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A study on the comprehension of irrational numbers

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Abstract

Following a theoretical introduction concerning the difficulties that people face for understanding the structure of the basic sets of numbers, we present a classroom experiment on the comprehension of the irrational numbers by students that took place at the 1st Pilot High School of lower level (Gymnasium) of Athens and at the Graduate Technological Educational Institute of Patras, Greece. The outcomes of our experiment seem to validate our basic hypothesis that the main intuitive difficulty for students towards the understanding of irrational numbers has to do with their semiotic representations (i.e. the ways in which we describe and we write them down). Other conclusions include the degree of affect of age, of the width of mathematical knowledge, of geometric representations, etc, for the comprehension of the irrational numbers.

Key words: Irrational and real numbers, learning mathematics

1. Introduction

The empiric comprehension of numbers by children is taking place during the pre-school age and it is based on their practical needs to distinguish the one among many similar objects and to count these objects (Gelman 2003). This initial approach of the concept of number helps children in understanding the structure of natural numbers. For example, it supports them to 'build" the "principle of the next of a given number" and therefore to conclude the infinity of natural numbers (Hartnett & Gelman 1998). It also supports the development of strategies for addition and subtraction based on counting (Smith et al. 2005), the comparison and order among the naturals, etc. The above approach is strengthened during the first two years of school education, where the natural numbers constitute a basic didactic target.

The decimals and fractions are introduced later, after the second year of primary school, while the negative numbers are usually introduced at the first year of high-school. Mathematically speaking, the set **Q** of rational is an extension of the set **N** of natural numbers that could be attributed to the necessity for subtraction and division to be closed operations. However **Q** is not simply a bigger set than **N**, but it actually has a completely different structure. In fact, while between any two natural numbers there exist at most finitely many other natural numbers (i.e. **N** is a discrete set), between any two rational numbers there always exists an infinite number of other ones (i.e. **Q** is an everywhere dense set).

It is widely known and well indicated by researchers that students face many difficulties for the comprehension of rational numbers (Smith et al. 2005). It seems that most of these difficulties have to do with a false transfer of properties of natural numbers to the set of rational numbers (Yujing & Yong-Di 2005, Vamvakousi & Vosniadou 2004 and 2007). For example, some students believe that "the more digits a number has, the bigger it is" (Moskal & Magone 2000), or that "multiplication increases, while division decreases numbers" (Fischbein et al. 1985). The idea of "the discrete" restricts also the understanding of the structure of rational numbers. In fact, many students believe that, as it happens for the natural numbers, the "principle of the next number" holds for the rational numbers as well (Malara 2001, Merenluoto & Lehtinen 2002).

Another characteristic of the rational numbers that possibly affects their understanding is that each one of them can be written in several ways, e.g. we can write $\frac{1}{2}=\frac{2}{4}$ =….=0,5. In fact, novices have the general trend to categorize objects by their surface rather, than by their structural characteristics (e.g. Chi et al. 1981), which for mathematics means that they have a great difficulty in understanding that different symbols may represent the same object (Markovitz & Sowder 1991). As a result of this many children consider that the different representations of a rational number correspond to different numbers (Khoury & Zarkis 1994, O'Connor 2001) and, even more, that decimals and fractions are disjoint to each other subsets of **Q**. The wrong perceptions about fractions and decimals have taken roots as habits even to elderly people, who usually consider fractions as parts of a set (e.g. $\frac{3}{4}$ of something), and decimals as being more similar to the natural numbers.

Such misunderstandings are sometimes strengthened by certain obscurities appearing in school books and therefore a great care is needed by authors to avoid them For example, in page 115 of the mathematics book for the first class of the high-school of lower level (Gymnasium) in Greece (Vandoulakis et al. 2008) we read: "Rational numbers are all the already known to us numbers: natural numbers, decimals and fractions, together with the corresponding negative numbers". This definition could make students to believe that all decimals are rational numbers, or that decimals and fractions are not related to each other rational numbers, etc. Notice that these wrong beliefs, when embedded, it is not so easy to be revised later.

It has been observed that students of all levels are not in position to define correctly the notions of rational, irrational and real numbers, neither they are able to distinguish among the integers and the above numbers (Hart 1988, Fischbein et al. 1995). In general, the concept of the rational number remains isolated from the wider mathematical knowledge of real numbers (Bryan 2005, Toepliz 2007).

2. Methods of constructing the real numbers

In order to develop the theoretical basis of our study, it helps first to attempt a brief presentation of the known methods of constructing the set **R** of real numbers. The methods in which one can extend **Q** to **R** include:

- The infinite decimal representation of real numbers
- The Dedekind's method
- The method of Cauchy

The first of the above methods will be discussed in the next section.

For the second method, we recall that a *Dedekind cut* (or simply cut) is defined to be an ordered pair (A, B) of non empty subsets A and B of **Q**, such that $A\cup B$=**Q**, $A\cap B=\Phi$, and $\forall a\in A,\ \forall b\in B\Rightarrow a<b$.

Notice that the definition of Dedekind cuts is actually an adaptation of the definition of the analogy among geometric magnitudes given by Eudoxus during the 4th century B. C. (Artemiadis 2000: p. p. 485-486), i.e. approximately 22 centuries earlier (Dedekind presented his ideas at the late 1800's)!

It is well known that for each Dedekind cut (A, B) there exists at most one rational q such that $q\geq a$, $\forall a\in A$ and $q\leq b$, $\forall b\in B$ and this happens, if, and only if, A has a maximal, or B has a minimal element. In this case (A, B) is called a *rational cut* corresponding to the rational number q.

However there exist Dedekind cuts not satisfying the above property. For example, if we take B={x∈ **Q**: x>0, $x^2>2$} and A=**Q**-B, it can be shown that (A, B) is a Dedekind cut, where A has not a maximal element and B has not a minimal element. This means that given x in A, there always exists y in A with y>x, and given x in B, there always exists y in B with y<x. It becomes therefore evident that, if we consider a new number, say a, such that $a^2=2$, we can approximate it as much as we want, either from the left, or from the right, by rational numbers. Cuts like the above are called *irrational cuts*.

When we define the rational numbers as quotients of integers, we face the problem that different quotients determine the same number (e. g. $\frac{1}{2}=\frac{2}{4}$). There is a similar equivalence among Dedekind cuts. In fact, we say that (A_1, B_1) is equivalent to (A_2, B_2), if given a_1 in A_1 we have that $a_1\leq b_2$ for all b_2 in B_2, and given a_2 in A_2 we have that $a_2\leq b_1$ for all b_1 in B_1.

The sum $(A_1, B_1) + (A_2, B_2)$ of two given cuts is defined to be the cut (A, B), such that for each b in B we can write $b=b_1+b_2$, with b_1 in B_1 and b_2 in B_2, while $A=\{q\in \mathbf{Q}:q<b, \forall b\in B\}$.

A cut (A, B) is called a *positive cut*, if A contains at least one positive rational number. The product $(A_1, B_1) \times (A_2, B_2)$ of two cuts, where at least one of them is positive, is defined to be the cut (A, B), such that for each b in B we can write $b=b_1b_2$, with b_1 in B_1 and b_2 in B_2, while $A=\{q\in \mathbf{Q}: q<b, \forall b\in B\}$.

Denote by F the set of all equivalence classes of Dedekind cuts and let x, y be in F. Then, if x is the class of (A_1, B_1) and y is the class of (A_2, B_2), the sum x+y is defined to be the class of $(A_1, B_1) + (A_2, B_2)$. Also, if at least one of x and y contains a positive cut (then is said to be a positive class), then the product x.y is defined to be the class of $(A_1, B_1) \times (A_2, B_2)$. Further, if both x and y are negative then x.y is defined to be the product (-x).(-y), while, if at least one of x and y is 0, then x.y is defined to be 0. It can be shown that "+" and "x" are well defined operations in F (i.e. independent from the choice of the representatives of the classes involved).

Then (F, +, x) becomes a field, known as the *Dedekind field* (Bagett 2006: Appendix). The Dedekind field .is defined (up to isomorphism)[1] to be the field **R** of real numbers.

For the third method, we recall that a sequence (a_n) of rational numbers is called a *Cauchy sequence*, if

$$\forall e\in \mathbf{Q},\ e>0,\ \exists n_o\in \mathbf{N}:\ \forall n,m\geq n_o \Rightarrow |a_n-a_m|<e.$$

In the set M of all Cauchy sequences we define addition and multiplication in the obvious way. Then M becomes a ring and the set I of all null sequences is an ideal of M. It can be shown that the factor ring M/I is a field. For each $q\in \mathbf{Q}$ we correspond the class of M/I where the constant sequence $a_n=q$ belongs. In this way **Q** is embedded (up to isomorphism)* to M/I. The field M/I is defined to be the field **R** of real numbers. Mathematically speaking this is a smart definition of **R**. Nevertheless our experience shows that many university students are not in position to approach in comfort the real numbers in this way, because the infinite processes and the concept of limit are always considered by them as sources of difficulties.

They are also known the following two methods of constructing **R** without making use of **Q**:

- The definition of **R** as the set of all equivalence classes of almost linear functions $\mathbf{Z}\rightarrow\mathbf{Z}$, where **Z** denotes the set of integers (S. Shanuel's definition)

We recall that a function f: $\mathbf{Z}\rightarrow\mathbf{Z}$, is called *an almost linear function*, if the set of numbers |f(m+n)-f(m)-f(n)|, m, n$\in$ **Z** is bounded. Further two almost linear functions f and g are called "equivalent", if the set of numbers |f(n)-g(n)| is bounded. A real number, say a, is defined to be the equivalence class of the function f(n)=[an], where [an] denotes the greater integer which is $\leq$ an. The sum and product of the real numbers corresponding to the almost linear functions f and g are defined to be the equivalence classes of the almost linear functions f + g and f o g respectively.

- The axiomatic definition of **R** as the unique (up to isomorphism) complete ordered field

We recall that an ordered field K is called *complete*, if every non empty subset of K that has an upper bound has a least upper bound (supremum) in K. It can be shown that there exists a complete ordered field (e.g. the Dedekind field) and that any two complete ordered fields are isomorphic (e.g. Bagget 2006: Chapter 1 and Appendix, or Mac Lane & Birkoff 1988). Thus **R** is defined to be the unique (up to isomorphism)* complete ordered field.

3. Obstacles appearing towards the understanding of the irrational numbers

As Fischbein et al. (1995) observe, little attention is paid to irrational numbers in school mathematics, which is mainly conceived as an ensemble of solving techniques and proving procedures of theorems. The

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[1] During the introduction of the "new mathematics" in school education (1960-1980) elements from theory of the basic algebraic structures (groups, rings, fields, vector spaces) were taught at the upper high-school level. Nevertheless the basic idea behind all these, i.e. the concept of isomorphism, which was in fact the new "message" of mathematics during the 20th century, was not taught (at least in our country, Greece). Thus students lost the opportunity to "receive" this message and, for example, to understand deeply why $\mathbf{R}^2$ and the field **C** of complex numbers is the same thing in practice.

Of course there is no doubt that the introduction of the "new mathematics" in school education was proved to be a complete failure (e.g. Kline 1990). However there is indeed a doubt about the reasons for which this happened. According to our opinion (and not only) one of the basic reasons was the complete absence of practical examples from teaching and cues - like the concept of isomorphism- about the usefulness of the new material taught..

idea of mathematics as a coherent, structurally organized body of knowledge is not systematically conveyed to students.

The irrational numbers are usually introduced at the second class of high-school of lower level and their comprehension presupposes the complete understanding of rational numbers by students. Therefore, if this has not been achieved yet (as it frequently happens), students are facing many difficulties in approaching this new kind of numbers. Apart from the above, it seems that there exist further inborn (scientific and cognitive) obstacles making the comprehension of irrationals even more difficult (Herscovics 1989, Sierpinska 1994, Sirotic & Zarkis 2007, etc). Nevertheless, studies that focus on the comprehension and didactic approach of the irrationals are very rare indeed.

Fischbein et al. (1995) made the hypothesis that possible obstacles for the complete understanding of the irrational numbers are the intuitive difficulties appeared also in the history of mathematics for the discovery of them, that is the perception of *incommensurable magnitudes* and the *"property of the continuous"* of the set of real numbers **R** (i.e. the fact that, although **Q** is an everywhere dense set, it cannot cover all the points of a given interval, as it happens with **R**). Although, as they accepted by themselves, the results of their experiments didn't seem to validate completely their hypothesis, they suggest that these intuitive difficulties ought to be projected rather, than to be ignored, or under-evaluated, by teachers, because in this way students could have a better approach to the concept of the irrational numbers.

Our hypothesis, although it takes seriously under consideration the above parameters, i.e. the incomplete understanding of rational numbers and the intuitive difficulties mentioned by Fischbein et al. (1995), as possible obstacles for the comprehension of the irrational numbers, it is based on a different argument. Namely, we claim that the main obstacle has really to do with the *semiotic representations of irrationals*, i.e. the ways in which we describe and we write them down.

From the brief presentation of methods for constructing **R,** attempted in the previous section, it becomes evident that the only compatible to the school mathematics method for presenting the irrational numbers to high-school students (and next to define **R**) is the use of their infinite decimal representation. The other methods are out of question, because they involve mathematics that is not taught at school.

For a successful introduction to irrational numbers in this way two prerequisites are necessary:

- First, students must already have understood that commensurable decimal (periodic) numbers and fractions are the <u>same</u> numbers written in different ways. For this, they must be able to convert in comfort periodic numbers, even with a non-periodic decimal part (mixed periodic numbers), to fractions and vice versa (i.e. to know that a fraction is a different expression of the division of two numbers).

Notice that in the mathematics book of the first class of Gymnasium in Greece the mixed periodic numbers are not defined (Vandoulakis et al. 2008: p.p. 135-136) and only two examples are presented (see unsolved Exercises 2 and 3 in p. 136), after the conclusion that every periodic decimal number can be written in the form of a fraction!

- The definition of the incommensurable (disproportionate) numbers must be stated with great care and austerity, in order to avoid unpleasant misunderstandings by students.

The teacher must have in mind (and transfer it to students) that given a finite approximation of a decimal number with infinitely many decimal digits, one could not be sure that this approximation corresponds to an irrational, when it could be a rational number with a long period. For example the number $\frac{144}{233}$=0,618025….. has a period of 232 digits.

It is characteristic that Fischbein et al. (1995: p.p. 32-33) in one of the questions of their experiments expected from students (high-school students and prospective teachers) to identify that 0,121221…. is an irrational number. But how one could decide about this, when not knowing the complete sequence of the decimal digits of the above number?

Sometimes students ask the following question: *"Which periodic number has the maximal period"?* This question gives a good opportunity to clarify all the above remarks and therefore we suggest that the teacher must pursue and "push" students to ask it.

For another related example, we refer to page 187 of the mathematics book for the second class of Gymnasium in Greece (Vlamos et al. 2007), wherefrom we read: "It can be shown that π is an irrational number, that is a decimal number with infinitely many decimal digits that are not obtained with a concrete process". According to the above statement 2,0013113111311113…… (see question 5 of our questionnaire in the next section) could be considered as a rational number, since its decimal digits are obtained with a concrete process!

The problems however are increased, when we arrive to the natural, but crucial, question (usually asked by students): *"Which numbers can be written in the form of an incommensurable decimal number?"*

At the lower high-school level students learn that this happens with the square roots of positive rational numbers that they cannot be exactly determined (i.e. they have not an exact price). Later, usually at the upper high-school level (which is called Lyceum in Greece), they learn that this also happens with the n-th roots, $n \in \mathbf{N}$, $n \geq 2$. However the converse is not true, since they are incommensurable decimal numbers that cannot be written as roots that they cannot be exactly determined, or in a more general expression they are not roots of an algebraic equation with rational coefficients. Thus we arrive to the concept of the *transcendental numbers*, for which, apart from some characteristic examples, like π and e, we don't know many details.

As a consequence of the above fact, it is rather inevitable to remain some blanks to students (and even to elderly people) towards the comprehension of the irrational numbers. The teacher of course must speak at some stage to students about the transcendental numbers, in order to complete the "puzzle" of real numbers. A good opportunity for this is given at the upper high-school level during the repetition of the already known sets of numbers and before the teaching of complex numbers.

4. The experimental study

The targets of the classroom experiment that we are going to describe below were the following:

- To check our basic hypothesis that the semiotic representations of irrationals is the main difficulty towards their understanding by students.
- To validate the existence of the obstacles mentioned also by other researchers, i.e. the incomplete understanding of the rational numbers and the intuitive difficulties with the perception of incommensurable magnitudes and the "property of the continuous of **R**".
- To investigate if other factors like the geometric constructions and the representation of irrational numbers on the real axis, the age, the width of mathematical knowledge of students, etc, affect the comprehension of the real numbers.

Our basic tool in this experiment was a questionnaire of 15 questions, properly designed with respect to the targets that we described above. The questionnaire was forwarded at the end of school year 2008-09 to 78 students being at the second class of the 1st Pilot Gymnasium of Plaka, in Athens, i.e. a few months after they had been taught the irrational numbers for first time. The above Gymnasium is considered to be one of the best public high-schools of lower level in the area.

The same questionnaire was also forwarded, approximately at the same time, to 106 students of the Schools of Technological Applications (i.e. prospective engineers) and Management and Economics of the graduate Technological Educational Institute (TEI) of Patras, Greece, being at their first term of studies. The majority of these students, according to their results at the exams for the entrance in tertiary education, corresponded to graduates of the secondary education of mediocre level. The students of TEI however, apart from their experience from secondary education, they attended also a brief recapitulation of the basic sets of numbers at their first term course "Advanced Mathematics".

In both cases the time given to students to complete the questionnaire was approximately one hour. We present below the 15 questions accompanied by some comments, and the percentages (with unit approximation) of the correct answers that we received, separately for Gymnasium and TEI. The answers were characterized as correct (C) and wrong (W). In cases of incomplete answers the above characterization created some obscurities, which however didn't affect significantly the general picture of students' performance.

Question 1: Which of the following numbers are natural, integers, rational, irrational and real numbers?

$$-2\ ,\quad -\frac{5}{3}\ ,\quad 0\ ,\quad 9{,}08\ ,\quad 5\ ,\quad 7{,}333\ldots,\ \pi=3{,}14159\ldots,\quad \sqrt{3}\ ,\quad -\sqrt{4}\ ,\quad \frac{22}{11},\ 5\sqrt{3}\ ,\quad -\frac{\sqrt{5}}{\sqrt{20}}$$

$$,\quad \left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right),\quad -\frac{\sqrt{5}}{2},\quad \sqrt{7}-2\ ,\quad \sqrt{\left(\frac{5}{3}\right)^{2}}$$

With the above question we wanted to check if students were in position to distinguish the category in which a given number belongs. The following matrix gives the percentages of wrong answers given by students:

	0 W	1-2 W	3-5 W	6-10 W	>10 W
Gymnasium	0	5	22	21	52
T. E. I	0	11	33	36	20

The most common mistakes were the identification of the symbol of fraction with rational and the symbol of root with irrational numbers. The failure of many students to recognize that all the given numbers were real numbers was really impressive. Notice that no students gave correct answers for all cases.

The following questions 2-5 were designed in order to check the degree of understanding of rational numbers by students.

Question 2: Are the following inequalities correct, or wrong? Justify your answers.

$\frac{2}{3} \prec \frac{14}{21}$, $\frac{2001}{1001} > 2$.

In this case we wanted to examine if students were able to check the order in **Q**. We considered as correct only the answers accompanied by a satisfactory justification. The percentages of the answers received are presented in the following matrix:

	2C	1C	2W
Gymnasium	50	49	1
T.E.I.	78	10	12

Notice that most students answered after converting the fractions to decimals. This means that they felt more comfortable to work with decimals rather, than with fractions.

Question 3: Which is the exact quotient of the division 5:7?

The quotient of the division has a period of 6 decimal points and therefore it was difficult for students to give the correct answer in the form of a periodic decimal number. The expected answer was that the exact quotient of the division is the fraction $\frac{5}{7}$. The answers received were the following (only 3 correct answers by students of TEI!):

	C	W
Gymnasium	14	86
T.E.I.	3	97

Notice that, in contrast to what happens in primary school (e.g. Dimou et al. 1984: p. 88), a fraction is not presented in the mathematics books of high-school in Greece as the exact quotient of the division of the numerator by denominator.

Question 4: Convert the fraction $\frac{7}{3}$ to a decimal number. What kind of decimal number is this and why we call it so?

Here we wanted to check if students were able to convert a fraction to a decimal number and to recognize a periodic decimal number of the simplest form, i.e. with a period of one digit. The answers received were the following:

	2C	1C	2W
Gymnasium	41	18	41
T.E.I.	73	22	5

Question 5: Are 2,8254131131131… and 2,00131311311131111… periodic decimal numbers? In positive case, which is the period?

The first is a mixed periodic number with period 131. The second is not a periodic number, although its decimal digits are repeated with a concrete process: 00, 13, 131, 1311, 13111, etc. No student noticed what happens in the second case, but the negative answer was considered as correct. This is a characteristic example supporting our hypothesis about the semiotic representations of irrational numbers.

The answers received were the following:

	2C	1C	2W
Gymnasium	23	49	28
T.E.I.	58	39	3

Question 6: Find the square roots of 9, 100 and 169 and describe your method of calculation.

Here we wanted to check if students were in position to calculate in comfort the square root of a square positive integer by using the corresponding definition. A difficulty was observed for the calculation of $\sqrt{169}$ .The answers are shown in the below matrix:

	3C	2C	1C	3W
Gymnasium	69	19	3	9
T.E.I.	75	25	0	0

Question 7: Find the integers and the decimals with one decimal digit between which lies $\sqrt{2}$. Justify your answers.

Our target here was to check the comfort of students for the approximate calculation of square roots. The difficulty of students of TEI to give a decimal approximation of $\sqrt{2}$ was impressive. The answers received are the following:

	2Σ	1Σ	2Λ
Gymnasium	59	15	26
T.E.I.	18	79	3

Notice that the known practical method (similar to division) for calculating the square root of a positive rational, although it was presented (without any explanation) in the old book of mathematics of the second class of Gymnasium in Greece (Papamichail et al. 1981: p.p. 154-155), it is not presented in the new one (Vlamos et al. 2007). We shall agree of course that the presentation of the above method without any explanation (the proper explanation is not in fact so easy to be given) doesn't help towards a better understanding of the concept of the square root, but some times it is very useful in practice, especially when we are not using a calculator.

Question 8: Characterize the following expressions by C if they are correct and by W if they are wrong: $\sqrt{2}$ =1,41 , $\sqrt{2}$ =1,414444…, $\sqrt{2} \approx$ 1,41, there is no exact price for $\sqrt{2}$.

The matrix of the given answers is the following:

	4C	3C	2C	1C	4W
Gymasium	17	41	34	4	4
T.E.I.	3	33	47	14	3

The low percentage of right answers in all cases reveals the difficulty that students have in writing correctly an irrational number, thus giving another strong support to our hypothesis about the semiotic representations of irrational numbers.

The following questions 9-12 were designed in order to check the perception of density of rational and irrational numbers by students in an interval with end points irrational numbers, or rational numbers in several forms (integers, fractions, decimals).

Question 9: Find two rational and two irrational numbers between $\sqrt{10}$ and $\sqrt{20}$. How many rational numbers are there between these two square roots?

The matrix of students' answers is the following:

	3C	2C	1C	3W
Gymnasium	14	14	46	26
T.E.I.	50	41	6	3

Answers involving only one rational, or irrational, number were considered as wrong. Many students (especially from high-school) answered that they are only finitely many rational numbers between $\sqrt{10}$ and $\sqrt{20}$.

Question 10: Find two rational and two irrational numbers between 10 and 20. How many irrational numbers are there between these two integers?

The matrix with the students' answers is the following:

	3C	2C	1C	3W
Gymnasium	24	22	31	23
T.E.I.	48	42	7	3

While there was no significant change to the answers of the students of TEI with respect to the previous question, for high-school students the correct answers increased here, where the end points of the interval were integers. This was expected, because, even after the introduction of the rational numbers, integers continue to play an important role in teaching and examples (Greer & Verschafel 2007).

Question 11: Are there any rational numbers between $\frac{1}{11}$ and $\frac{1}{10}$? In positive case, write down one of them. How many rational numbers are there between the above two fractions?

Many students, especially from high-school, answered that there is no rational number between $\frac{1}{11}$ and $\frac{1}{10}$, because these are two successive (!) fractions (improper transfer of the corresponding property of natural numbers to fractions). On the other hand a considerable number of students converted the given two fractions to decimal numbers in order to arrive to the correct answers. The given answers are the following:

	2C	1C	2W
Gymnasium	23	24	53
T.E.I.	68	30	2

Question 12: Are there any rational numbers between 10,20 and 10,21? In positive case, write down one of them. How many rational numbers are in total between the above two decimals?

Here again many students considered 10,20 and 10,21 as successive decimals (!), thus transferring improperly the corresponding property of natural numbers to decimals. Nevertheless the correct answers increased with respect to the previous question, which means that students had achieved earlier a better understanding of decimals rather, than of fractions. Students, who answered correctly, were orally asked why there exist infinitely many rational numbers between 10,20 and 10,21. A typical answer was: "*After 10,20 there exist 10, 201, 10,209, 10,2099 and as many other digital numbers, as we want, smaller than 10,21*". The above answer is an argument of density, but it is only a special case. The matrix with the students' answers is the following:

	2C	1C	2W
Gymnasium	44	31	25

T.E..I.	71	27	2

Question 13: Characterize the following expressions as correct or wrong. In case of wrong ones write the corresponding correct answer.

$\sqrt{3+5}=\sqrt{3}+\sqrt{5}$, $\sqrt{3\cdot 7}=\sqrt{3}\cdot\sqrt{7}$, $\sqrt{\frac{2}{9}}=\frac{\sqrt{2}}{3}$, the unique solution of the equation $x^2=3$ is $x=\sqrt{3}$, $\sqrt{\left(1-\sqrt{17}\right)^2}=1-\sqrt{17}$

The target here was to investigate students' ability of using correctly the properties of irrational numbers. The matrix with the answers received is the following:

	5C	4C	3C	2C	1C	5W
Gymnasium	3	19	35	29	14	0
T.E.I.	20	64	9	6	1	0

The majority of correct answers were given for the first 2 cases, while the majority of wrong ones were given for the fourth case.

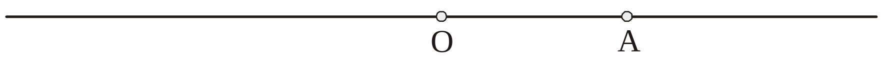
When students were asked why they considered that $x=\sqrt{3}$ is the unique solution of the equation $x^2=3$, they asked: *"Because we know that the square root of 3 is a positive number, such that $x^2=3$"*. In fact, this is exactly the definition of the square root of a positive number given in page 41 of the school book of mathematics of the second class of Gymnasium in Greece (Vlamos et al. 2007).

It becomes therefore evident that the restriction of the definition of the square root to positive numbers only creates a confusion to students, although in the school book (Vlamos et al. 2007) they are some hints about the solution of equations of the form $x^2=a$, (e.g. Exercise 6 in page 44: Find the positive numbers satisfying the equations $x^2=9$,………, $x^2=\frac{100}{81}$, Exercise 4 in page 48: Solve the equations $x^2=0$, $x^2=5$, $x^2=-3$, $x^2=17$, etc).

One could claim that the above restriction is necessary in order to be able to consider $f(x)=\sqrt{x}$, $x>0$, as a function of x. Nevertheless we can pass through it easily, if we accept that a positive number x has two square roots: A positive one, which is symbolized by $\sqrt{x}$ and its opposite $-\sqrt{x}$, which is of course a negative number. This is accepted so in several counties, including England.

We believe that the rejection of the negative square root is an arbitrary restriction that imposes unnecessary difficulties for the teaching of roots of higher order not letting students to approach the roots as an inverse process of the process of raising to a power. For example, it is completely unreasonable to accept that $\sqrt[3]{-8}$ does not exist [as it happens in the book of mathematics of the first class of upper high-school level in Greece (Andreadakis et al. 2007: p.p. 44-45)], mainly because we always expect from roots to be positive numbers!

Question 14: In the real axis of the below figure line segment OA represents the unit of measuring the lengths. Construct, by making use of ruler and compass only, the line segments of length $\sqrt{2}$ and $\sqrt{3}$ respectively and find the points of the given axis corresponding to the real numbers $\sqrt{2}$ and $-\sqrt{3}$.



Here we wanted to investigate the students' ability to construct incommensurable magnitudes and to represent irrational numbers on the real axis. The answers of students of TEI were really an unpleasant surprise. Nobody constructed $\sqrt{3}$ correctly, only two of them constructed $\sqrt{2}$ and only one found the point corresponding to it on the real axis! On the contrary, the high-school students, recently taught the corresponding geometric constructions, had a much better performance. The above phenomenon could be considered as one of the negative consequences of neglecting the teaching of Euclidian Geometry at the upper classes of high-school (Lyceum) in Greece (e.g. Voskoglou 2007).

The matrix with the answers received by students is the following:

	3C	2C	1C	3W
Gymnasium	27	20	13	40
T.E.I.	0	1	1	98

Question 15: Is it possible for the sum of two irrational numbers to be a rational number? In positive case give an example.

Here the superiority of correct answers of the students of TEI was impressive, obviously because of their greater experience. Correct answers were considered only these accompanied by an example.

The matrix of the given answers is the following:

	C	W
Gymnasium	5	95
T.E.I.	80	20

Next, and in order to achieve a statistical representation of the experiment's data, we proceeded to the graduation of the 78+106=184 in total completed questionnaires by allowing 2,5 units to question 14, 2 units to questions 1 and 13, 1,5 unit to questions 5, 7 and 8 and 1 unit to each one of the other questions (20 units in total). The above distribution of units to each question was decided before carrying out the experiment, according to its difficulty and the estimated mean time needed to be answered by students. The matrix of frequencies of the marks obtained is given below, separately for high-school and TEI students and in total:

Mark	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Gymn.	0	0	1	5	9	8	9	5	6	6	3	8	5	2	4	3	1	3
T.E.I.	1	2	0	4	0	5	8	20	17	9	12	13	5	7	3	0	0	0
Total	1	2	1	9	9	13	17	25	23	15	15	21	10	9	7	3	1	3

The means obtained, with approximation of two decimal digits, are 9,41 for high-school, and 9,46 for TEI students. The great accumulation of marks of TEI students between 8 and 12 is remarkable, while the marks of high-school students have a more uniform distribution. All these can be observed better through the matrices of frequencies of marks and the cyclic diagrams (pies) of the percentages of marks presented below, separately for students of Gymnasium and of TEI.

I) Gymnasium

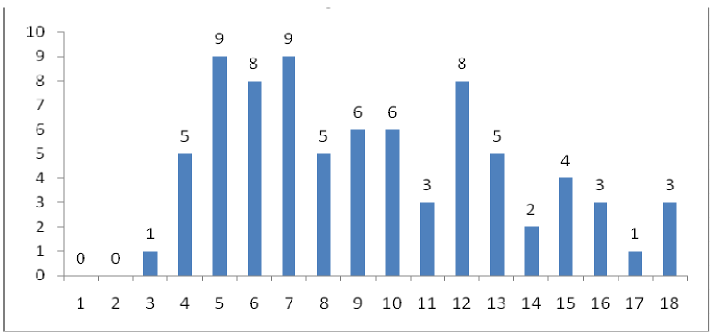


Figure 1a: Frequencies of marks

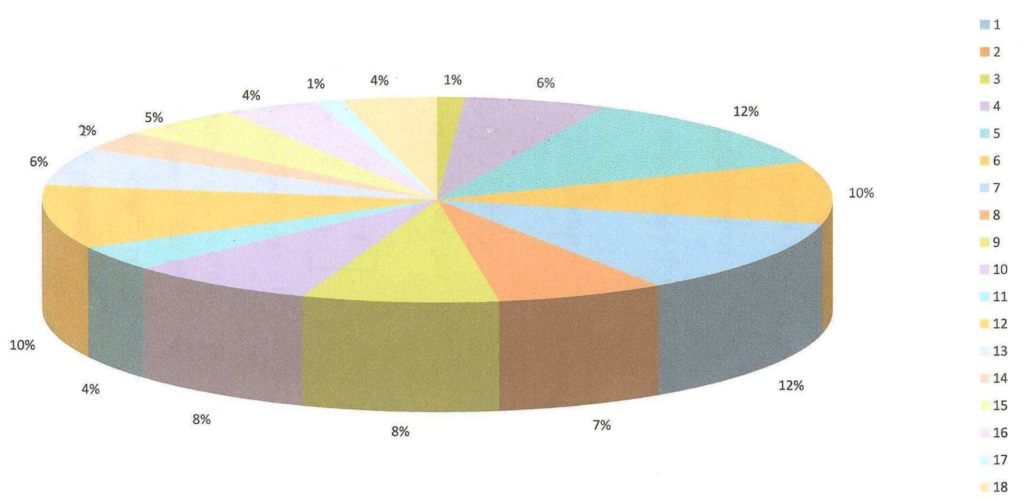


Figure 1b: Cyclic diagram of the percentages of marks

II) T. E. I.

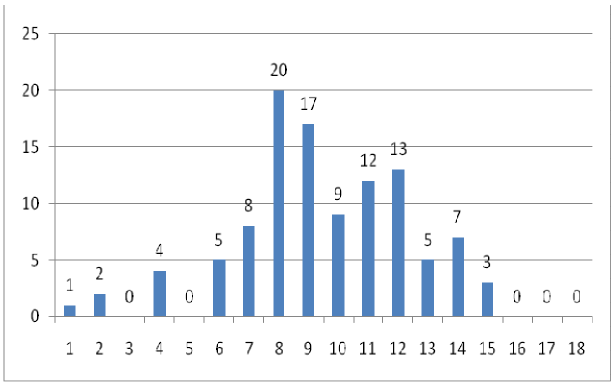


Figure 2a: Frequencies of marks

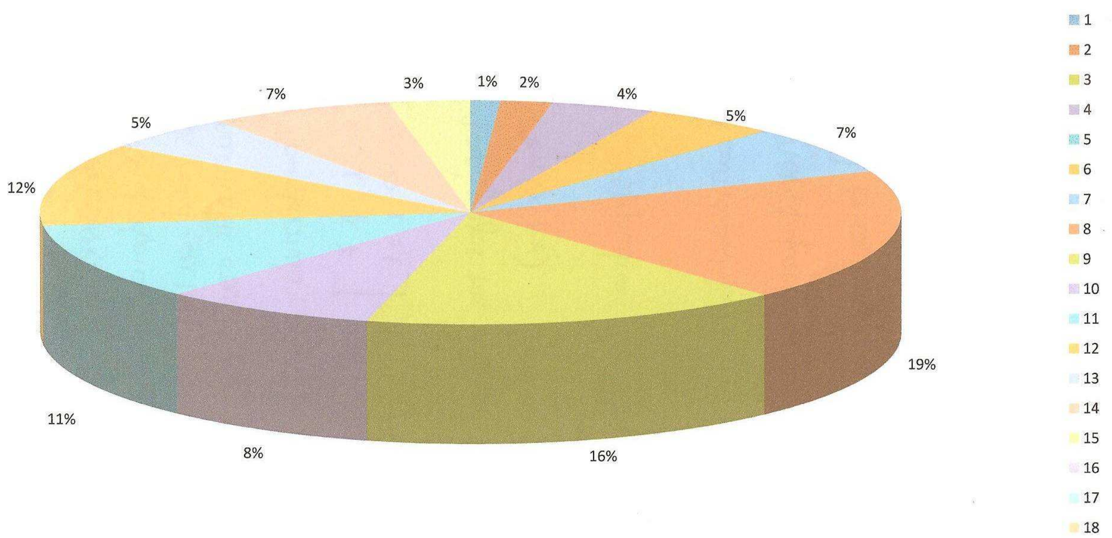


Figure 2b: Cyclic diagram of the percentages of marks

Finally, calculating the first and third quarters (Q_1=7 and Q_3=12 respectively) and the median (M=9) we constructed the “Five Number Summary” shown in Figure 3, that gives a concrete view of the distribution of frequencies of marks of our total sample.

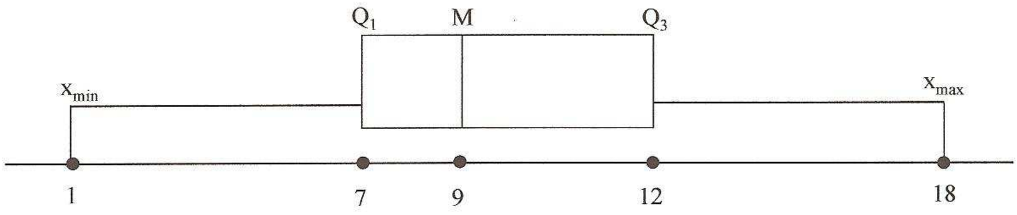


Figure 3: The “Five Number Summary” of the total sample

We observe that the median is lying to the left side of the orthogonal, which means that there exists an accumulation to the low marks.

5. Conclusions and didactic suggestions

The general conclusions obtained through the evaluation of our experiment’s data are the following:

- The understanding of rational numbers was proved to be incomplete by many students (questions 1-5 and 9-12). In general students worked in more comfort with decimals rather, than with fractions (questions 11, 12, etc). Further, students who failed to give satisfactory answers to questions 1-5 and 9-12, failed also in answering satisfactorily the rest of the questions. This obviously means that, the incomplete understanding of rational numbers is in fact a great obstacle for the comprehension of irrational numbers.
- Our basic hypothesis about the intuitive difficulties with the semiotic representations of irrational numbers seems to be validated (questions 5, 8, 13, etc).
- The density of rational and irrational numbers in a given interval doesn’t seem to be embedded properly by a considerable number of students, especially by those of high-school (questions 9-12).
- The students of TEI showed a complete weakness to deal with processes connected to geometric constructions of incommensurable magnitudes and to the representation of the irrational numbers on the real axis (question 14). However this didn’t prevent them in answering satisfactorily the other questions.
- It seems that the age and the width of mathematical knowledge affect in a degree the comprehension of the real numbers. In fact, although the majority of the TEI students corresponded to mediocre graduates of secondary education, the superiority of their answers was evident in most of the questions (apart from 3, 7, 8 and 14). However this superiority was not illustrated by the means of the marks obtained. The reasons for this were the almost complete failure of the TEI students in answering question 14, whose graduation was the greatest one (2,5 units) and the superiority of high-school students in obtaining high marks (15-18).

We underline also the following didactic suggestions, which, according to our opinion, could improve the comprehension of real numbers by students.

- The definition of rational numbers must be given when students have already clarified that fractions and periodic decimal numbers are the same numbers written in different ways. For this, they must embed that the result of the division of the numerator by the denominator of a fraction is always a periodic number, and vice versa they must be able to convert in comfort, by using the proper equations, a periodic number to a fraction, even when it contains a non periodic decimal part (mixed periodic number).
- The concept of incommensurable (non periodic) decimal numbers must be the basis for the introduction to the irrational numbers and the definition of **R**. Their definition must be given with a great care by teacher: if the infinite decimal digits of a decimal number are repeated with a concrete process, this does not mean that it is necessarily a periodic number (e.g. second case of question 5 of our questionnaire). Further students must understand that, given a finite approximation of a decimal

number with infinitely many decimal digits, one cannot be sure whether or not this approximation corresponds to an irrational number, even if its decimal digits "seems" to be repeated in a random way, since it could be a rational number with a long period.

- It is also important for students to embed that in many cases the writing of an irrational number as a non periodic decimal number is the only way to express it. In other words, there exist irrational numbers that cannot be written, in an alternative form, as roots of rational numbers that they cannot be exactly determined (e.g. π and e). However the formal distinction between algebraic and transcendental numbers must be presented at the upper classes of high-school (Lyceum), when students have already acquire a good experience of real numbers, and before the introduction to the complex numbers.
- A great care is needed by teacher in order to resolve the confusion, created to students with respect to the existing solutions of the equation $x^2=a$, $a>0$, by the definition of the square root as a positive number (see comment in question 13 of our questionnaire). At Lyceum also it must be clarified that the symbol $\sqrt[n]{x}$, $n=2k+1$, has meaning (and a negative price) when $x<0$ (e.g. $\sqrt[3]{-8}=-2$).

We shall close with some open questions on the teaching of real numbers that require additional research and properly designed experiments in order to be answered:

- In the cultural environment of ancient Greek mathematics, even after the introduction of Euclid's axioms, it seems that geometric figure was the basis for "unfolding" mathematical thought. It helped towards the production of conjectures, of fruitful mathematical ideas and explanations. The use of auxiliary lines, the visual redesign and new modified figures obtained convincing proof arguments resulting to a more complete mathematical reasoning. On the contrary, in our contemporary society the numerical culture dominates over the geometric one. As a natural consequence, in school mathematics the numerical thought is much more in use than the geometric one, since the teaching of mathematics is mainly based on formulas and calculations. However we are convinced that an early and excessive "arithmetization" wounds the geometric intuition. In fact, a rich experience of students with geometric forms, before they have been introduced to numerical thought and analytic proofs, is not only useful, but necessary indeed (Kosyvas & Baralis, 2010). Under this sense, an open question is how useful it could be for the comprehension of the irrational numbers by students to support our teaching methods by geometric constructions of certain such numbers and their representation on the real axis. Our experiment didn't succeed in giving a clear indication about this, since the almost complete failure of the TEI students to deal with question 14 didn't prevent them in answering correctly the other questions.
- Of course they are irrational numbers that cannot be constructed by ruler and compass (e.g. $\sqrt[3]{2}$, π, e and many others). However we correspond also points of the real axis to such numbers in an axiomatic way, which usually is not justified clearly to students. Therefore another open question has to do with the proper use of "approximate mathematics" for teaching and learning the real numbers. During the last Panhellenic Conference on Mathematics Education of the Greek Mathematical Society (Halkida, 19-21 November, 2010) a high-school teacher told us that she was embarrassed when she was asked by a student the following question: *"Are there any circles whose length of circumference is a rational number"?* The student added that the length of the circumference of a circle with radius $R=\frac{1}{\pi}$ should be 2 metric units. The problem is that R in this case (and therefore the corresponding circle) cannot be geometrically constructed.
- Finally another important open question (of course one could find more) is how we can drive forward and improve as teachers the intuition of density of rational and continuity of real numbers by students.

References

Andreadakis, S., et al. (2007), *Algebra (A΄ General Lyceum)*, 7th Edition, Didactic Books Publishing Organization, Athens (in Greek).

Artemiadis, N. K. (2000), *History of Mathematics (Under the Scope of the Mathematician)*, Academy of Athens (in Greek).

Bagget, L. W. (2006), *Analysis of functions of a single variable: A detailed development*, University of Colorado.
Available at: http://spot.colorado.edu/~bagget/analysis.book.pdf.

Bryan, M., (2005), Students' Early Mathematical Representation Knowledge: The Effects of Emphasizing Single or Multiple Perspectives of the Rational Number Domain in Problem Solving, *Educational Studies in Mathematics*,, **60**, 37-69.

Chi, M. T. H., Feltovich, P. J. & Glaser, R. (1981), Categorization and representation of physics problems by experts and novices, *Cognitive Science*, **5**, 121-152.

Dimou, G., Zervas, J. & Broumas, C. (1984), *My Mathematics (5th Class of Primary Schoo) - Part I*, 1st Edition, Didactic Books Publishing Organization, Athens (in Greek).

Fiscbein, E., et al. (1985), The role of implicit models in solving problems in multiplication and division, *Journal for Research in Mathematics Education*, **16**, 3-17.

Fiscbein, E., Jehiam, R. & Cohen, D. (1995), The concept of irrational numbers in high-school students and prospective teachers, *Educational Studies in Mathematics*, **29**, 29-44.

Gelman, R. (2003), The epigenesis of mathematical thinking, *Journal of Applied Developmental Psychology*, **21**, 27-33.

Greer, B. & Verschafel, L. (2007), Nurturing conceptual change in mathematics education. In S. Vosniadou, A. Baltas & X. Vamvakousi (Eds.), *Reframing the conceptual change approach in learning and instruction* (pp.319-328), Oxford: Elsevier.

Hart, K. (1988), Ratio and proportion. In: Hiebert, L. & Behr, M. (Eds.), *Number Conceptsand Operations in te Middle Grades* (v. 2, pp. 198-219), LEA Publishers.

Hartnett, P. M. & Gelman, R. (1998), Early understanding of number: Paths or barriers to the construction of new understandings? *Learning and Instruction*, **18**, 341-374.

Herscovics, N. (1989), Cognitive obstacles encountered in the learning of algebra. In Wagner, S. & Kieran, C. (Eds.), *Research issues in the learning and teaching of algebra* (pp. 60-86), Reston, VA: National Council of Teachers of Mathematics.

Khoury, H. A. & Zarkis, R. (1994), On fractions and non-standard representations: Pre-service teachers' concepts, *Educational studies in Mathematics*, **27**, 191-204.

Kline, M. (1990), *Why Johnny cannot add: The failure of modern Mathematics*, translated in Greek, Vanias Publ., Thessaloniki.

Kosyvas, G. & Baralis, G. (2010), Les strategies des eleves d'aujourd'hui sur le probleme de la duplication du carre, *Reperes IREM*, **78**, 13-36.

Mac lane, S. & Birkoff, G. (1988), *Algebra*, 3d Edition, Chelsea Publ. Company, NY.

Malara, N. (2001), From fractions to rational numbers in their structure: Outlines for an innovative didactical strategy and the question of density. In: J. Novotna (Ed.), *Proceedings of the 2nd Conference of the European Society for Research Mathematics Education, II* (pp.35-46). Prague: Univerzita Karlova v Praze, Pedagogicka Faculta.

Markovits, Z. & Sowder, J. (1991), Students’ understanding of the relationship between fractions and decimals, *Focus on Learning Problems in Matematics*, **13(1)**, 3-11.

Merenluoto, K. & Lehtinen, E. (2002), Conceptual change in mathematics: Understanding the real numbers. In Limon, M. & Mason, L. (Eds.), *Reconsidering conceptual change: Issues in theory and practice* (pp. 233-258), Dordrecht: Kluwer Academic Publishers.

Moskal, B. M. & Magone, M. E. (2000), Making sense of what students know: Examining the referents, relationships and modes students displayed in response to a decimal task, *Educational studies in Mathematics*, **43**, 313-335.

O’Connor, M. C. (2001), “Can any fraction be turned into a decimal?” A case study of a mathematical group discussion, *Educational studies in Mathematics*, **46**, 143-185.

Papamichail, D., et al. (1981), *Mathematics (2nd class of Gymnasium)*, 4th Edition, Didactic Books Publishing Organization, Athens (in Greek).

Smith, C. L., Solomon, G. E. A. and Carey, S. (2005), Never getting to zero: Elementary school students’ understanding of the infinite divisibility of number and matter, *Cognitive Psychology*, **51**, 101-140.

Sierpinska, A., (1994), *Understanding in Mathematics*, Falmer Press, London.

Sirotic, N. & Zarkis, R. (2007), Irrational numbers: The gap between formal and intuitive knowledge, *Educational Studies in Mathematics*, **65**, 49-76.

Toeplitz, O. (2007), *The Calculus: A Genetic Approach University*, The University of Chicago Press.

Vamvakoussi X. & Vosniadou, S. (2004), Understanding the structure of the set of rational numbers: A conceptual change approach, *Learning and Instruction*, **14**, 453-467.

Vamvakoussi X. & Vosniadou, S. (2007), How many numbers are there in a rational numbers’ interval? Constraints, synthetic models and the effect of the number line. In Vosniadou, S., Baltas, A. & Vamvakoussi, X. (Eds.), *Reframing the conceptual change approach in learning and instruction* (pp. 265-282), Oxford: Elsevier.

Vandoulakis, J., et al. (2008), *Mathematics (A΄Gymnasium)*, 2nd Edition, Didactic Books Publishing Organization, Athens (in Greek).

Vlamos P., et al. (2007), *Mathematics (B΄Gymnasium)*, 2nd Edition, Didactic Books Publishing Organization, Athens (in Greek).

Voskoglou, M. Gr. (2007), Formalism and Intuition in Mathematics: The Role of the Problem, *Quaderni di Ricerca in Didattica (Scienze Mathematriche)*, **17**, 113-120.

Yujing, N. and Yong-Di, Z. (2005), Teaching and learning fraction and rational numbers: The origins and implications of whole number bias, *Educational Psychologist*, **40(1)**, 27-52.