COMENIUS UNIVERSITY BRATISLAVA
FACULTY OF MATHEMATICS AND PHYSICS
DEPARTMENT OF DIDACTIC MATHEMATICS

MATHEMATICS, PHYSICS AND MUSIC

TO ELABORATION OF A DIDACTIC SITUATION

IN SECONDARY SCHOOL

(PUPILS AGED 14-18)

Doctoral Thesis by

DANIELA GALANTE

ADVISOR: Prof. FILIPPO SPAGNOLO

Introducing

All the time, mathematics presented and presents tight links with the real world. We can say that historically most of mathematics theories have started from the need to quantify, to measure, to describe, to schematize and to rationalize particular aspects of reality.

Recently besides, mathematics have assumed a constant greater relevance not only in the field of physics, engineering, economics, but also for numerous other disciplines, once considered “far” from mathematics, such for example chemistry, biology, medicine, social sciences, and art disciplines.

A modern teaching of mathematics can’t absolutely ignore the bonds with reality, if it doesn’t want to be only a vain exercise of formal skills meant to be forgotten at the end of the educational path.

So that it needs to teach how to know “implicit” mathematics in the different situations of the real world; it is necessary to bring up the students to choose by themselves the right variables to solve a particular problem, stimulate them to build appropriate mathematical patterns and to estimate if that pattern is right or not according to the situation which they mean to represent in a schematic form.

In these last years, under the push of the evolution of mathematical thought, a lively debate has been opened among teachers and researchers on the necessity to bring about changes to the mathematics programs, both to adapt them to the new needs of technological development, and to give students of secondary schools an idea of peculiar aspects to modern mathematics research.

It can be supposed that something is going to change in mathematics and physics teaching not only in its contents, but in its teaching methods too.

The main subject of the duality content-method implies the cultural role of mathematics and the renewal relating to the content can be considered as a revisiting of its applications and interdisciplinary aspects, according to the latest development of the discipline.

So we can say that geometrical transformations have a very important role if we think about simmetry in art, in nature and in the structure of crystals and of molecules.

The object of geometrical transformations is to show that we can study geometrical figures in a way different from Euclidean geometry. In fact according to Felix Klein’s (1849-1925) proposal put forward in the Erlangen Program in 1872, it is possible to study geometrical figures by introducing the concept of transformations and by finding out their features on the basis of those factors which come out as invariant. However it’s appropriate to say that this new method of studying geometrical figures is not antithetical to that of Euclidean geometry but is complementary to it and inclusive of the previous method; as it is known, Klein was able to give us a complete situation which had been built previously through the introduction of the concept of group of transformations.

In this case my thesis emphasizes the interdisciplinary aspect between geometrical transformations and music.

This research, the title of which is “Mathematics, physics and music, the elaboration of a didactic situation in secondary school**”,** comes out of these preliminary remarks. For a correct treatment of the subject I have divided the work into three parts each one divided in two chapters.

In the first part I investigate the presence of isometries in the history of music. In Chapter 1 since in music the “artistic” component is strictly connected to the “scientific” one I introduce the geometrical transformations in the language of music; in Chapter 2, I analyze the composition techniques based on the geometrical transformations used, in the dawning tonal system, by J.S.Bach in his *Music Offering*; in Chapter 3, I explore the integral seriality of P. Boulez in *Stuctures I* for two pianos, a composition in which the author gives up his creating “self” to deliver himself to the pitiless rationality if the geometrical transformations summarized into matrixes, where the number contains the sound in its complete dimension.

In the second part, since the knowledge of our body and the working of our brain help us to understand how to favour the understanding of didactic experiences, I introduce the course of sound: in Chapter 4 I speak about the sound and its characteristics from the physical point of view, in Chapter 5 I deal with the subject of perception and analysis of sound passing through the ear the acoustic ways and the auditory areas of the brain; in Chapter 6 I analyze the learning mechanism and the motivational system of reward.

In the third part I introduce the didactic experimentation, the data collected and the results obtained.

From general point of view this research contribute to clearing questions connected to the representations of isometrical knowledge and the way of problem solving from the students’ point of view and it is a contribution to the didactics of mathematics through the interdisciplinary aspect of the study of geometrical transformations with music.

With this consideration I have formulated the following hypotheses of research:

H1 *In musicians students (music liceo-conservatoire) the constant study of a musical instrument creates unconscious potentialities which are translated into strategies and methodologies for the solution of problems concerning isometries differently from non musicians students (pedagogical liceo).*

H2 *Students possessing a knowledge of the musical rhythmic structures have a greater ability in recognizing the rhythm of geometrical forms for the construction of objects in comparison with those who do not have such knowledge.*

To test the two hypotheses I have carried out a didactic experimentation at Liceo Statale “Regina Margherita” of Palermo where I singled out two different samples of students: students of the Music Liceo associated to the State Music Conservatoire “Vincenzo Bellini” of Palermo and students from the Social-Psicological-Pedagogical Liceo.

Besides eight couplet of students of the third classes where involved for couple interviews with the task of writing down the observations drown after a common agreement with registration of the protocols of the same interviews.

Therefore the experimental research is based on a comparison between the students of a liceo who study music at the conservatoire and the students of a liceo who do not study music at the conservatoire but who, however, have a base theoretical musical knowledge.

This research has focused on spontaneous conceptions regarding the geometrical transformations in general and their connection to music. The experimental research was carried out following the stream of the *theory of situations* by Guy Brousseau.

Both samples examined where given four blocks of questions. The first one is formed by six questions about the geometrical transformations in the plane without any analytical reference, present in any textbook of the first two years of upper secondary school; the second one is formed by five questions about the geometrical transformations with analytical reference in the Cartesian plane. The third one is a problem concerning the reconstruction of a mosaic through the identification and the iteration of geometrical figures; finally the fourth set is formed by five questions concerning the application of geometrical transformations to melodic bits of tune.

In a double entry chart “students/strategies” for each student I have shown with value 1 the strategies used and with value 0 the strategies not used. The collected data where analyzed in a quantitative way by using the implicative analysis of the variables of Regis Gras through software CHIC 2004.

In Chapter 7 I introduce the methodology of research, the theoretical frame of reference and the a-priori analysis.

In Chapter 8 I develop the quantitative and qualitative analysis of data.

In the appendix I enclose the score of the *Musical Offering* by J. S. Bach, the score of *Structures I* for two piano by P. Boulez, the test, the statistical charts elaborated with CHIC 2004 and the protocols of thee interviews.

FIRST PART: THE ISOMETRIES IN THE MUSIC HISTORY

Chapter 1. The Geometrical Transformation in the music language

The world of music presents two strictly connected components: the “artistic” one and the “scientific” one. On one hand, there is the emotional aspect which involves both those who play and those who listen; on the other hand there are aspects essentially founded on physical-mathematical bases which is necessary to analyze to understand how the musical event is fulfilled, how we can repeat it and what are the laws which influence it. With this work I show how it is possible not only to see the probable applications of geometrical transformation but also to listen to the effect they can have over a melody through the musical aid which makes the study of geometry more interesting.

Some transformation of a melody used in the composition technique quite correspond to a translation in music this kind of transformation is called *transposition* In the field of music reflection is another technique frequently used to develop a melody and in music is the *inversion* (*reflection* along the *x* axis), the *retrograde* (*reflection* along the *y* axis) or *retrograde of inversion* (*reflection* along the origin). At the end the *rotation* also belonging to the music language through the *modulation*.

Translation

- For the translation along the x axis the transformed melody is played after a moment of silence
- For the translation along the y axis the transformed melody is played at another height (over or under) the original one keeping the sequence of intervals unchanged.

Reflection

- For reflection as regards the x axis intervals in the original melody are played in an inverse direction. In music this kind of transformation is called *canon through a contrary movement or inversion* or also *mirror canon* when the original and the reflected melody start at the same time.
- For reflection as regards the y axis the reflected melody is composed by the same notes at the same height as the original one in a sequence of sounds moving backwards: the melody starts from the last note of the original melody to conclude with the first one. In music this transformation is called *retrograde* or *crab canon*.

- For symmetry in relation to the origin The reflected melody is built by a simultaneous composition of the reflection both as regards the x axis and as regards the y axis: in practice there is an inversion of the intervals and the melody moves backwards. In music this transformation is called *inverse retrograde*

Rotation

- Also in this case, the mathematical concept of rotation can be recognized by the approach with music in its double aspect: theoretical and auditive.

Chapter 2. The Geometrical Transformations in the "Musical Offering" by Johann Sebastian Bach (1685-1750)

Geometrical transformations have been used in compositional techniques since the XII century when polyphonic music was in its infancy and the success of the tonal model, together with the development of counterpoint, promoted in the XVII and XVIII centuries a further affirmation and spreading of the composition process based on the principles of geometrical transformations.

Johann Sebastian Bach summarizes in his enormous vocal and instrumental production all the style characters of the transition period which marks the passage from the baroque age to the modern one. He was the first among the great European composers to understand the endless compositional opportunities coming out of the application of the emerging tonal system, together with the richness of the composition process based on the use of the geometrical transformations in the new system.

The works created by J. S. Bach during the last decade of his life stand out for the extraordinary concentration through which he explores once and for all the potentialities of the technique and style of counterpoint[1], with a result of perfection and richness of the polyphonic design unparalleled in all the history of music.

Among the works of this period the M*usical Offering* is the one of the most significant examples of the scientific application of geometrical transformations. The *Musical* Offering owes its birth to one of the rare occasions of honorific titles in Bach's life. In May 1747 during a visit to his son Carl Philip Emmanuel at Potsdam, the king Frederic the Second who was a very competent musician and a talented flute player, gave him a musical theme to develop in counterpoint manner in one of the court pianos. Some month after the visit Bach sent the king a copy of his *Musical Offering*.

[1] From Latin *punctum contra punctum*, i.e. point against point, that is note against note, the counterpoint is the art of to superimpose two or more melodic lines. The born of counterpoint is connected with those of polyphony and the counterpoint principle can be used occasionally or rigorously applied to whole compositions.

It is composed by a three – voice *fugue* (*ricercar*[2] *), a* six-voice *fugue*, two *canons* and a *trio* based on a unique theme the *Tema Regium*: it is the *given theme*, i.e. the theme on which is based the whole sonorous architecture of the composition which with its incisive, magnificently round melody, and its energetic rhythm offers excellent potentialities, for the construction of the counter subject[3] and of the unlimited opportunities for variations, and it is in this sense that Bach mainly explores it. This work has a theoretical – scientific nature so that it doesn't have a specific instrumental[4] destination (like the *art of fugue*).

In this work we can find a severe application of geometrical transformations and the analysis isn't artificial because Bach intentionally used them when he composed his work. In fact, to face the hard problems given by the development of *Tema Regium* he did it in the spirit of the 15th and 16th centuries Flemish masters , who had a strongly logical and creative technique and compositional discipline. The result is musical and mathematical, the events flow rapidly and they are severely controlled and the composition is always expressive and original through imagination and creativity. The *Tema Regium*, which is always present is the common denominator of a kaleidoscopic variety of forms.

In particular in the *"Canon à 4"* are emphasized the translations along the x and y axes; in the *"Canon à 2"* (*Quaerendo invenietis*) Bach uses the reflection along the x axis, while in the *"Canon à 2"* (Cancrizans) one finds the reflection along the y axis; in the end in the "*Canon à 2"* (per tonos ascendenteque modulatione ascendat Gloria Regis) the great composer makes another translation along the y axis and especially the rotation through continuous modulations by means of the scales arranged in a succession of ascendant fifths.

After Bach and until the threshold of the third millennium, in the various musical genres, first the translation, then the reflection and in the end the rotation have been the composition processes adopted by many composers and ultimately with the help of music it is possible not only to see the applications of the geometrical transformations but also to listen to the effect they have over a melody.

In this way the aspect of the unity of culture in the diversity of knowledge is concretely understood.

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[2] the word ricercar is present in the original edition and it is due to the two 3 and 6 parts fugues.

[3] The fugue is a complex polyphonic musical form and articulate which is based on the principle of imitation. It is divisible in three fundamental parts: *exposition*, *development*, *stretto*. During the exposition the different voices present the theme (subject). After the exposition of the subject is concluded the first voice proceeds with another musical theme, the counter subject, while the successive voice performs the subject called answer and so on.

[4] In the edition we are looking at, it is indicated one of the possible choices for the performing praxis.

Chapter 3: The Geometrical Transformations in *Structures I* for two pianos by Pierre Boulez (1925)

This work, which is connected to a previous article (Galante 1999), considers the use of geometrical transformations in *Structures I* of Pierre Boulez (Montbrison Loire 1925), he is a French composer and conductor, who also studied mathematics; the aim is to offer a contribution to the decoding, understanding and interpretation of that part of contemporary piano solo repertoire of difficult fruition, for those who perform and for those who listen, owed just to his exclusive mathematization.

The tonal system, which had dominated for about two centuries the western composition technique, had arrived to an irremediable saturation in the late nineteenth-century.

It was urgent to look for new musical possibilities and the years between the nineteenth and twentieth centuries ,characterized just by this anxiety of renewal, were then detected by contemporaries as a period of "Modern Music".

The ways to escape from tonality were many. The border of the "New Music" can be traced back to about the year 1910 , when Schömberg and his school wrote the first compositions conceived in the field of a new system, dodecaphony, based on the 12 notes of the chromatic scale (which are not in a hierarchical relation among them, as it happened in classical harmony); their compositions were characterized by the series principle. This principle is based on a succession rigorously pre-arranged and invariable of sounds, called series: the only relation which ties sounds among them depends on the fundamental rule by which is not possible to repeat a note until the other 11 are used. Keeping the relations of interval belonging to the series, the same disposition of sounds can be changed through the processes of inversion (reflection along the x axis), retrograde (reflection along the y axis), retrograde of inversion (reflection along the origin).

The generation come out of the horrors of the second world war wanted to deny all the musical language which had preceded and start again from zero.

The most radical operation in this direction was made by Pierre Boulez. In 1952, in a famous article called *Schömberg is dead*, he accuses the dodecaphony of Schömberg to have limited itself to serialize heights and he invites, developing some intuitions of Webern (1883-1945), to extend to other components of sound (duration, intensity, pitch) the composition technique of serial music, generating in this way the composition principle of integral seriality. Boulez, in the first of the three parts of *Structures I* for two piano wants to reach the "zero grade" of the composition, cancelling completely not only the emotional subjectivity of the author, but as far as possible also his compositional acting: given some elements of start (and also those not of his creation, but taken from the Mode of Messiaen[5], Boulez applied geometrical transformations through a very rigorous numeric grill expressed in the form of a matrix which imposes all the composition choices on him.

SECOND PART: THE WAY OF THE SOUND

Chapter 4: Sound properties and their perception

The sounds which are around us are made by many sonorous concomitant waves. In general we can say that sounds are generated starting from a fundamental frequency of prevalent intensity to which are added sounds of frequency first double, then triple, etc…. of much lower intensity and hardly perceivable. Each one of these frequencies takes the name of H*armonic*. The knowledge of the acoustic phenomenon of harmonic sounds had a great importance in the construction of scales and of tonality and it became the natural basis on which all the laws of harmony developed.

In fact if we superimpose one upon the other harmonically the first six sound of any harmonic series we will obtain a perfect major accord. The disposition of the accord so obtained fully responds to the principal harmonic laws which establish how in the redoubling of the notes of an accord it is first of all to be preferred the fundamental one, then the fifth, and last the third (here in fact the fundamental one is found three times, the fifth twice, the third once).

From this it can be inferred that the perfect major accord, the basis of the modern tonality, is a direct emanation of the acoustic phenomenon of harmonic sounds, whose practical application opened the way to the transformation of the counterpoint polyphony of the Sixteenth Century into harmony of the following centuries.

The ear consists of three basic parts - the outer ear, the middle ear, and the inner ear. Each part of the ear serves a specific purpose in the task of detecting and interpreting sound. Moreover the inner ear serves to transform the energy of a compressional wave within the inner ear fluid into nerve impulses which can be transmitted to the brain

Chapter 5: Perception and sound analysis

Knowing our body and brain functioning helps us understanding how to encourage the comprehension and the storage of teaching experiences during mathematics teaching – learning activities, and, in this case, through interaction with musical language.

[5] The French compositor Olivier Messiaen (Avignone, 1908-Parigi 1992) was an important ring of conjunction among the historic vanguards and the <<new music>> of the second after war. Teacher of P. Boulez, to Damstadt, in the 1949-50 wrote *Mode de valeurs et d'intensitès*, one of the *Quatre etudes de rytme* for piano.

The activity of sound perception and the one of decoding and analyzing musical language combined with mathematic language could be analyzed from different points of view. In this chapter, I'll try to analyze it from a neurophysiologic aspect.
Undoubtedly, psychology is essential for teaching activity. But to reach some results it needs to combine behaviour studies with a parallel study about what happens in the brain.
For this reason it's important to understand how learning processes happen, from a biological point of view[6]. As it is shown in the experimental section, in this work I demonstrate that there are some mathematics learning ways, in this case some geometrical transformations, that are not used and that involve all human body's senses.

Chapter 6. Learning process and recompense

Between memory and learning there's a level difference, because memory is a requirement of material structures, and learning is a complex function of the organisms, resulting from their interaction with the environment.
William James (1842 – 1910) introduced the idea of two different components of the memory: Primary Storage (short-term memory) and Secondary Storage (long-term memory). Short-term memory is used to store data for a short period of time; it has been demonstrated that these data can contain at maximum seven components, for example seven notes, seven numbers, seven objects.
Long-term memory is used to store bigger amounts of data for long periods of time. The repetition of the data cause the passage from the short-term memory to the long-term memory, that is the consolidation. After that, the recollection is indelible
Human brain is constantly busy in exchanging energy and information with the external world (physical, social and cultural environment), and it can be considered as a *motivate* system which has dispositions apt to receive knowledge, to explore the world and to classify it in categories (*appetitus noscendi*).
Motivational status have three particular duties:

- guidance: they lead the behaviour to precise goals;
- activating: they augment the vigilance status and push the individual into action;
- organizing: they favour the action to goal's achievement, putting together single behaviour components in coherent sequences.

[6] Ferreri Mario, *L'evoluzione come apprendimento, l'apprendimento come evoluzione* in G.R.I.M. –Quaderni di ricerca in didattica- n.2, 1991, pp.87-148.

THIRD PART: THE SPERIMENTATION

Chapter 7: Methodology, theoretical frame of reference and analysis of data

The aim of this research is to verify if the constant study of a musical instrument creates unconscious potentialities which are translated into strategies and methodologies for the solution of problems related to isometries.

In fact, among the principal functions that the study of music is able to perform, besides the mere knowledge function, the linguistic-communicative function, the cultural, critical, aesthetical and affective function, a cognitive one is recorded because music exercises and develops the capabilities of thought: the productive-imaginative thought in the first place (in the activities of sound production) but also the analytical, logical and inferring thought (in the activities of reflection and interpretation).

Research methods

Research method is the one of G. Brousseau's teaching situations.

Experimental phases are:

- Teaching problem formulation: we learn geometrical transformations through geometrical language and musical language; through this we can see and hear them.
- Research aim formulation: favouring a learning process in the students, through interdisciplinarity, view as a unity of culture in the diversity of knowledge.
- A priori analysis of problem – situation, taking into account[7]:
    1. the epistemological representation of both mathematical and musical concepts;
    2. the historical-epistemological representation of the same concepts (interfering time variations)
    3. foreseeable students behaviour regarding the situation – problem.
- Research hypothesis.
- Creating instruments apt to falsify the hypothesis, with the creation of an experimental system through the preparation of:
    1. questionnaires;
    2. double interviews with the duty to write their common consideration written after a common agreement (interviews protocols registration)
- Experimental data analysis: experimental data correlation for the a priori analysis:

  Quantitative analysis on questionnaires problems.

[7]This variation of a priori analysis with epistemology and history contributions is of F. Spagnolo, *Insegnare le Matematiche nelle scuole secondarie,* La Nuova Italia, 1998

Application of:

1. descriptive statistic;
2. implicative statistic analysis of R. Gras (1997, 2000), using CHIC 2004 software;
3. qualitative analysis of the protocols to single or double interview, etcetera.

- Documentation and communication of research results.

Considering that we formulated the two following hypothesis research:

H1 *In musicians students (music liceo-conservatoire) the constant study of a musical instrument creates unconscious potentialities which are translated into strategies and methodologies for the solution of problems concerning isometries differently from non musicians students (pedagogical liceo).*

H2 *Students possessing a knowledge of the musical rhythmic structures have a greater ability in recognizing the rhythm of geometrical forms for the construction of objects in comparison with those who do not have such knowledge.*

In order to verify these two hypothesis, we realized an experimental teaching in Liceo Statale “Regina Margherita” in Palermo, where two different samples of students have been identified:

- Students from Music Liceo, connected to Conservatorio di Musica di Stato “Vincenzo Bellini”: 70 students, aged from 14 to 16 (classes I and II);
- Students from Liceo Socio-psico-pedagogico: 70 students, aged from 14 to 16 (classes I and II).

Furthermore, eight couples of students from the III classes have involved (four for each course) with the duty to write their common consideration written after a common agreement with a interviews protocols registration[8].

So, the experimental research is based on a comparison between Liceo’s students studying music at the Conservatorio and Liceo’s students who don’t study music at the Conservatorio but who have a basic theoretical-musical knowledge. This research focused on spontaneous concepts about geometrical transformation and their connection with music.

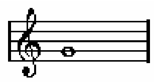
[8] Find protocols in annexes E, page 150

THE TESTING AND SOME RESULTS

I proposed four sets of questions to both samples examined: the first two are about classical exercises on geometrical transformations present in any textbook for the first two years of upper secondary school; the third one is a problem regarding the reconstruction of a mosaic through the identification and iteration of geometrical figures and finally a last set of exercises regarding the application of the geometrical transformations in melodic tune bits. I would like to concentrate some first conclusions above all on the pupils' behaviour adopted towards the last set of questions. It is to be précised that both samples hadn't yet carried out in the classroom the study of geometrical transformations and this allowed me to pick their spontaneous conceptions on the subject.

The set of questions proposed is the following one:

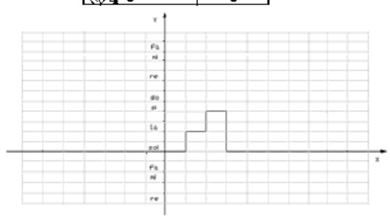
<<Let's consider a plane (x,y) and put the time on the x axis, which corresponds to a sequence of beats which have constant intervals (for example those ones produced by a metronome) and on the y axis the height of sound from the lowest to the highest. In this way any melody can be represented by a law f so that $y = f(x)$. After that let's choose as unit of measurement the second and match it to the musical crotchet figure (metronomic speed ♩ = 60) for the x axis and the semitone[9] tempered for the y axis; in this way we can have the graphic representation through little squares which simultaneously indicate the duration of each sound that is how they flow through time (on the x axis) and the height they have according to a tempered scale (on the y axis). Moreover, in musical writing notes written on the stave receive their names and indicate their height thanks to the use of the clefs[10]: for example to the treble clef corresponds the G note in the second line of the stave.



Treble clef or G clef

As consequence, if we consider as starting point of our system of reference, that is y = 0, the height of the correspondent sound to a G, the following melody: 

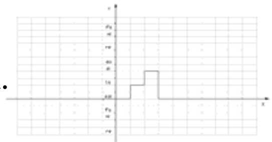
Is represented in a Cartesian plane in the following way:



[9]It's the distance between any sound of the tempered scale and its immediate subsequent, either in ascending sense or descending one. It is the shortest interval of our musical system and it corresponds to the half of a tone.

[10]They are graphic symbols that fix the position of all the sounds in a stave related to a sound fixed before them..

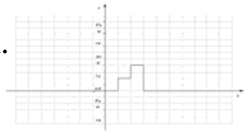
On the basis of these suggestions try to complete the following charts.



A) In the following Cartesian plane you see the original melody represented.

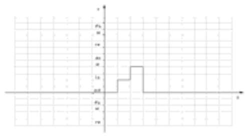
Draw, in the same Cartesian plane, this melodic tune bit:

Identify if there is a translation or a reflection of the original melody in relation to the x or y axis or to the origin of the axis. Give reasons for your answer.



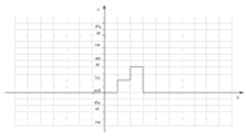
B) In the following Cartesian plane you see the original melody represented.
   Draw, in the same Cartesian plane, this melodic tune bit:

   Identify if there is a translation or a reflection of the original melody in relation to the x or y axis or to the origin of the axis. Give reasons for your answer.



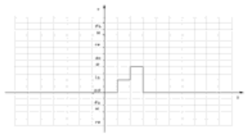
C) In the following Cartesian plane you see the original melody represented.
   Draw, in the same Cartesian plane, this melodic tune bit:

   Identify if there is a translation or a reflection of the original melody in relation to the x or y axis or to the origin of the axes. Give reasons for your answer.



D) In the following Cartesian plane you see the original melody represented.
   Draw, in the same Cartesian plane, this melodic tune bit:

   Identify if there is a translation or a reflection of the original melody in relation to the x or y axis or to the origin of the axis. Give reasons for your answer.



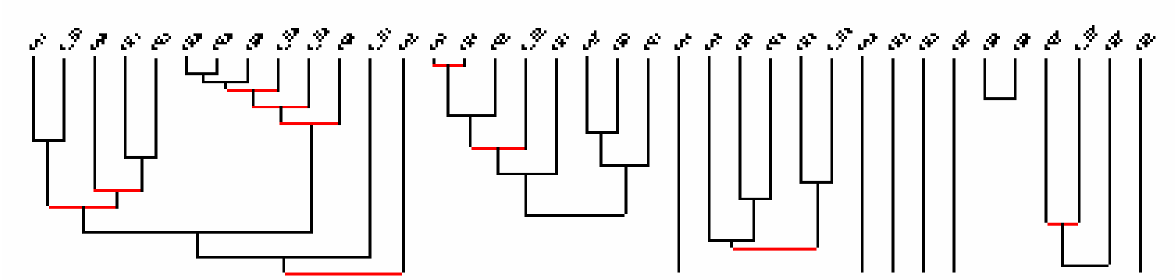
E) In the following Cartesian plane you see the original melody represented.
   Draw, in the same Cartesian plane, this melodic tune bit:

Identify if there is a translation or a reflection of the original melody in relation to the x or y axis or to the origin of the axes. Give reasons for your answer.>>

ANALYSIS OF DATA

The set of questions was met with great interest and enthusiasm by both samples of pupils because they were made curious by the matching of geometrical transformation with music. The students who have elementary music knowledge[11] preferred to look for solutions in the field of music rather than in that of geometry, for example in the first exercise they said there was a translation because there is a pause. The sample of the musician students, in particular, used the term transposition to indicate the translation because in music translating a melody means moving it in time and height, therefore these students identified correctly the term transposition as a synonym of translation. In a double-entry pupils-strategies chart, for each student I have indicated with value 1 the strategies used and with value 0 the unapplied strategies. The collected data were analyzed in a quantitative way, using the implying analysis of the variables of Regis Gras by means of the Chic 2004 software.

Observing the following Chart of Similarities regrouping the two samples examined

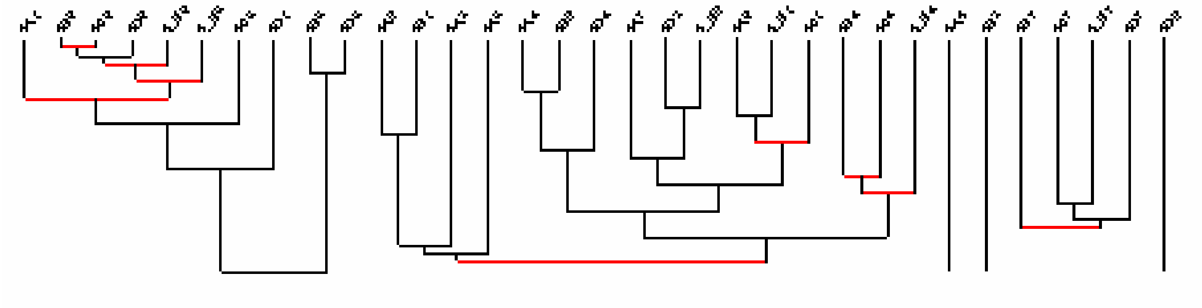


and analyzing all the data collected from the two samples, four typologies of main strategies emerge:

- Identifies the translation in relation to the x axis and the translation in relation to the y axis (i.e. answers the A) and B) questions correctly) but confuses the concept of *translation* with *reflection* in C) and D) questions although answers the E) question correctly.
- Identifies the reflection in relation to the x axis and the reflection in relation to the y axis (i.e. answers the C) and D) questions correctly) but confuses the concept of translation with reflection in A) and B) questions .
- Draws the chart but does not say if there is a reflection or a translation;
- Does not draw the chart of the tune bit but affirms there is a reflection

[11] In the Italian school system a little music theory is studied only in junior middle schools, while professional study of music is entrusted to state music conservatoires.

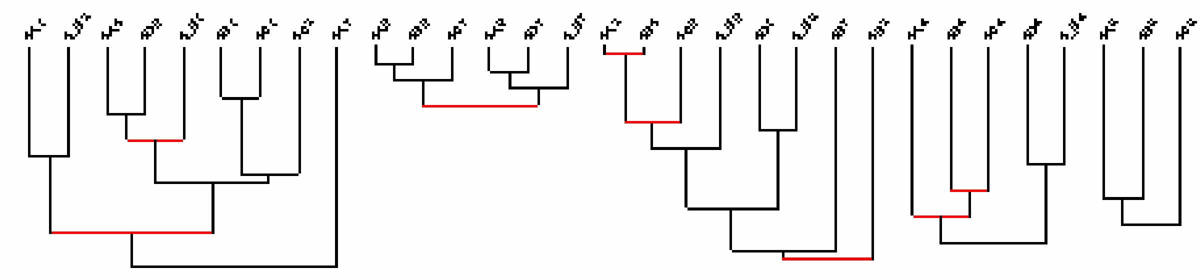
From this first quantitative analysis I have stressed that in general a concept mistake is present between the terms translation and reflection both for musicians and non musicians.
To trace possible different behaviours I analyzed both samples separately and from the analysis of the following chart of similarities came out that the non-musicians sample



chose three typologies of main strategies:

- Identifies the translation in relation to the x axis (4A1) and the reflection in relation to the y axis (4D1), but considers the other charts as *identities*, that is neither *translation* nor *reflection*;
- Identifies the translation in relation to the y axis (4B1) and the reflection in relation to the x axis (4D1) and the reflection in relation to the origin (4E1) and is anyhow able to draw the chart but confuses the concept of *translation* with *reflection*;
- Does not draw the chart of the tune bit but affirms there is a translation.

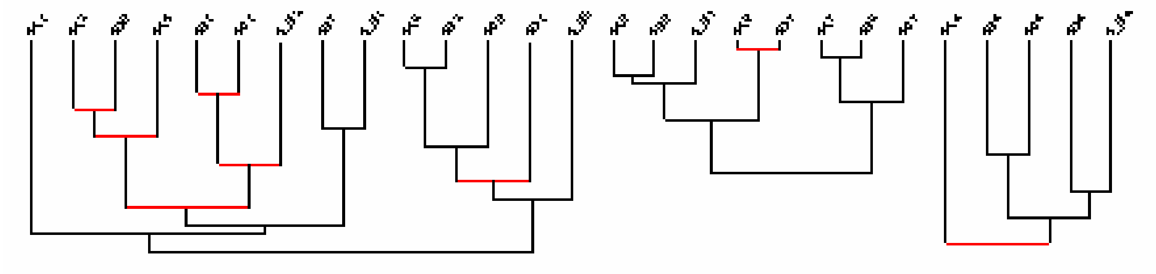
From the analysis of the chart of similarities of the musicians sample five typologies of main strategies emerged:



- Identifies the translation in relation to the x axis (4A1), the translation in relation to the y axe (4B1), the reflection in relation to the x axis (4C1) and the reflection in relation to the origin and is anyhow able to draw the chart but confuses the concept of *translation* with *reflection*;
- Always confuses the concept of *translation* with *reflection*;

- Confuses the concept of *translation* with *reflection* but is able to identify the reflection in relation to the y axis;
- Draws the chart but does not say if there is a reflection or a translation;
- Does not draw the chart of the tune bit but affirms there is a reflection.

Since the musicians sample is formed both by instrumentalists (winds and strings) and pianists I analyzed the sub-sample formed by pianists only and from the analysis of the following chart of similarities we can see that they chose three typologies of main strategies:



- Identifies the translation in relation to the x axis (4A1), the translation in relation to the y axe (4B1), the reflection in relation to the x axis (4C1), the reflection in relation to the y axe (4D1) and the reflection in relation to the origin (4E1) but confuses the concept of *translation* with *reflection*;
- not draw the chart of the tune bit but affirms there is a reflection or a translation but confuses the concept of *translation* with *reflection*;
- Draws the chart but does not say if there is a reflection or a translation.

CONCLUSIONS

Whit this work, I highlighted the interdisciplinary aspect of the geometrical transformation study with music; because, started as creative and intellectual activities since human origins, they are always strictly connected.

In the first section I examined the presence of isometries in History of Music, because they were used in composition technique of the XII century, when polyphonic music started been known, and they had a key role in the musical language development.

I focused on Johann Sebastian Bach, because he sums up, in his huge instrumental and vocal production, all the stylistic features of the transition period, marking the passage from Baroque to modern age.

I examined the *Offerta Musicale,* which is one of the most significant example of the isometries' scientific applications in music.

The work has a theoretical-scientific nature, so without a specific instrumental destination. We find in it a strict application of the isometric transformations with a non-contrived analysis, because Bach used them intentionally in his work. The outcome is both mathematics and musical, an events flow strictly controlled, where, through imagination and inventive, composition is always original and expressive, where the *Tema Regium* , always present, is the common denominator of a kaleidoscopic variety of shapes. In particular, in the *Canon à 4*, translations along x and y axes are emphasized; in the *Canon à 2 Quaerendo invenietis*, Bach uses the reflection as regards the x axis; whereas in the *Canon à 2 Cancrizans*, we have the reflection as regards the y axis; finally, in the *Canon à 2 per tonos Ascendenteque Modulatione ascendat Gloria Regis*, the composer realises another translation along y axis and, above all, the rotation through continuous modulations, with scales ordered in succession of ascendant fifths.

The counterpoint art, and consequently the use of geometrical transformations of the main theme, went through a decline period since the end of the XVIII century till the end of the XIX century. We have to wait till the XX century to have again some examples of geometrical structures applied to music. Geometrical transformations use was completely renewed by the dodecaphonic school of Vienna and by the vanguards after that.

I examined the geometrical transformations use in Pierre Boulez' s *Structures I* , which represents a significant answer to the strong need of a musical language innovation of the twentieth-century owed just to its almost exclusive mathematization. After World War II's horrors, the relation between the *World* and the *Individual* was full of incommunicability and the artist perceived an isolation stronger and stronger; for this reason there was a choice of *expressing oneself* through the rude rationality of the *number* , which involves all the sound features.

In the second part of the work, I examined the sound route, considered as a wave created through objects vibrations and propagates through a means from a place to another, together with music, forms our daily experience and has, generally, a big relapse in the learning process. Furthermore, our body knowledge and the brain functioning help us understanding how to encourage comprehension and storage of teaching experiences during the activity of mathematics teaching-learning, and in this case, through the interaction with musical language.

Sound perception activity and encoding and analysing musical language with mathematics language can different points of view; in this work, I examined the neurophysiologic aspect.

As we know, the learning process is a complex function of the organisms, deriving from their interaction with the environment. In this work I demonstrated that there are some mathematics learning ways, and in particular, geometrical transformations learning ways, that are not used and that involve different human body senses, the hearing included.

From the study of associative learning through hippocampus-amygdale system and other prefrontal areas has been demonstrated that sensory inputs are never monomodal inputs, like only hearing, because they exist in a parallel way; so, at the same time, we have the visual input, the auditory input, the touching sensory input, the smells and the taste. In particular, in the musician, the learning process takes place through all the body, not only through the auditory sensory input. When a child plays an instrument or sings, the musical information he receives is not only connected to space and time context, but to the pleasure too. The more inputs the information has, the easier we learn. So, we understand why mathematics taught at the same time with music is better understood; we just need to use multimodal inputs information. Moreover, if the child (or boy) likes playing music, he'll be apt to study mathematics and to get through his aversion to this subject, thanks to his love for music.

At cerebral level information are not divided, they're not serial (serial is only the signal that goes from periphery to centre); all signals are in parallel. Therefore, in a learning process, imitating hippocampus system and prefrontal areas system, we give the information multimodal inputs, for example through interdisciplinarity, so that it will be easily understood. Interdisciplinarity realizes the concrete application of this way of learning.

The explored contents of the first and second part of the research are the bases of the teaching testing presented in the third section of the work.

From a general point of view, this research contributes to clarifying questions regarding isometric knowledge representations and problems resolution system from students point of view, and it's a contribution to mathematics teaching through interdisciplinary aspect of geometrical transformation study with music.

The research methodology follows the Guy Brousseau's Theory of Situations, and, in order to verify the two hypotheses I realized a teaching testing at Liceo Statale "Regina Margherita" of Palermo, where two different samples of students have been identified: Students from Music Liceo, connected to Conservatorio di Musica di Stato "Vincenzo Bellini"; Students from Liceo Socio-psico-pedagogico. Furthermore, eight couples of students from the III classes have involved (four for each course) with the duty to write their common consideration written after a common agreement with a interviews protocols registration.

The experimental research is based on a comparison between Liceo's students studying music at the Conservatorio and Liceo's students who don't study music at the Conservatorio but who have a basic theoretical-musical knowledge. This research focused on spontaneous concepts about geometrical transformation and their connection with music.

I proposed four sets of questions to both samples examined. In a double entry table "students/strategies", for each student we indicated with 1 the used strategies, and with 0 the strategies not-applied. The recorded data have been analysed in a quantitative way, using the implicative analysis of Regis Gras' variables through the CHIC 2004 software.

Analysing the answers given for the first two questionnaires, we notice that both samples had troubles in identifying and recognizing symmetries and tried to answer the questions using only mathematics language.

Through the quantitative analysis is evident that both samples identify the recursiveness and the rhythm of the shape sequence; analysing the samples separately we see that Liceo Pedagogico students can identify part of the mosaic, whereas most of the students of Liceo Musicale identify the recursiveness of the whole draw.

This happens because the musician, playing a lot with music rhythm while studying a musical instrument, fixes in his mind the technique of recursive repetition; so that, after identifying the starting cell, he's automatically able to rebuild the rhythm structure of the whole mosaic.

Therefore we can affirm that most of the ability of the sample of musicians of identifying the mosaic shapes rhythm lies in the rhythmic structure of the musical language, acquired through the constant study of a musical instrument and that unconsciously creates the mental structures apt to acquire and recognize the recursiveness and the rhythm of the geometrical figures.

The set of questions was met with great interest and enthusiasm by both samples of pupils because they were made curious by the matching of geometrical transformation with music. The students who have elementary music knowledge preferred to look for solutions in the field of music rather than in that of geometry, for example in the first exercise they said there was a translation because there is a pause. The sample of the musician students, in particular, used the term transposition to indicate the translation because in music translating a melody means moving it in time and height, therefore these students identified correctly the term transposition as a synonym of translation. In general, we found a bigger interest, from both samples, for musical exercises, this involved a higher ability in identifying isometries using both geometrical and musical languages.

In order to solve the questions, both samples adopted 4 main strategies typologies. Through a quantitative analysis, generally, we found, for both musicians and non-musicians, a concepts mistake with words "*translation*" and "*reflection*". In order to identify possible different behaviours, I analysed the samples separately. I noticed that the sample of Liceo Pedagogico used 3 main strategies typologies, whereas the sample of the Liceo Musicale adopted 5 main strategies typologies. As the sample of Liceo Musicale is formed by instrumentalists and pianists, I analysed the subgroup formed by pianists only. Through the analysis we see that it used 3 main strategies typologies.

Through a final analysis is evident that both students from the Liceo Pedagogico and from the Liceo Musicale make a conceptual error with the words "*translation*" and "*reflection*" (which could be considered as a "misconception"), this is also revealed in the first two sets of questionnaire, strictly geometrical. Through the analysis of the pianists subgroup emerged that this "misconception" is less present, due to the piano features; in fact, this is an instrument that makes the pianist use both hands symmetrically, making both translations and reflections, acquired unconsciously through neuro-tendon receptors of superior limbs.

Regarding instruments players (woodwinds and strings), through the data analysis emerged that they find more troubles in identifying the reflection as regards the y axes; in fact, while pianists can identify the "y axes" through the piano central Do, the other instrumentalists don't have this reference.

As regards the qualitative analysis, we referred only the fourth questionnaire, proposed, as we already said, to 8 couples of students of the two samples (4 couples each study course) through interviews.

Through the analysis of the behaviour of the couples of Liceo Socio-psico-pedagogico, emerged that students associate the word "reflection" to "identical", and the word "translation" to "shifted". The adopted strategy is to draw the graphs of the melodic incise and compare them with the graph of the original one. Through the analysis of the behaviour of the couples of Liceo Musicale, generally, emerged that students identified correctly the Translations and the Reflections, and that in order to find a solution, we observe that both musical and geometrical language were used.

Finally, we can affirm that with the music's help it's possible, not only to see not only the possible applications of the geometrical transformations, but we can also hear the effect these could have on a melody; this, in my opinion, makes the study of geometry definitely more fascinating.

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