed model, is a clear indicator of a lack of didactic awareness on the part of the teacher and a sure recipe for failure.

2) At times a fraction is a quotient, a division not carried out, such as *a*/*b*, which should be interpreted as *a* : *b*; in this case the most intuitive interpretation is not that of part/whole, but that we have *a* objects and we divide them in *b* parts.

3) At times a fraction indicates a ratio, an interpretation which corresponds neither to part/whole nor to division, but is rather a relationship between sizes.

4) At times a fraction is an operator

5) A fraction is an important part of work on probability, but it no longer corresponds to its original definition, at least in its ingenuous form.

6) In scores fractions have a quite different explanation and seem to follow a different arithmetic.

7) Sooner or later a fraction must be transformed into a rational number, a passage which is by no means without problems.

8) Later on a fraction must be positioned on a directed straight line, leading to a complete loss of its original sense

9) A fraction is often used as a measure, especially in its expression as a decimal number.

10) At times a fraction expresses a quantity of choice in a set, thereby acquiring a different meaning as an indicator of approximation.

11) It is often forgotten that a percentage is a fraction, again with particular characteristics.

12) In everyday language there are many uses of fractions, not necessarily made explicit, e.g. for telling the time (“A quarter to ten”) or describing a slope (“A 10% rise”), often far from a scholastic idea of fractions.

The noetics and semiotics of fractions

The term "***noetics***" refers to conceptual acquisition and thus within the school environment to conceptual learning.

The term "***semiotics***" refers to the representation of concepts through systems of signs.

Both are of extraordinary importance in Mathematics.
On the one hand any form of mathematical activity requires the learning of its concepts. On the other it is impossible to study the learning in Mathematics without referring to semiotic systems.

It is important to bear in mind that the concepts of Mathematics do not exist in concrete reality. The point P, the number 3, addition, parallelism between straight lines, are not concrete objects which exist in empirical reality. They are pure concepts, ideal and abstract, and therefore , if we want to refer to them, they cannot be “empirically displayed” as in other sciences. In Mathematics concepts can only be ***represented*** by a chosen semiotic register.

As a matter of fact, in Mathematics we do not work directly with objects (i.e. with concepts), but with their ***semiotic representations***. So semiotics, both in Mathematics and in Mathematics Education, is fundamental.

To represent a given concept there are many possible registers.

Passing from one representation to another within the same register is called “transformation by treatment”, while a change of semiotic representation into another register is called “transformation by conversion”.

In 1993 Duval called attention to a cognitive paradox hidden within this topic. We shall see that as regards the didactics of fractions this is an extraordinary important issue.

The fraction is a concept thus its learning is within noetics. As such, it cannot be concretely displayed. We can operate with a one-whole, an object, a cake, dividing it and obtaining a part. But the result is not the mathematical “fraction”, only the “fraction of that object”. Working with the semiotic register of concrete operations, we have shown a semiotic representation, not the concept.

The distinguishing features that characterize the different objects are chosen– the act of dividing, the cake (continuous), the surface (continuous) of a rectangle, the set of balls (discrete), the formal writing with its specific names – treatment transformations (few) and conversion transformations (many) are continuously carried out, taking for granted that, if the student is capable of reproducing them, the teaching has been successful, the learning achieved and the concept constructed.

Difficulties in the learning of fractions and Mathematics Education

Research has illustrated some errors which are "typical" in students the world over. Research has thoroughly and precisely studied and listed them. Below we summarize the most important.

1) Difficulty in ordering fractions and numbers written in the decimal notation.
2) Difficulty with operations between fractions and between rational numbers.
3) Difficulty in recognizing even the most common schemata.

4) Difficulty in handling the adjective “equal”.
5) Difficulty in handling equivalences.
6) Difficulty in handling the reduction to minimum terms.
7) Difficulty in handling non standard figures.
8) Difficulty in passing from a fraction to the unit that has generated it.
9) Difficulty in handling autonomously diagrams, figures or models.

Research has highlighted these typical errors and has classified them, but without using modern Mathematics Education considered as Learning Epistemology in the specific case of fractions, thereby turning over its point of view, using the results of the copious research that has been conducted over the past 40 years.

This need has pushed me to consider the main research topics into Mathematics Education and come back to the previous classical research into fractions under this point of view, looking for didactic and not mathematical motivations of these “typical errors”. The following list considers but a few of the issues involved.

Research has highlighted that the origins of difficulties lie in some of the main topics of present Mathematics Education and more precisely:

1. Didactic contract
2. Excessive semiotic representations
3. Prematurely formed images and models
4. Misconceptions
5. Ontogenetic, didactical, epistemological obstacles
6. Excess of didactic situations and lack of a-didactic situations

I will outline only one of these features.

Epistemological obstacles

Many of the things to be learnt concerning fractions can be considered as true epistemological obstacles and are easily recognizable in the history and in the practice of teaching.

a) The reduction of fractions to minimum terms has for long been a specific object of study in history, as is shown by the fact that the Egyptians who cultivated fractions for many centuries used only fractions with unitary numerators.
b) The passage from fractions to decimal numbers required over 4,500 years of Mathematics.
c) The handling of zero in fractions has often created enormous problems in history, even for illustrious mathematicians.

I hope that this summary has given a picture of a research that has involved 6 years of my life and that is fully testified in the thesis that I handed over as my presentation for this Ph.D. session.

ĎAKUJEM ZA VAŠU POZORNOSŤ A TRPEZLIVOSŤ