How the sense of mathematical objects changes when their semiotic representations undergo treatment or conversion

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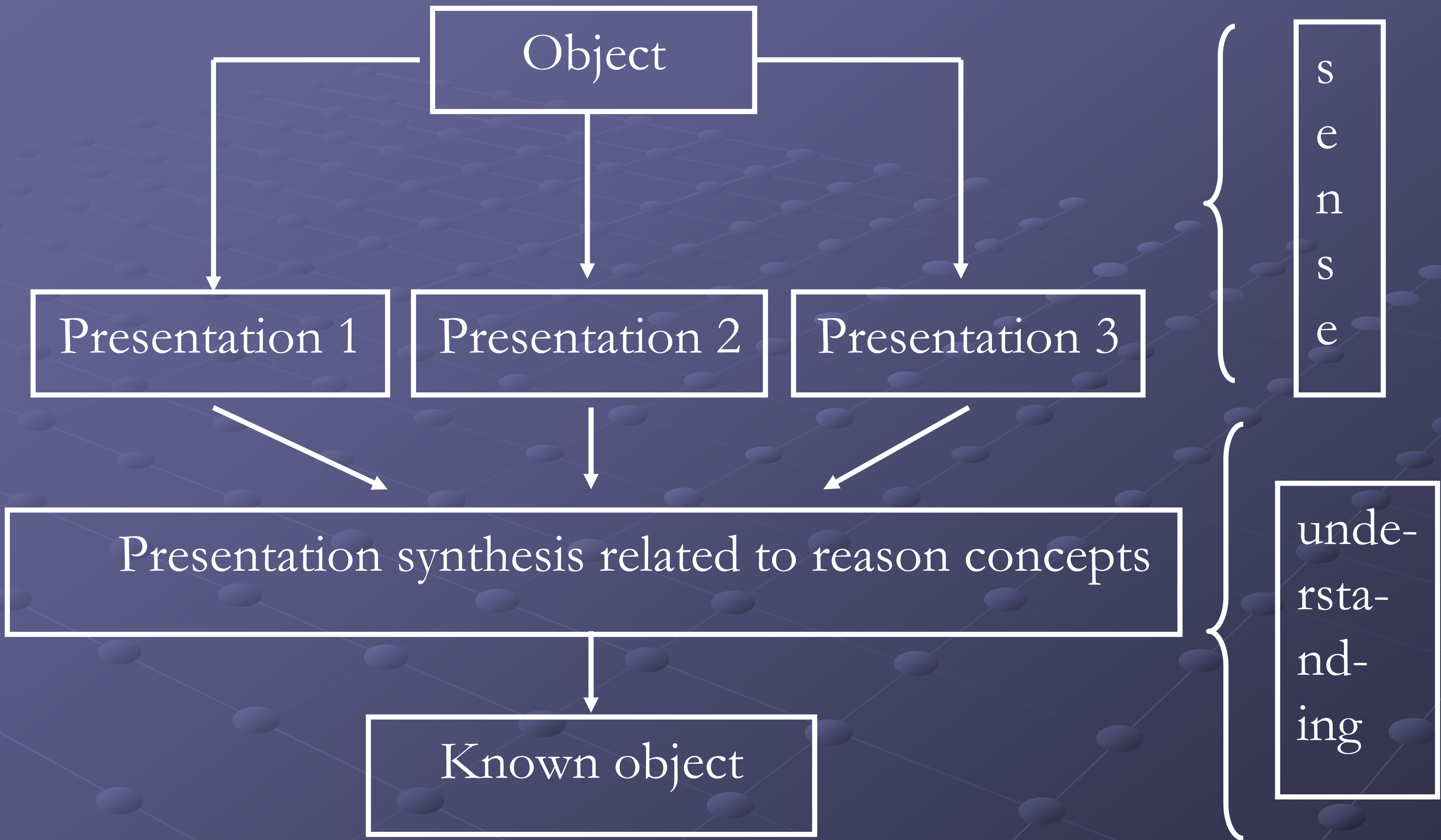
1. Preliminary remarks

It often happens, at any school level, in mathematical situations that can also be very different between each other, that we are surprised by a statement that suddenly reveals a missed conceptual construction regarding topics that instead appeared thoroughly acquired.

We will give a roundup of examples that we found in the past years and we will try to give one of the possible explanations of this phenomenon, analysing in particular an example.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2. Mathematical object, its shared meaning and its semiotic representations: the narration of an episode

2.1. *The episode*

In a fifth class (pupils aging 10 years) of an Italian Primary School, the teacher has conducted an introductory lesson in a-didactic situation concerning the first elements of probability, in which the pupils construct, with at least the use of some examples, the idea of "event" and "the probability of simple events". As an example, the teacher uses a normal die with six faces to study the random results from a statistical point of view. From this emerges a frequency probability which is, however, interpreted in the classical sense. The teacher then proposes the following exercise:

Calculate the probability of the following event: the result of an even number when throwing the die.

Pupils discuss in groups and above all sharing strategies devised under the direction of the teacher decide that the answer is expressed by the fraction 3/6 because «the possible results are 6 (at the denominator) while the results that render the event true are 3 (at the numerator)».

After having institutionalised the construction of this knowledge, satisfied by the result of the experience and the fact that the outcome has been rapidly obtained and the pupils have shown considerable skill in handling fractions, the teacher proposes that, on the basis of the equivalence between 3/6 and 50/100, it is also possible to express the probability by writing 50% and that this is indeed more expressive, since it means that the probability of such a result is a half, in terms of the generality of all possible events which is 100.

A pupil observes that «so we can also use the [fraction] 1/2», and the proposal is verified through the explanation of the pupil, rapidly accepted by all and once again institutionalised by the teacher.

2.2. *Semiotic analysis*

- ***semiotic register***: natural language: probability that the result of throwing a die is an even number
- ***semiotic register***: the language of fractions: 3/6, 1/2, 50/100
- ***semiotic register***: the language of percentages: 50%.

2.3. *The sense shared via different semiotic representations*

Each of the preceding semiotic representations is the signifier which follows from a preceding single meaning (Duval, 2003).

- ***conversion***: from the semiotic representation expressed in the natural language register to the written form 3/6
- ***treatment***: from the written forms 3/6 and 1/2 to 50/100
- ***conversion***: from the written form 50/100 to 50%.

2.4. *Required previous knowledge*

In the episode considered several types of knowledge, apparently well-constructed, interact:

- knowledge and use of fractions
- knowledge and use of percentages
- knowledge and use of the event: the result of throwing a die is an even number.

Each of these is manifest in the unitary and shared practices of the class.

2.5. ***Sequel to the episode: the loss of a shared sense caused by semiotic transformations***

At the end of the sequence the pupils are asked if the fraction 4/8 can be used to represent the same event, since it is equivalent to 3/6. ***The answer is negative, unanimous and without hesitation***.

Even the teacher, who had previously handled the situation with confidence, asserts that «4/8 *cannot* represent the event because a die has 6 faces and not 8».

Pressed to consider further the question, the teacher adds: «There are not only dice with 6 faces, but also dice with 8 faces. In that case, yes, the fraction 4/8 can represent the result of throwing a die is an even number».

3. A symbolism for semiotic principles

In other studies we have already used the following definitions and symbols (D'Amore, 2001, 2003a,b, and elsewhere):

semiotic $=_{df}$ representation realised via a system of signs
noetic $=_{df}$ conceptual acquisition of an object.

Hereafter we will use:
r^m $=_{df}$ m^{th} semiotic register
$R^m_i(A)$ $=_{df}$ i^{th} semiotic representation of concept A in the semiotic register r^m
(m = 1, 2, 3, …; i = 1, 2, 3, …).

characteristics of the semiotic: ***representation – treatment – conversion*** [imply different cognitive activities]

concept A to be represented → choice of distinctive features of A

REPRESENTATION of A [$R^m_i(A)$] in a given semiotic register r^m

transformation of representation ***TREATMENT***

new representation ($i \neq j$) [$R^m_j(A)$] in the *same* semiotic register r^m

transformation of register ***CONVERSION***

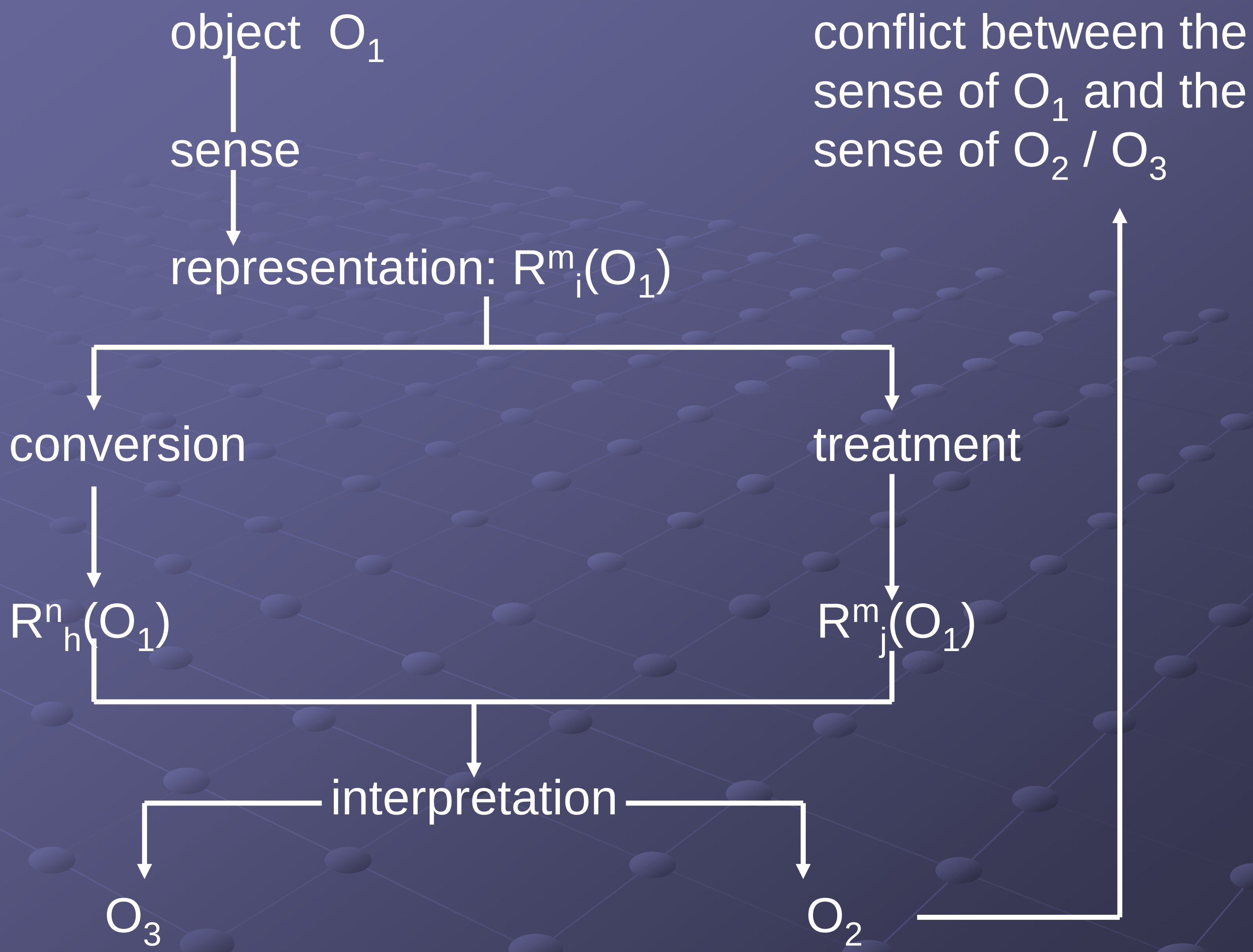
new representation ($h \neq i$, $h \neq j$) [$R^n_h(A)$] in a *different* semiotic register r^n ($n \neq m$)

($m, n, i, j, h = 1, 2, 3, \ldots$)

4. Let's turn back to the episode

- There exists a mathematical object O_1 to represent: the probability that the result of throwing a die is an even number;
- a *sense* is ascribed to the object on the basis of a presumable shared experience which is part of a social practice shared in the class;
- a semiotic register r^m is chosen in order to represent O_1: $R^m_i(O_1)$;
- a treatment is effected: $R^m_i(O_1) \rightarrow R^m_j(O_1)$;
- a conversion is effected: $R^m_i(O_1) \rightarrow R^n_h(O_1)$;
- $R^m_j(O_1)$ is interpreted and the mathematical object O_2 is recognised in it;
- $R^n_h(O_1)$ is interpreted and the mathematical object O_3 is recognised in it.

What is the relationship between O_2, O_3 and O_1?


object O_1
sense
representation: $R^m_i(O_1)$
conflict between the sense of O_1 and the sense of O_2 / O_3
conversion
treatment
$R^n_h(O_1)$
$R^m_j(O_1)$
interpretation
O_3
O_2

In our example:

- object O_1: the probability that the result of throwing a die is an even number;
- sense: the shared classroom experience under the supervision of the teacher leads to the conclusion that the sense of O_1 is that described by the pupils and desired by the teacher: many possible outcomes and many outcomes consistent with the event;
- choice of a semiotic register r^m: rational numbers Q expressed as fractions; representation: $R^m{}_i(O_1)$: 3/6 ;
- treatment: $R^m{}_i(O_1) \rightarrow R^m{}_j(O_1)$, i.e. from 3/6 to 1/2 ;
- treatment: $R^m{}_i(O_1) \rightarrow R^m{}_k(O_1)$, i.e. from 3/6 to 4/8;
- conversion: $R^m{}_i(O_1) \rightarrow R^n{}_h(O_1)$, i.e. from 3/6 to 50%;
- $R^m{}_j(O_1)$ is interpreted and the mathematical object O_2 is recognised in it;
- $R^m{}_k(O_1)$ is interpreted and the mathematical object O_3 is recognised in it;
- $R^n{}_h(O_1)$ is interpreted and the mathematical object O_4 is recognised in it.

What is the relationship between O_2, O_3, O_4 and O_1?

In some cases, (O_2, O_4), identity of the signifier is recognised, thus indicating previously-constructed knowledge which permits this recognition. There is one single, shared sense. In another case, (O_3), the identity is not recognised, in that the interpretation is or seems to be different and so the sense of the object (meaning) O_1 has been lost.

Duval too treats the question of different representation of the same object (Duval, 2006).

It is not necessarily the case that the loss of sense occurs only as a result of conversion. As we have seen in our example, the loss is caused by the treatment from 3/6 to 4/8. The teacher's interpretation of 4/8 did not consider a plausible object the very same O_1 derived from the shared sense which had led to the representation 3/6.

The same experiment conducted with older students and even trainee teachers shows that if the treatment from 3/6 to 4/8 is an example of loss of sense, the loss is even greater with the treatment from 3/6 to 7/14; while it is decidedly less in the conversion from 3/6 to 0.5.

5. Conclusion

What we would like to emphasize here is how the sense of a mathematical object is more complex than it is considered within the usual pair (object and its representations). There are semantic links between pairs of this kind:
(object, its representation) – (object, its *other* representation)

The phenomenon described can be used to complete the picture proposed by Duval of the role of the multiple representations of an object in understanding it and also to break the vicious circle of the paradox. Every representation carries with it a *different* "subsystem of practices", from which emerge *different* objects (previously called O_1, O_2, O_3 y O_4). But the articulation of these objects within a more general system requires a change of perspective, a movement into another context in which the search for a *common structure* is a part of the system of global practices in which distinct "partial objects" play a role

The progressive development of the use of different representations undoubtedly enriches the meaning, the knowledge and the understanding of the object, but also its complexity. In one sense the mathematical object presents itself as unique, in another as multiple.

What is then the nature of the mathematical object? The only reply would seem to be “structural, formal, grammatical” (in the epistemological sense) together with “global, mental, structural” (in the psychological sense) which we as subjects construct within our brains as our experience is progressively enriched.

Clearly these considerations lead to potential future developments in which ideas, apparently diverse, will work together to search for explanations for phenomena concerning the attribution of sense.

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