

3. La créativité et les activités mathématiques

Pour expliquer le succès au travail et dans la vie courante dans ce vingt-et-unième siècle, on évoque fréquemment la créativité mathématique et l'innovation. Enseignants, didacticiens des mathématiques, mathématiciens, chercheurs, parents et les élèves eux-mêmes, ont tous à coeur le développement de la créativité mathématique et le succès en mathématiques. Ceci soulève quelques questions: Qu'entend-on par créativité dans l'enseignement et l'apprentissage des mathématiques? Est-ce que tous les élèves peuvent faire preuve de créativité ou est-ce réservé à seulement quelques-uns? Comment pouvons-nous développer ou stimuler la pensée créatrice dans la classe de mathématiques? Comment harmoniser la créativité et le développement d'habiletés? Comment stimuler la créativité à l'extérieur de la classe? Y a-t-il un lien entre la créativité et les compétitions mathématiques, entre la créativité dans la classe de mathématiques et à l'extérieur de la classe? Comment mesure-t-on la créativité dans la classe de mathématique?

3. Creativity in mathematical activities

Mathematical creativity and innovation are often cited as critical to success in work and in life in this twenty-first century world. Teachers, mathematics educators, mathematicians, researchers, parents, and students themselves all have a stake in learning how best to nurture and support this development of mathematical creativity and the realization of mathematical promise. Some of the questions to be investigated in this strand are : What does creativity mean in the process of teaching and learning mathematics? Is this something for all students or only for a few? How might we develop or stimulate creative thinking in the mathematics classroom? How does this balance with skills training? How might mathematical creativity be stimulated outside the classroom? How is this related to mathematical competitions? What is the role of technology in the development of mathematical creativity within and outside the classroom? How does this fit with high-stakes, standardized or standards-based tests?

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Inviting Creativity to Math Class: Open-Ended Projects in a Middle School Classroom Heather Gramberg Carmody

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Introduction

Each student seated in a math classroom has a unique way of viewing the world and working through mathematics. They come with different insights, abilities, struggles and needs. One method of incorporating this diversity is the development of open-ended projects. Openended projects offer the opportunity for differentiation and high levels of engagement that go beyond solving a set of predetermined problems. Over the last several years, I have developed a structure for open-ended math projects that helps me meet the needs of a variety of middle school students. The students take great ownership of the mathematics, and their enthusiasm is contagious. The inclusion of these projects has allowed me to invite a great deal of creativity to math class.

Opportunities for Excellence

Students have a variety of needs when learning mathematics. Gifted and creative students have a need for depth of application (Hirsch and Weinhold, 1999). All students need to have opportunities to reflect upon their work and refine it (Koshy, 2001). To mirror true experts in the field, students need an opportunity to let ideas develop and form (Hadamard, 1945). Finally, many students need the opportunity to discuss their thinking with peers and adults (Hirsch and Weinhold, 1999).

To engage students in meaningful ways teachers need to adopt many roles. The teacher becomes one who discovers talent. High quality math experiences are not reserved for a particular group of students. Appropriate teaching and activities can allow many students to demonstrate advanced thinking. Teachers need to provide opportunities for students to reveal their abilities (Greenes and Mode, 1999). It is only through intentional classroom activities that students' creativity and potential can emerge. "Pupils who are given a diet of repetitive exercises and closed problem-solving exercises from text books are unlikely to show their true potential. A child who is capable of detecting patterns and generalising will only do so if suitable activities are provided." (Koshy, 2001, pg. 22). Once talent is noticed, teachers need to provide support so students move beyond what is familiar into areas of challenge and creativity (Mann, 2006). This type of teaching requires a high level of flexibility. Teachers need to "appreciate creativity and to enjoy the unpredictability of working with divergent thinkers" (Greenes and Mode, 1999, p. 122).

The need for flexibility also applies to the classroom setting. Classrooms that encourage mathematical creativity are learner centered, have an emphasis on independence, favor flexibility over structure or chaos, and are open to new innovations (Wheatley, 1999). The students work independently towards solutions and new understandings of mathematics (Hirsch and Weinhold, 1999). Zemelman, Daniels and Hyde made a number of suggestions for classroom structure including that "assessment be an integral part of teaching", that class time is spent "developing problem situations that require applications of a number of mathematical



ideas", and that teachers focus on "using multiple assessment techniques, including written, oral, and demonstration formats" (1998, pg. 105).

Project Setting

The following project structure has been applied repeatedly in seventh grade mathematics classes. A series of eight projects were developed as an attempt to address the variety of needs and to restructure learning. They address the National Association for Gifted Children (NAGC) curriculum standards and the National Council for Teachers of Mathematics (NCTM) standards. They have proven effective in large urban districts as well as small private schools. Sometimes my students have been in completely heterogeneous groups. However, in other settings they were in distinct groups after an extensive placement process. These math projects fit the needs of a wide array of children. Students who were classified as gifted have completed these assignments. Some were labeled twice exceptional, exhibiting gifted characteristics as well as a variety of special needs. Students with learning disabilities, communication disorders, those with autism, and those with behavioral disorders have completed these projects.

Project Purposes and Goals

Project Purposes

Moving beyond traditional curriculum takes time and thought. It is important to know the purpose of the time and activities. These projects first seek to provide engaging content in order to increase student motivation and learning. Challenging tasks yield higher cognition, metacognition and motivation (Diezmann and Watters, 2005). Secondly, the projects are designed to give opportunity for students to move beyond being passive subjects of math curriculum to creative thinkers who view the world around them through the lens of mathematics. Sheffield suggests that there is a continuum in mathematics of "innumerators, doers, computers, consumers, problem solvers, problem posers, [or] creators" (1999, p. 43). As students work through the long term activities, they move towards the more creative end of the spectrum. Finally, the projects serve to provide opportunity for student-driven differentiation. Current research suggests the importance of differentiating students' work in terms of content, process, product or epistemology (VanTassel-Baska and Stambaugh, 2006).

Project Goals

Any new student project should be developed with several goals in mind. The first is to encourage students to apply mathematics to real life (NCTM, 2000). Secondly, these projects should encourage students to move beyond the basics of mathematics and to allow them to explore meaningful applications. These projects are not simply to have a fun activity. They must move past "unimportant knowledge" to a deeper exploration of the world around them (Renzulli, 1982). Next, these projects should encourage students to have independence and autonomy in their mathematical learning. The proposal stage and the opportunities for peer collaboration allow students to experiment and move beyond what is comfortable without the fear of failing. The hope is that these projects allow for "responsible risk taking" (Costa, 2001). The final goal of the projects is to require students to reflect upon their work, and to refine their projects. Authentic learning and complex mathematics are very rarely achieved on the first try. Just as students are taught to refine their writing through a sequence of editing different drafts, they need



to learn how to revise their work in mathematics. The goal is to build in time for planning, testing and creating quality projects.

Format of Projects

Although the specific mathematical content changes between projects, the format remains the same. The following factors are always included:

- Extensive mathematical computation
- Algebraic notation
- Written expression
- Visual or graphic representation
- Project proposal
- Peer collaboration
- Evaluation of complexity

The extensive mathematical computation is the anchor for the projects. Students should get as much, if not more, practice on essential skills when completing these projects as they would when working through traditional curricula. The algebraic notation is important in developing abstract thinking and reasoning. The written expression in projects allows students to articulate their discoveries. Mathematics becomes much more than a completed list of problems. The inclusion of visual representation in each project allows students to integrate multiple ways of viewing mathematics. The artistic strengths, creativity with words and strong interpersonal abilities that some students have can lead to a deeper understanding and explanation of mathematics. As mentioned before, each project contains a mandatory proposal well before the deadline. Students are given time to daydream and ask some of the "what if" questions at the beginning phases of a project. In addition to collaboration with the teacher, students also work with each other. Many of the projects involve days where students bring in their works in progress to get reactions and suggestions from peers. Finally, a component of the grading evaluates the complexity of students' work. Students reflect upon and discuss the level of challenge and complexity in their own work. In my experience, students will push themselves to heights far beyond the accepted curriculum or standards.

Questions to ask when developing a project

With all of these goals and components in mind the projects begin to evolve. When I am considering a new project, the following questions help to provide the structure:

- What topic is essential to the curriculum? Or, what topic is a cornerstone of the course?
- What are some meaningful and real world applications of this topic?
- What applications are appropriate for students of this age and level of mastery to explore?
- How can I include mathematical computation, algebraic notation, written expression, visual representation, peer collaboration, evaluation of complexity, and opportunities for reflection and revision?
- What choices can I offer in terms of content or product?
- What resources do I need to find for the project? Am I ensuring that all students have equal opportunities for success in terms of required resources?



- What amount of time will students need to produce quality work? How can I structure a timeline that allows for creativity and rigor?
- Are there adequate opportunities for differentiation to accommodate various student needs?
- Does this project encourage a deeper understanding of mathematics than students would otherwise have?

Conclusion

The increase in the diversity of our students and the complexity of their needs can be a rich addition to a mathematics classroom. The challenge for teachers is to find a way to include students' backgrounds and creativity in a way that allows for rigorous mathematics. The idea of open-ended projects has allowed me to look beyond textbooks and fun activities when I plan for my students. This structure provides students the opportunity to excel and flourish. The elements of collaboration and reflection enrich the culture in a classroom. These long term projects invite creativity to math class, and the result is well worth the risk of moving beyond what is familiar.



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In classroom Mathematical activities for families

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Abstract

Previous research shows that students who their families are getting involved in the school practices and activities are getting better performances in mathematics than other students. The current research lines in mathematics education (CEMELA, Harvard, Includ-ed) suggest that to create opportunities and spaces for families to participate into the dynamics of the schools are a way to fight against situations of inequity that, sometimes, are barriers for learning of mathematical activities with the families are discussed. The paper offers a look on the role that families play in the process of the creation of the activities, how they approach this process, what are the difficulties (and how they are managing them), and the impact that their prior experiences have in the process of learning mathematics.

Key words: dialogic learning, family involvement, mathematical activities, and "the other mathematics"

Introduction

Family training by itself is not enough to have success in learning mathematics. However, prior research claims that students with difficulties in mathematics have better performances in this matter as far as their families start to get involved in the school dynamics (Kliman & Mokros, 2001). As Sheldon & Epstein (2005) declare: "encourage parents to support their children's mathematics learning at home was associated with higher percentages of students who scored at or above proficiency on standardized mathematics achievement tests".

This paper addresses the analysis of the outcomes achieved in a research project focused on teachers' mathematics training from the lens of the work with families. The objective is to analyze the kind of practices and activities that families are carrying out to help their children to learn mathematics. We explore this question from a didactical point of view. Then a reflection on the impact of the mathematics teaching and learning is provided. Finally, the paper concludes with some contributions drawing on the discussion of our data.

Family training and mathematics education

There are several experiences about how families get involved in the mathematics education¹. Hoover-Dempsey & Sandler (2005, 1997) distinguish three different ways to categorize how families participate in the mathematics education of their children: *modelling* (that include

¹ Examples may be visited at two special issues focused on this topic: *Mathematics Thinking and Learning*, and *Adult Learning Mathematics: International Journal* (special issue).



their influence on students' attitudes and routines of study); *reinforcement* (embracing all the families' behaviours that encourage the school success of their children); and *direct instruction* (which means all the situation in which a member of the family take courses to learn mathematics and be able to help their children at home with the homework).

There are different experiences around the world about this kind of practices. For example, in the United States centres of research such as CEMELA, or MetroMath are very well known. The Harvard Family Project is also a famous program in this field. In Europe there is an incipient interest for this type of work. Although there are not many experiences working with families, is important to point out examples such as the Learning Communities developed by CREA, or Parents' Nights, which is a case from Sweden.

This field has a huge importance because as Kliman & Mokros (2001) claim: "parents have great potential to influence children's mathematical learning" and "parental support is necessary for successful implementation of reform mathematics programs" (p. 1).

Methodology

This paper discusses dada coming from a research titled: Teacher training for a family mathematics education in multicultural contexts". It is a research project funded by the Research Department of the Catalan Government (AGAUR, *Agència Catalana de Gestió d'Ajuts Universitaris per a la Recerca*).

The goal of this project is to improve the quality of the teaching practices carried out in Catalonia, through the intervention on family training. The concrete objectives include: (1) to identify the elements and educative strategies in the work with adult people in the field of mathematics education, from a multicultural lens; (2) to create training resources addressed to teachers of mathematics in adult education, in order to promote equity and opportunities for all to learning mathematics; and (3) to offer resources for a teacher training of quality, connected to real classroom-situations, in order to promote a inclusive family training in mathematics education.

In order to achieve all these objectives, we carried out a case study (Stake, 1995), that includes several workshops of mathematics addressed to the families, carried out in two schools, according to the research work developed by CEMELA in the United States, and CREA in Spain and Catalonia.

A total of 4 workshops of mathematics for families are carried out during 2008. The workshops have been conducted in two different schools: an elementary school placed in a city south of Barcelona, and a middle/high school located in a working-class neighbourhood in Barcelona. Around sixty persons were involved in these 4 workshops, from different countries of origin, including Catalonia, Morocco, Equator, Romania, Czech Republic and Armenia.

Table 1: Steps to create the workshops of mathematics.

Step 1: To meet with a person from the main	Step 2: To ask the teachers' staff which curricu-
office of the school (like the principal), to ex-	lum of mathematics they are implementing in the
plain the project.	school, how they organize the contents, and what
	kind of difficulties they notice among the students.



Step 3: To talk with the families that have	Step 4: To design a body of activities including
children in the school in order to figure out what	all the voices and demands from both, teachers and
are the troubles that their children need to face	parents.
learning mathematics, in order to reach an	
agreement about what to include in contents of	
the workshop.	
Step 5: To implement the activities in the works	hops. Among the topics included there are fractions,
equations, number sense, basic operations (addit	ion, subtraction, multiplication and division), and
problem solving	

Data collected is qualitative. Videotapes, field notes, discussion groups, and in-depth interviews were collected. The methodological approach used has been the critical communicative methodology (Gómez, Sánchez, Latorre, Flecha, 2006).

Discussion

Activity 1: The Candy Jar

This is an ice breaker activity. This kind of activities are useful because facilitate to "build a community sense". When a new workshop starts, in many occasions people don't know each other, or if so, it is not in an educative *milieu*, where variables such as motivation, safety, role played inside the group, etc., are really important (Goffman, 1959).

Lave & Wenger (1991) talk about "community of practices" to analyze this kind of situations. We further study this issue in another paper (Díez-Palomar, Prat Moratonas, 2009). The evidences that we have collected during this project show that as soon as this "community sense" is already built, the results in terms of mathematics learning are better.

The activity about *The Candy Jar* is a very well-know activity, already used in other contexts (MAPPS project –Math for Parents-, Tucson, October 2004). It works with a jar of glass full of candies. Participants in the workshop may take the jar in their hands, they can look at it, analyze it, but they cannot open the jar and count the candies. They must figure out the total number of candies by estimation. The person who provides the closest estimation to the result, the real number, gets the jar as a prize.

Counting and guessing	Two families took the jar in their hands; they turn it around several times, counting the candies through the glass. Then they presented a tentative number.
Approximation estimating the vol- ume of the jar	Most of the families took the jar, they turned it up and down several times, and they identified "layers" of candies. Then they counted the number of can- dies that they saw through the glass (a perfect cylinder), and then they multi- plied that number for the total of layers that they already counted from the bot-

Table 2: Strategies used by families to solve the activity.



	tom to the top of the jar.
Prior experience	A family just came up with a random number, drawing on their prior experi- ence related to the capacity of this kind of jars.

Activity 2: Solving 'ax+b = c' equations

This activity was about how to solve several equations with one unknown factor. The families directly proposed the topic because it is one of the difficulties that most of them have in order to help their children to solve their homework. Next quote shows the kind of dynamics that occurred in the group:

(Context: We are in a classroom placed in a high school. Parents are working with first grade equations with one unknown. Now the topic "how to solve an equation" is showing up. The facilitator solves the problem using a method, and one mother claim that her daughter uses other way to do it. At this point the facilitator explains the method used by the daughter. She has dived the chalkboard into two columns: on the first one there is the method used by the facilitator –which is the one known by the mother-

	Proje
$2 \times + 5 = 40 - 3 \times$	2x + 5 = 40 - 3x
Z× = 40-3×-5	Zx+5-5=40-3x-5
2x + 3x = 40 - 5	2x = 35 - 3x
5x = 35	28+3-35-2-2
$x = \frac{35}{5}$	5 25
x = 7	35
- 	x = <u>5</u>
\sim	× =)

; on the second one the facilitator wrote the daughter' method –which is the one used by teachers and children at the school-).

Facilitator: How it is going? Good?

Mothers: yes... very good (the mom who asked the question is the one who speaks louder). Mother: We didn't understand it at home.

Facilitator: eh?

Mother: I didn't understand it like this at home; this that you have explained to us my daughter used to say "mom, we wrote this here", and I say "where do you put this?" because I know it in the other w... in the old way (a noise in the background is heard, like admitting she is right) and I was not able to understand it because there is no explanation on the text book. Facilitator: But, now did you get it?

Mother: (Some mothers admitting on the background are heard) Kind of, but what happens is that here is so easy... but to me... (She starts to laugh and makes gestures with her hands to say that sometimes the activities are difficult).

Facilitator: ... well... this is the same... but you have to go to....

Mother: (At the same time) now you're getting it, because, because...

Facilitator: (At the same time) to everybody.

Mother: she explains that she does it that way, but I don't know how to explain it...

In this activity appears as aspect that is really common while working with families: the conflict that sometimes emerges between parents and children, because they use different strategies



to solve the same problem. The use of activities like this one is useful to discuss all this differences. Through the dialogue the families learn how teachers solve the equations in the school nowadays, and also they share their different ways to solve this kind of activities.

Conclusions

Data collected during the research shows that families have different ways to solve mathematical problems. To solve the conflicts that sometimes arise between parents and children, a possibility is to build what we call "spaces of dialogue" where everybody could include their voice (parents, children and teachers as well). Doing that it is possible to reach an agreement that has as benefit the improvement of the quality of the help that families are able to offer their children in solving mathematics (homework). In addition, the existence of these spaces also allow the appearance of different ways to solve the activities, so they become learning spaces where different approaches are valued. As Hoover-Dempsey & Sandler (2005) claim: the opportunities for family participation condition their involvement in the education of their children.



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Challenging pattern tasks with geometric transformations in elementary teachers training: tessellations, polygons patterns and kaleidoscopes

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Abstract

Studying mathematics through patterns is an opportunity for students of all levels to develop mathematical knowledge connecting different subjects like geometry/geometric transformations and algebra. In a teacher training course an exploratory study was developed in the context of two disciplines: one of mathematics and other of mathematics didactics. Pre-service teachers worked with challenging tasks looking for geometric transformations that underlie the patterns constructed. According to pre-service teachers the options made in this teacher training course seem to be adequate to the challenge future teachers will face in schools.

Resumé

Étudier les mathématiques à travers des structures est une occasion pour les étudiants de tous les niveaux de développer la connaissance mathématique en reliant différents sujets comme la géométrie/les transformations géométriques et l'algèbre. Dans un cours de formation de professeurs on a mené une étude exploratoire dans le contexte de deux disciplines: les mathématiques et la didactique. Les étudiants du cours de professeurs on travaillé avec des tâches défiantes en recherchant les transformations géométriques qui sont à la base des structures construites. Selon les étudiants, les options choisies dans ce cours de formation de professeurs semblent être adéquates au défi que les futurs professeurs trouveront dans les écoles.

Keywords: patterns, geometric knowledge, geometric transformations, teacher training.

Considered mathematics as the science of patterns (Biggs e Shaw, 1985; Devlin, 2002; Goldenberg, 1998; Mottershead, 1985; Orton, 1999) it was the start point for our work. Patterns gave many opportunities for the study of mathematical concepts and the development of mathematical process as problem solving, communication, reasoning and proof.

At Portugal we have new national mathematics programme at elementary levels, (grades 1-9) where patterns, algebraic thinking and geometric transformations have a more emphasis and a change of a paradigm. So it is important that mathematics education pay attention to this changes. In this study we are concerned with patterns and geometric transformations.

The most widespread connection of patterns with mathematical concepts it is at algebraic field. At elementary school algebra is approached through recognition of pattern and consequent looking for generalization in an informal or formal way depending on the level of students. Most of the patterns studied are those that we call growing pattern.

The type of pattern that we are now interesting are someway different from those that we studied to develop algebraic thinking: the geometric patterns. Patterns that we can find. everywhere around us from flowers to building, passing through clothes and make use of concepts and properties of geometric transformations.

To provide a definition of pattern it is difficult. We will used that from Sawyer (1995) when he says that a pattern is any kind of regularity that can be recognized by the mind. But when we study geometric pattern the idea of repetition it is important, but is not enough, as Jean Orton (1999) refer, it includes ideas about shapes recognition, congruence and similarity. Includes also another idea from Zusne (1975) that this kind of pattern is like a configuration consisting of several elements that somehow belong together.

Teacher has an important role in the classroom as mediator between students and knowledge. They need to organize the teaching and learning in a way that they can get involved in mathematics. As patterns are transversal features of mathematics, children need to work with them in all contents. Pre-service teachers need to study and work with patterns, to develop competences in this subject in their teacher training course to do the same in their classrooms and to aid students to observe, to conjecture, to investigate, to communicate, to recognize invariants and look for patterns (Hefendehl-Hebeker, 1998).

The exploratory study

If it is our understanding of mathematics as the science of patterns and as they appear in all mathematical contents pre-service teachers need to work with patterns in different subjects with different perspectives, since mathematical to didactical ones.

Geometry is an important subject to be worked in schools because it allows us to perceive our real world, it's necessary also to interplay between concrete and abstract side of geometry and this is a challenge to a teacher. Besides our students have low performance levels in items related to geometry. Visualization in mathematics is in renaissance, but little pedagogical efforts seems to be invested to implement it. Like Hershkowitz (1998) said, may be that some mathematics teachers possess a naive assumption that the human beings are born with visual thinking abilities that are used when necessary and school don't need to develop them.

Dreyfus (1990) says that visualization plays an important role to student reasoning. Besides this visual reasoning has low status because it is assumed as a preliminary stage of reasoning process. So, mathematics learning must include programmes that compel students to think visually and they can develop this ability through experiences in situations that require such

kind of thinking. To foster visualization it is needed to develop the "geometrical eye" of students and teachers (Fujita & Jones, 2002).

Nowadays in Portugal, as it was referred before, we have a new Mathematics National Programme (ME, 2007) that emphasizes the study of rigid motions/geometric transformations since early grades (grade 1-2). To implement this programme pre-service and in-service teachers need to be prepared in this subjects. As teachers educators, in an elementary teacher training course, we must be aware to these new approaches of our programme, so future teachers prepare their own students with solid mathematical knowledge for their critic adult roles in an information-rich society.

We design an exploratory study which intends to describe and to analyze the work developed by pre-service teachers when they construct and look for patterns in rigid motion/geometric transformations. In particular we want: (a) to promote the development of mathematics knowledge on geometric transformations and its didactics for elementary grades (1-6); (b) to develop competences in observing; noticing patterns; conjecturing; testing and refining conjectures; justifying; proving; and (c) to adapt and construct didactical materials to explore this subjects. According to these goals we focus on a exploratory qualitative approach. Data has been collected through problem solving tasks, observations and interviews. Data analysis was in a holistic, descriptive and interpretative way.

Participants and context. The study we will present is included in a large research project *Mathematics and Patterns: perspectives and curricular experiences of students and teachers.*¹ The project is being developed with two main groups: students (including pre-service teachers) and teachers. We work with some students of pre-service teacher training course (grades 1-6) at their last year of mathematics preparation in two disciplines, Geometric Transformations and Didactics of Mathematics. In these disciplines we want to promote the development of mathematical and didactical knowledge, particularly on geometric transformation, and of competences in observing and noticing patterns, conjecturing, testing and refining conjectures, justifying and proving. To achieve these goals we need to organize a learning environment so that pre-service teachers can solve and explore problem tasks, and also to get involved in class discussions. During some classes was used an application of dynamic geometry. This one made possible that features which maintain invariant, gain more easily relevance and may contribute to discover patterns and to formulate conjectures. After that students need to understand why the conjectures are true and then, proof could appear plain of significance.

¹ The project is supported by FCT (PTDC/CED/69287/2006) and is coordinated by prof. Isabel Vale.

Some examples of tasks proposed. In the mathematics discipline of Geometric Transformations pre-service teachers began to study the geometric transformations like translations, rotations, reflections and glide reflections. They worked with the definitions and explored and analyzed properties of the transformations. After this kind of work they were challenged with the following task: "Imagine you are a designer in a factory that



makes wallpaper. A client want a new wallpaper made by a motif he chooses. Please create new wallpaper patterns proposals with this motif'. This is the motif brought by the client. The preservice teachers created different wallpaper patterns and were asked to identify the geometric transformations that underlie each one of the patterns created.

To do so they need to look to the patterns and "see" the geometric transformations, that are not always explicit. This kind of task needs, in the sense of Fujita & Jones (2002), the "geometrical eye" and develops visualization competences of the pre-service teachers. Firstly pre-



service teachers looks to the patterns using a global apprehension of the figures, but after that they went on the analysis using a punctual apprehension of the figures, relating the original point and the corresponding transformation point.

Pre-service teachers identify different kind of geometric transformations, like translations, rotations, reflections and glide reflections, as is shown in figures.

Secondly they identified too some of their compositions, for example, the composition of two half turns with different centres, the

composition of two reflections of parallels axis, the composition of two reflections of concurrent axis and they conjecture about the existence, in each case, of just one geometric transformation equivalent of the

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compositions. To test the conjectures with more examples, pre-service teachers used a

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environment of dynamic geometry (Geometer's Sketchpad). After testing the conjectures they looked for a proof to each case.



When pre-service teachers look for features that repeat in a problem; when they intend to understand the reasons for the repetition and observe the relations that maintain invariant when all around changes; when they intend to understand and explain the reasons that justify these relations; then they develop their mathematics comprehension and competences in a way close that

of mathematicians. Pre-service teachers also develop a more dynamic conception about mathematics and they find out that patterns are underlying to all mathematics.

In the discipline of *Didactics of Mathematics*, after the initial study of geometric transformations in the mathematics discipline, pre-service teachers have to look for this topic in a didactic perspective for 1-6 grades students. So they were firstly challenged to explore real images of embroideries, pottery, floor and wall tiles, tessellations, friezes and to identify and to explore the geometric transformations involved. The goal of these kind of tasks was to aware preservice teachers that, like Orton (1999) said, geometric patterns are present in everyday life. They also analyze a didactical learning sequence for study geometric transformations and tasks that can be used with elementary schools students. We can illustrate some examples used during these classes.

Pre-service teachers are asked to construct a motif, that must not have reflection axes, using a square tile. After they must to make friezes and tessellations and identify the geometric transformations that underlie the pattern constructed.



The figures show one student proposal presented. The motif chosen was made in a transparency to be more easy to verify the geometric transformations used. This kind of material revealed more easier to work with young children because of it versatility.

Another proposal it is to investigate the rotational symmetries of the geometric forms bellow. The exploration presents a very good way to work we elementary school students. With this materials they can analyze more easily the existence of rotations with, for example, a quarter turn and half turn because the geometric forms were inscribed in a "portractor".





Pre-service teachers are also challenged to work with *pentominoes*. First of all they solve the follow problem: "To the party of school it will be need to put together five small squares tables. Show the different arrays of putting together the tables, knowing that two squares need to have a common side". To solve this problem pre-service teachers constructed all pentominoes. Observed the twelve pieces and discovered which kind of symmetry has each pentominoes.

Tessellations offer rich classroom interactions that allow students to realize, for instance, that the tiled floor, their desk rest on, is a tiled plane. The mathematics associated with geometric transformations further connects students to their mathematics education. So this content provide a rich opportunity for pre-service teachers to connect and motivate geometric learning in their future classrooms. So pre-service teachers were asked if all the pentominoes tessellate the plane. The answer is "No". They thought that only pentominoes like "I or X" do it. To home work they were asked to experiment if they can get a tessellation using each one of different pentaminoes. They were enthusiastic because they concluded that all pentominoes could tessellate the plane and they conclude also that it's possible to put together the same pieces in different ways, as it is shown in figures.



After that they look for geometric transformations used in each of their tessellation. The pre-service teachers also constructed *polygons patterns* gluing different colored paper rectangle. The polygons patterns are generated using the principles: (a) a simple rule is used to

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overlap the rectangles; (b) the angle of overlap rotation of the consecutive rectangles is a factor of 360° (so that after a fixed number of rectangles the pattern is complete and symmetrical).

The figure represents four polygon patterns constructed with rectangles. Each new rectangle is overlapped so that one of its diagonals



coincides with one diagonal of the previous rectangle.

After that, by observing these polygons patterns it is possible to explore geometrical relations between several shapes that emerge, like classifying polygons, analyzing the congruence or similarity of triangles, quadrilaterals, hexagons, and so on. Moreover of these concepts related to geometric transformations it's possible to explore, for example, counting, area and perimeter.

To apply the knowledge acquired we proposed to pre-service the construction of a *kaleidoscope*. Nowadays are available several types of kaleidoscopes that delight us when we turn it and look. They are made with mirrors, triangular prismatic mirrors and a lens multifaceted that originate several and complex images. We proposed to our pre-service teachers the constructions of simple kaleidoscope. They used reflector paper to construct triangular prisms. Those triangles could be equilateral or isosceles and acute or obtuse ones. The figures show different phases of the construction of an kaleidoscope.





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We propose this task because kaleidoscopes are applications of the reflections and constitute an attractive material for elementary school students construct and explore these subjects.

We thought that the tasks presented in this teacher training course allowed pre-service to be more confident to face similar tasks with their own future students because they experimented the doubts, the uncertainty, the feeling of not to know the answer and not to know how to explore the situations presented and they gained more confidence in their own knowledge to face "new" and challenging situations.

Some conclusion

After this exploratory study and with preliminary results it's possible to say that pre-service teachers revealed more interest to explore geometric transformation in the context of frieze and wall paper patterns they construct by their own, because this is a "challenging task" and become more confident and aware to face the implementation of the new mathematics programme to elementary school; they learned more deeply to recognize geometric transformations and their compositions in a context; despite they revealed some difficulties in visualizing "I have difficulties in visualizing; at the beginning I need to use the motif in paper and turn, flip and slide it to "see" if the geometric transformations I had imagine are or not real" and in justifying and constructing proofs for their own conjectures, they defend that "it is important get involved in tasks like these ... to make proofs because despite proving is a very difficult task ... how can I explain this ... proof compel us to think a lot and we will be more conscious about our knowledge and our difficulties".

We expect that our instruction with pre-service teachers it will be a way for they follow and engage youngsters in their classrooms. According to pre-service teachers opinions the options made in this teacher training course are adequate to the challenge future teachers will face in schools. They defended that they can use the learning sequence proposed for this subject with his own children's in elementary schools.

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Posing and Incorporating Ethnomathematical Problems for School Mathematics: Providing Teachers with Required Prospective Tools

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Introduction

The education system challenges for 21st century committed the teacher to understand how the societal surroundings and communication affect his pupil's perceptions and foreknowledge. Consequently, the orientation in teachers' education, and so naturally, in the math teachers' education, would be developing teacher's comprehension of day's tendency. These include, inter alia, teacher's capability to bridge between the academic common math knowledge and the knowledge roots in the culture and society his pupils come from. Being the case, teaching, classroom activity, contents and tasks in mathematics lesson should be adapted to the socio-cultural background of the classroom population. The question is therefore this: how this purpose can be realized?

Ethnomathematical Program (D'Ambrosio, 2006), incorporated as a part of syllabus in the 'History of Mathematics' Educational College course, allowed us to give a number of answers to aforementioned question. D'Ambrosio launched Ethnomathematic at the NCTM, 1985, while he explained the conception of socio-cultural basis of mathematics education, and interpreted ethnomathematics as corpora of knowledge derived from practices, such as counting, weighing, measuring, and etc. Ethnomathematics represents practical mathematics in both western and developing societies, and has been identified within the areas of architecture, ornamentation, weaving, sewing, agriculture, kinship' relations, spiritual and religious practices.

Prospective teachers (**PT**), representatives of two unlike cultures background, Jewish and Bedouin, that comprise the college course population, learn about and experience the mathematical practices in their own culture. Previous study in the subject, conducted in the course, demonstrated that Ethnomathematics in mathematics teacher education could influence the teachers' professional development so that they incorporate humanistic and social aspects of mathematics, and conceive the perception of mathematical concepts as a mix of abstract interpretation in academic mathematics and comprehension of practical math rooted in their culture. As a recommendation it was written that the educators' mission is to lead the PT's experiences so that they are likely to link classroom encounters with mathematics content and pedagogy (Katsap & Silverman, 2008). The ongoing study in the course focused on examination of PT's training in posing ethnomathematical problems for school mathematics, and clarification of this problem's nature. The program was carried out by qualitative-oriented study, while the collection and analysis of the data (observations, feedback-questionnaire, discussions and interviews) follows the method of the Grounded Theory.

Ethnomathematical problem posing: the process and how PT have done with

Development of the ability to understand mathematical activity in the cultural mathematical practicum and acquirement of learning-teaching skills in posing problems in ethnomathematical context, are among the goals of the ethnomathematics program in the course. The training proc-



ess model was constructed based on the union of three theoretical frameworks: ethnomathematical approach (D'Ambrosio, 2006), method of problem posing (Brown and Walter, 2005), and conceptualization of the model, 'Preparing Teachers for a Changing World' (Darling-Hammond and Bransford, 2005). The term problem posing was introduced by Friere. This method of education is based on creativity and encourages genuine thinking effort and action that influence reality. Later, Brown and Walter (ibid) gave the concept a new meaning and interpreted it as a 5 stages process which guided the learner to understand and solve the math problem through investigation. When we discuss the discipline of mathematics, our main attention focuses on the development of the learners' ability to solve problems. The problems these pupils become exposed to during the lessons are of dual origin: teachers and books. Since there are not books containing ethnomathematical problems, the teachers should be trained to pose these problems by themselves. As a part of the program duties in the course the PT's had to conduct a field investigation involving screening and identification of the components embedded in mathematical practice found in texts and tales of their own society and culture. Following this they have posed ethnomathematical problems.

Accordingly to these theoretical frameworks 5 stages for ethnomathematical problem posing process were constructed as follows:

Stages in PT learning process towards ethnomathematical problem posing

a) Study and identification of the practical mathematics used in the socio-cultural community the teacher belongs to.

b) Examination of the context and bridging between formal mathematical knowledge and the common mathematical knowledge in the community practice.

c) Planning of original math lessons combining ethnomathematical contents.

d) Posing ethnomathematical problems.

e) Strengthening professional responsibility and commitment to the community at a time of delivering the problems for common use.

Searching in the sources and posing of ethnomathematical problems, whose ethnic content matches the practical math existing in the community, customs and tradition, was the method that induced the PT's to integrate the subjects mentioned during the course already in the forth-coming lessons in their schools. The motive behind the desire to integrate these problems in the Math teaching in the school was showing their care for the pupils, as expressed by one of the PT's:

"... I think that solving ethnomathematical problems constitutes much greater contribution for the pupils than solving math problems they're familiar with at math lessons. First of all, with these problems the pupils enrich their vocabulary, since the vocabulary in these problems is not used routinely by them [the pupils]. Beside that, the contribution of ethnomathematical problems stems from the possibility to find essence in the problem's substance and understand that nothing is obvious... The pupil cannot say: "what do I need it for" regarding this problem. I think that these problems would cause the pupils to internalize mathematical concepts and forge the contexts. It tempts me to think and learn about my cultural lives and borrow examples from life and environment.... during the course I was able to 'come full circle' for things I did not know how do they happen, and what are their sources. I think that experiencing ethnomathematical problem posing would change the way pupils perceive the subject of mathematics."



Orey & Rosa (2004) raised perceptions about multi-cultural education and multi-cultural mathematics that can be useful as interpretation in the discussion about ethnomathematics and professional development of teachers in Brazil and USA. They summarize various researches showing that multi cultural perspective encourages the intellectual development of the teacher and enables him to learn mathematical contents through numerous experiences referring to cultural, historic and scientific development of mathematics.

PT's expressions enable to identify how exploration and scrutiny of the ethnomathematics' subject helped them to comprehend the academic and pedagogic potential offered by it. Following are 5th grade PT quotations:

"... Already from the glance at the subject of Jewish rugs it could be seen how much knowledge can be produced, for mathematics lesson purposes, from the study of this subject. This is something that previously escaped our attention and was considered as unimportant... Even when the subject of 'rugs' did surface during a math lesson, we did not link the rug to culture and mathematics. It was just an 'example'. Now however, if I'll bring rug pictures to the class, I would ask the pupils to begin with the study of the geometric figures in them. Then I would send them to look for matchless on the cultures that made the rugs, which is already a different interesting approach in teaching, to identify the differences and variety in their geometric figures. At the next stage, I'll ask them to apply their conclusions on the study of the differences between our and other cultures, as reflected in the geometric figures on the rugs..."

This is the place to mention that PT's approached the integration of ethnomethematical contents with great enthusiasm, and many of them thanked the course for its contribution in bringing them closer to the roots and sources of the society they grew in.

In one case, a Bedouin PT told how changing of the word 'first perform', appearing in order of operations (calculations inside parentheses first perform) with the word ruler, in a story expressing hierarchy of rulers in the Bedouin family, helped him to "reach a state of 100% memorizing by the pupils the rules of order of operations". This case study took place in a school after this PT learned the method of ethnomathematical problem posing. According to him, his studies in the college inspired him to have an idea he would not have thought about before. He notes:

"I was surprised and very satisfied with myself and the class that was so connected to my story. To pose a problem that did not exist before, a problem so tangible and pretty, that helped the pupils to remember the rules."

The Bedouin PT experimenting permits him to supplement an additional way for teaching/learning Mathematics in the school that looked the Mathematics as spiritual endowment belonging to a particular society's, traditions and customs, and opens to the pupils a window to a mathematical world with a cultural way of expression.

What did the PT learn during the program?

The process underwent by the PT's in the Ethnomathematics program can be diagnosed as fostering of mathematical literacy. As said above Ethnomathematical problems are unavailable in contrast with mathematical ones, and the mathematical situation taken from mathematical practice in the PT's own culture was not presented as a formal given of the mathematical problem. The field investigation included clarification and identification of notions, which are mathematical concepts present in the texts and/or deeds, and stories of the PT's own people. While filtering the collected data and choosing the subject matter from mathematical practice

the PT's began adapting the raw material: rewriting, reconstruction and decoding. The following citation by one PT describes the way she went through:

"... I began searching the verses and couldn't understand anything, there were words with mathematics, but how does it all come together to establish a link? ... I saw there was an ocean of knowledge - the problem was how to look for it. There are verses which can be turned into a problem with equation or adding and deduction problems... it was like a verbal problem that had to be translated into mathematical language, and these problems can be posed easy. There were texts however, that required really scrupulous digging for words to find the math in them. Usually, these texts could be converted into geometry problems. In other problems the difficulty was that there was not a single comprehensive sentence and I had to reinvent it myself. In texts concerning inheritance for example, which can be converted into "percentage" problem, there is a certain formula how to calculate the percentages and the text had to be invented..."

Wrap things up expose to view that in order to pose an ethnomathematical problem the PT's had to acquire new skills, which they did not need previously, and put into practice both mathematical and language literacy. Moreover, the PT's provided direct and indirect confirmation of the aforementioned assumption and stated that they have indeed felt the professional change they've went through following the program. The cases described above, and others, which occurred during the program, testify how the investigation, deeper delving into the ethnomathematical subject - matter and experience of ethnomathematical problem posing brought the PT's toward the comprehension of the academic and pedagogic potential it offers. In the post course feedback-questionnaire at the end of the course all PT's pointed out that they do inter-weave ethnomathematical discussion in math school teaching in this way or another. The citation below reflects these feelings:

"As a teacher I have acquired a new method for lesson structuring. Now I have another option for planning a lesson and teaching-learning methods in the mathematical class... I can give them (children) information about the culture and tradition as well as mathematical knowledge... This is a meeting with sorts of problems that enable to understand and link between mathematics and ways of life of the pupils in their culture... It tempts me to think, learn about my cultural life, import examples from life and environment - I am creating an interdisciplinary lesson."

Ethnomathematical problems and the outcomes

While the planning the ethnomathematics program was chosen an 'ethnomathematical problem' (**EMP**) as a major outcome of it. Furthermore, was fix on that the EMP hold unique structure, and permeated with the awareness that mathematics assimilated its essence from human development and socio-cultural influence of the environment.

Ethnomathematical problem is a mathematical problem in which the verbal text uses a narrative to describe mathematical practices present in the customs, traditions and daily experiences of different socio-cultural groups. The numerical value of the problem solution must be examined in social context (Katsap, 2008).

The text itself and phrasing of the tasks in it in EMP's differ from those appearing routinely in math-study books, and revolves not only on mathematical information, but also contains philosophical-social one. The number of words, in which the mathematical essence hides in numbers, symbols, geometric forms, etc, is significantly lesser than that of other problems, thanks to



the descriptive abundant information the problem should include.

The EMP has a permanent structure, where the problem's text is built from two parts. *The first part contains a prelude*, which is an information segment of narrative nature and is associated with culture, tradition or customs of the teacher's own culture. *The second part of the text contains two types of questions in the task.* One of them is mathematical questions whose solution addresses mathematical objects and structures necessitating investigation and/or proof. The other type refers to non-mathematical questions, such as those referring to social issues and whose answers satisfies the needs of society or facilitates the common mathematical practise in this society. The text of the problem preserves two principles. Firstly, the questions of both types and the prelude would have common content subject, and secondly, the EMP's text would have a narrative form. Narrative is explained in the literature as a "process or action during which the story is relayed in writing or orally". The choice of the 'narrative form' of expression is based on the nature of ethnomathematics itself, where the narrative enables actual bridging between the "ethno" and the "mathematics" in the text of the problem.

The works done by man and requiring mathematical knowledge encompass a variety of human activities occurring on daily basis in various cultures and societies dwelling in the same area. This variety enables PT's to find a topic they hold dear, and to create in a college class a multiplex of teaching units with ethnomathematics contents, all of which deserving integration in the teaching of mathematics in the school. The study analyzed the nature of ethnomathematical problems posed by the PT's in the course.

Family and Commu- nity Area:	Man & Place Area:	Religion Area:					
Bedouins	Jews	Bedouins only	Jews and Bedouins				
1. Legacy and in-		1. Settlement and its physical	1. Shapes of structures, portals				
heritance	X	geography	and items particularly identified with synagogues/mosques.				
2. Kinship	2. Kinship	2. Residential employment: men, women	2. Numbers in the Holy Scrip- tures designated to assist in the deciphering of the meaning and/or symbolise religious values.				
3. Justice-making	Х	Х	3. Verses describing problem solution with mathematical tools originating in the Holy Scriptures.				
Х	4. Symbols which are shapes and/or salient religious features, becoming recognizable socio- cultural signs.						

The areas and the contents put forward by the PT's

The conclusions suggested by observation of topics selected by PT's for their EMP's (see table above), reveal that all contents of the *Religion Area* and <u>Kinship</u> contents in the *Family and Community Area* were common to both Jews and Bedouins sectors. Whereas the remaining contents in the *Family and Community Area*, as well as *Man & Place Area* contents were identified only among the PT's from the Bedouin sector.

The puzzling fact is the absence of the following area: A person as an individual and his pri-



vate dealings. Last can attest to that <u>the PT's from both sectors perceive the field of ethno-</u><u>mathematics and EMP's as a subjective mutual-community substance in which man does not ex-</u><u>ist by his own.</u>

The research findings suggest that the EMP's posed by PT's can be characterised by seven statements:

The ethnomathematical problem:

- a) Reflects reality and the environment where the teacher who posed problem lives in.
- b) Imbued in words which are symbols and images belonging to the teacher's own culture.
- c) Interlaces questions from mathematical and socio-cultural areas into one set of tasks.
- d) Is linked to a variety of mathematical subjects.

e) Transfers the problem solver from the world of abstract mathematics into the mathematical daily practices acquired by his people via various channels: occupation in a specific area of interest, fulfilling of rites, preservation of community customs, etc.

f) Forces the problem solver to identify with the matter of the text and through this to play an active role in the text completion by solving the problem and providing the answers it needs.

g) Bridges between the contemporary world of the problem solver and his roots and people.

EMP's constitute a unique position in the teaching of verbal problems in mathematics, which invites the solver of mathematical problems to feel and experiences himself as an integral part of his community trough language whose mathematical meaning is a part of the tradition of his own culture.

Summary

The findings indicate that PT's learning/teaching occurred during the process of posing and incorporating ethnomathematical problems in school mathematics, contributed to them acquisition of new learning-teaching-pedagogic skills, fostering of mathematical literacy and reestablishment understanding the purpose of mathematical problems. The main manifestation of the PT's successful functioning were the awakening sentiment of kinship to the cultural mathematical heritage found in the community the PT belong to.

Recommendations

In the instructions to PT's embarking upon a collection of materials necessary to compile EMP's it is desired to include a list of questions each of them should ask him toward the problem posing.

"In the EMP I've posed:

1. Does the topic I've selected serve the pupils from my own culture?

2. How the prelude assists understanding of the mathematical and socio-cultural questions appearing in the next stages of the problem?

3. Is there an identifiable a mutual link and common language between socio-cultural questions and mathematical questions of the problem?

4. It there a solution of the problem limited by a set of values, and if so, what are the values, and why they must be limited?

5. Do in the text of the problem contains mathematical concepts that are unique to the mathematical practice in my own culture, and if so, what are the parallel mathematical concepts in the western culture?"



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Appendixes Ethnomathematical Problems

Problem 1: The rural setting of water and pasture

Read the prelude before you and answer the following questions:

Prelude

The Bedouin settlement can exist in a place that has water sources and sheep pasture. This is the reason the new settlement is erected at reasonable walking distance from these resources. The settlement also cannot be located too close to the well, which is a source of water to all passers-by, who can stop and drink water or give it to their horses, camels or sheep. Therefore, the erection of the tents in the new settlement takes these limitations into consideration and erects them at a distance that leaves sufficient space to movement between the settlement dwellings and the water source. Moreover, it is taken care that the pasture would be within the boundaries of the settlement.

The task

a) The distance between the water source and Bedouin settlement is 1 km. It is known that a women leaving the settlement in the direction of the water source with one jar requires one hour to reach the water source and return and that every additional jar prolongs the walking time 1.5-folds. The jar contains 30 litres. Calculate the time needed for women that left her home in the settlement to go forth and back if she carries 3 jars of water.

b) Describe what you've learned about the rural setting of the water source and pasture from the prelude information.

Problem 2: Omer Count

Read the prelude before you and answer the following questions:



Prelude

In the Jewish tradition the Omer count is a 49-day counting from Passover to Pentecost. It has seven cycles of 7 days each, and on the fiftieth day the Pentecost (Shavu'ot) is celebrated, named after the seven weeks (shavu'a) of the count. For religious persons, this period is characterised by relative absence of happiness and joy – avoidance of shaving and haircutting, no weddings, avoidance of listening to music (the lenient approach endorses music without musical instruments) or dancing. The origin of the commandment is Leviticus 23: And you shall count from the next day after Sabbath, from the day I've brought you, the Omer: seven Sabbaths shall thee count, until the next day after Sabbath you shall count fifty days and made a new sacrifice to the Lord".

The task

a) The beard growth rate of a certain person is 2.5 mm per week. What shall the length of his beard be at the end of the Omer Count?

b) The hair length of a 10 year-old child is 10 cm. During the month (4 weeks) his hair reached the length of 6 cm. What would his hair length be at the end of the Omer Count?

c) A person is used to go to the sea on daily basis. During the Omer Count days he avoids going to the sea. What is the percentage of the days' amount in relation to the all year he does not go to the sea?

d) Think how mathematical knowledge helped religious Jews to observe the commandments.

Problem 3: Mosque - a place for worship and mathematical arts

Read the prelude before you and answer the following questions:

Prelude

A Mosque is characterized by multiple usages of arts characterized by a variety of geometric patterns. Each of the mosque's structure parts: the dome, turret, windows and the niche are plentiful with decorations. Typically, these decorations feature Arab writing or plant motives. According to Islamic laws no human or animal images are allowed. The door is the entrance into the mosque. Moslem art creativity found its expression in wood etching and ironwork, as well as in decoration and ornamentation of the mosque's doors, characterized by various colorful geometric patterns.

The task

a) Describe what you've learned about the rural setting of the water source and pasture from the prelude information.

b) Which geometric patterns can be identified on the doors and decorations of the mosques in the picture below?

c) Copy the geometric forms and patterns you've identified in the picture to your notebook. If you can identify symmetry in the copied doors and patterns, and if so, list the symmetry types.

d) Think about the artists that embellished the mosques, and conceive in the mind did they have to learn geometry? Discuss this question with your friends.

Problem 4: Women's Section

Read the prelude before you and answer the following questions:

Prelude

An old Jewish custom says that men and women should never sit together in order to avoid distraction. The custom triggered a separate section for women in each synagogue, known as the women's section. This name stems from the Jewish Temple's, and the Gemara expounds what was the great repair installed:



that it was surrounded by a balcony and the women see from above and men from below. The porch women sit in has a high parapet and usually a curtain, through which they peep on what happens in the men's section.

The task

a) What do you think is felt by the women attending a synagogue? There are three rows in the women's section and each one has 5 seats.

b) In how many ways it is possible to arrange: Five women in a row? The five women, one mother and four daughters, when the mother has a permanent seat - first on the right?

c) In how many ways it is possible to arrange in the women's section 2 mothers and 5 daughters in one row: When the mothers sit next to each other? When the mothers don't sit next to each other? The five daughters sit next to each other?

d) Pose two additional problems on the subject and write down their solutions.

Problem 5: Rape or a Rape Attempt

Read the prelude before you and answer the following questions:

Prelude

In the case of crimes against women, such as abduction, rape or attempted rape - the punishment is very severe. These offences are seen as violation of family honor. Bedouins compare rape to a situation of breaking a clay pottery, which cannot be repaired. A girl cooperating with the abductor and does not resisting loses her rights in the trial. As for women that was taken by force or raped, she must bring at least three witnesses to support her testimony, those that heard her cries and rushed to her aid. In this case, any person rushing to the aid of the assaulted girl receives a reward, granted by the judge. The sum of the reward is sentenced according to the number of steps the person made back and forth. Except this, there is the fixed payment to the girl's father, at the sum of 1000 dinars.

The task

a) Do you think the traditional punishment, is indeed, just? Explain.

b) If you were the judge, would you sentence the rapist to the same punishment and adjudicate the same sum as compensation?

c) The judge awarded 10 dinars for every step made by the man who came to aid the girl. What is the sum to be paid to the man who went 110 meters in each direction, if it is known that every 2 meters are equal to 3 steps?

d) Pose two additional problems on the subject and write down their solutions.

Problem 6: Hanukkiyya I have....

Read the prelude before you and answer the following questions:

Prelude

The Hanukkah Lamp (Hanukkiyya) is constructed from nine branches: eight branches for the burning candles and the ninth for the 'sexton' (shamash), a candle serving to light the other candles. In the Jewish tradition, the eight candle branches symbolize the eight days of the Hanukkah miracle. In comparison, the Menorah (Candelabrum), one of the Jewish symbols since the days of the Temple, appears seven branches. As argued in the Talmud, it might be a result of the controversy concerning the prohibition of using a lamp resembling the Candelabrum of the Temple. In a 'kosher' Hanukkah lamp the eight branches must be in a straight line, at uniform height and at equal spacing, while the branch for 'sexton' should differ from the others in some way.



The task

a) A DIY store sells 15 cm iron poles at the price of 10 NIS each. Members of the Yehuda Ha-Maccabi family came to store to buy iron poles for a Hanukkiyya. Help them to calculate the budget required to build the lamp.

b) Draw a Hanukkiyya and calculate the number of iron poles required, and the price to be paid for them.

c) Propose two additional Hanukkiyya designs.

d) The Ha-Maccabi Family also decided to construct a Candelabrum with branches of similar height to those in the Hanukkiyya. Knowing that the Candelabrum and Hanukkiyya differ from each other, point out the difference and the reason behind the appearance of each one of these Jewish symbols?

e) Calculate the number of iron poles required to build the Candelabrum you've designed, and the purchasing expenses of the Ha-Maccabi Family.



La communauté virtuelle CASMI encourage-t-elle plus de créativité dans les solutions d'élèves?

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Abstract

Many researchers suggest that the use of rich and complex mathematical tasks to schoolchildren may give them opportunities to develop problem solving abilities such as generating multiple and possibly original solutions and strategies, to reason mathematically, to making conclusions, justifying them and to communicate mathematical ideas (Sheffield, 2008). This vision does not however seem to be making its way into the classroom. In fact, many authors say that the pedagogy used in mathematical classrooms tend to focus more on what students reproduce instead of how they think, which would foster students' natural curiosity (Meissner, 2005). Could the rapidly grown virtual online learning communities help bring this vision with teaching practices giving new opportunities to develop mathematical creativity among students?

In our exploratory study, we will analyze a case of the CASMI science and mathematics interactive learning internet-based community (<u>www.umoncton.ca/casmi</u>) created by a team of researchers from the Université de Moncton (Freiman, Manuel & Lirette-Pitre, 2007). We will first look how rich are the problems posted on a bi-weekly basis on the website according to the list of characteristics that we found in the literature on what could be defined as a good mathematical problem. Also, we will analyze a collective solution space of selected problems as it has been suggested by Leikin (2007) looking at its potential in terms of mathematical creativity by examining the fluency, the flexibility and the originality of each solution.

Further, we will explore if there exists a relationship between the richness of the problems posted and the potential mathematical creativity found in the collective solution space. We assume that solutions will be more creative when problems are rich. We will present some preliminary results of our study during the conference, followed by recommendation on the use of potentially rich online tasks by teachers in order to develop more creative approach to mathematics in their students.

1. La résolution de problèmes en salle de classe : contexte actuel

Plusieurs chercheurs mentionnent qu'en proposant aux élèves de problèmes riches et complexes en mathématiques, on leur permet de générer des solutions multiples ou de stratégies originales et ainsi apprendre à raisonner, justifier des conclusions et communiquer mathématiquement des résultats (Sheffield, 2008). Ces diverses habiletés ne sont pas innées chez une personne et ne se développent pas automatiquement ; pour un enseignant, il devient



alors important d'apporter de changements réels dans leurs pratiques afin de cultiver et de nourrir la créativité mathématique chez les élèves

Les auteurs soutiennent que la pédagogie utilisée en salle de classe devrait permettre aux élèves de développer leurs propres stratégies de résolution de problèmes et de chercher de solutions originales au lieu de mettre l'accent sur l'habileté des élèves de reproduire les procédures enseignées. Selon Meissner (2005), cette pratique courante risque de nuire à la curiosité naturelle des apprenants et leur enthousiasme envers les mathématiques à mesure qu'ils grandissent. De plus, étant habitués à chercher une réponse en appliquant une stratégie apprise, les élèves s'adaptent difficilement aux enseignants qui tentent de développer la créativité mathématique dans leur salle de classe (Messiner,2000).

Le développement des espaces virtuels au cours de dix dernières années a permis d'enrichir les pratiques d'enseignement de mathématiques en proposant une vaste gamme de problèmes en ligne (Klotz, 2003). La communauté virtuelle CASMI (Communauté d'apprentissages scientifiques et mathématiques interactifs, <u>www.umoncton.ca/casmi</u>) contient des centaines de problèmes potentiellement riches et complexes qui ont été résolus par de milliers d'élèves membres de la communauté. Reprenant la définition de l'espace collectif de solutions multiples à un problème mathématique (Leikin, 2007) dans le contexte de la communauté virtuelle CASMI, nous allons étudier l'originalité, la flexibilité et la fluence dans les solutions des élèves en lien avec la richesse des problèmes mathématiques. Afin de construire un cadre de référence permettant cette étude de cas, nous allons tout d'abord nous pencher sur les caractéristiques de la créativité mathématique. Ensuite, nous allons analyser en profondeur le concept d'un problème riche en mathématiques. Nous allons également décrire l'espace virtuel collectif de solutions qui est la communauté CASMI tout en précisant nos choix théoriques et méthodologiques.

2. Le développement de la créativité mathématique et les problèmes riches

Les chercheurs ont différentes visions par rapport au concept de créativité mathématique, ce que reflète une multitude de définitions qui se trouvent dans la littérature recensée. Tandis que Runco (1993) tient compte de la relation entre la pensée convergente et divergente, d'autres auteurs comme Haylock (1997) et Krutetskii (1976) tiennent compte de la fluence (trouver plusieurs réponses à un problème), la flexibilité (utiliser plusieurs stratégies pour résoudre un problème) et l'originalité (produire des solutions uniques) des solutions aux problèmes mathématiques. Deux éléments semblent toutefois se trouver dans la majorité des définitions :

- la créativité et l'innovation sont souvent citées avec la résolution de problèmes, la pensée critique, la communication et la coopération comme étant des connaissances et des habiletés cruciales et des expertises que les élèves ont besoin pour avoir du succès dans leurs vies dans la société de nos jours ;

- il faut un environnement particulier pour que la créativité apparaisse (se développe).



En effet, les chercheurs comme Cline (1999), Sheffield (2003), Freiman (2006) reconnaissent que la créativité mathématique se développe lors de la résolution des tâches riches en mathématiques prenant souvent la forme de problèmes ouverts et de situations-problèmes. Les problèmes ouverts semblent permettre de se déconnecter du stéréotype que chaque problème possède seulement une réponse en incitant les élèves en trouver différentes en utilisant leurs propres stratégies (Klavir & Hershkovitz, 2008). En procédant ainsi, les élèves peuvent trouver différentes stratégies pour résoudre un problème, trouver différentes possibilités de réponses, développer des stratégies originales, et prendre des risques, toutes des caractéristiques de la créativité mathématique ressorties par des chercheurs (Mann, 2005). L'auteur affirme que l'utilisation de problèmes mathématiques ouverts permet aux élèves d'atteindre automatiquement les premiers stades de la créativité mathématique. Étant donné que la résolution de problèmes ouverts semble favoriser la créativité mathématique, le site CASMI peut-il donner l'occasion aux élèves de développer leur créativité mathématique? Dans la section suivante, nous allons présenter brièvement cette communauté virtuelle.

3. Le CASMI : une communauté virtuelle à potentiel créatif ?

Créé en 2006, le site Internet CASMI est une communauté d'apprentissage virtuelle pour les mathématiques, les sciences, les échecs et la lecture. Ayant pour but d'augmenter chez les élèves les expériences positives en mathématiques, d'améliorer les habiletés de résolution de problèmes chez les jeunes à travers des défis de grande taille en utilisant les technologies et de favoriser l'apprentissage par la collaboration, la communauté virtuelle CASMI devient un exemple d'environnement favorisant des apprentissages plus riches (Freiman, Manuel, & Lirette-Pitre, 2009).

Selon une étude menée auprès les élèves participants au projet (Freiman & Manuel, 2007), ces derniers ainsi que leurs enseignants semblent apprécier les défis que leur posent les problèmes affichés sur le CASMI en leur permettent d'améliorer leurs habiletés de résolution de problèmes. De plus, les élèves semblent être motivés à résoudre les problèmes et apprécient le déroulement du site et les rétroactions qu'ils reçoivent. Pourtant, les raisons de cette bonne appréciation ne sont pas encore connues. Entre autres, comme bien de chercheurs cités ci haut, ceci pourrait être attribué à la richesse de problèmes proposés en ligne et aux possibilités qui s'offrent aux élèves d'être plus créatifs. Qu'est-ce qu'un problème riche? Les problèmes posés dans la communauté virtuelle CASMI sont-ils riches?

Mais quelles sont les caractéristiques de problèmes riches? Les solutions sont-elles plus créatives lorsque les problèmes sont les plus riches? Ces questions émergent ainsi en nous permettant de mieux cibler les objectifs de notre recherche. Les participants vont-ils toujours utiliser les mêmes stratégies lors de la résolution des problèmes sur le CASMI? Ou vont-ils plutôt développer leurs propres stratégies? Trouverons-nous des stratégies originales? Un problème potentiellement plus riche, crée-t-il un espace virtuel collectif de solutions et de stratégies plus originaux? En étudiant ces éléments plus en détail, nous pourrons déterminer s'il existe des liens entre la richesse des problèmes posés dans le CASMI et la créativité



mathématique retrouvée dans les solutions soumises par les membres. Les retombées de cette recherche pourront servir à donner des pistes aux enseignants des types de problèmes qui ont le potentiel de développer davantage la créativité mathématique chez les élèves.

4. Choix méthodologiques

Notre recherche se déroule en deux étapes. Tout d'abord, nous analysons la richesse des problèmes qui sont posés dans la communauté virtuelle CASMI. Suite à notre analyse approfondie de la littérature, nous avons ressorti plusieurs critères mesurables et observables qui caractérisent la richesse d'un problème. Une grille d'analyse contenant ces critères a été créée et validée. Par la suite, notre deuxième étape est d'étudier le potentiel de la créativité mathématique de l'espace virtuel collectif de solutions de ces problèmes. Les composantes de la créativité mathématique que nous utilisons dans cette analyse sont la fluence (nombre de bonnes réponses trouvées au problème), la flexibilité (nombre de stratégies différentes utilisé pour résoudre le problème) et l'originalité (la fréquence minime des réponses et stratégies et de la communication mathématique des solutions). Étant donné la nature exploratoire de notre recherche et l'espace limité de cet article, nous allons illustrer notre analyse préliminaire à l'aide d'un exemple.

4.1 Exemple d'un problème

Depuis la semaine où nous vous avions proposé de jouer avec les palindromes, notre mascotte CASMI a composé une énigme avec des lettres de son nom. Il a tout d'abord pris un nombre à cinq chiffres et il l'avait additionné avec un autre nombre qui contient les mêmes chiffres, mais dans l'ordre inverse. Ensuite, il a calculé la somme. Finalement, il a remplacé tous les chiffres sauf un par les lettres de son nom (une lettre remplace un chiffre; les mêmes lettres remplacent les mêmes chiffres). CASMI te lance un défi de trouver quel chiffre se cache derrière la seule lettre qui ne se trouve pas dans son nom.

CASMI +IMSAC ARARA

ce problème aurait pu être plus riche en présentant un contexte réel. De plus, il ne demande pas à l'élève de faire des choix entre différentes options (comme, par exemple, choisir le meilleur plan d'achat). En somme, nous considérons le problème moyennement riche par rapport à notre modèle. Nous pouvons donc s'attendre que l'espace collectif de solutions soumises par les élèves à ce problème sur le site CASMI nous donne quelques solutions créatives. Nous avons donc examiné cet espace en appliquant les critères d'originalité, de fluence et de flexibilité.



4.3 Potentiel de créativité mathématique de l'espace collectif de solution

Un total de 37 solutions a été soumis pour ce problème. Nous en avons enlevé 8 car elles étaient incorrectes, ou ne contenaient pas suffisamment d'informations pour les évaluer.

Fluence : Le problème a demandé de trouver la valeur de la lettre R, la seule qui ne se trouvait pas dans le mot CASMI. Un total de 5 réponses différentes a été trouvé, soit les valeurs de R égales à 3, 4, 7, 8, et 9. La plupart des jeunes (85,7%) ont trouvé une seule réponse. 7,2% d'élèves ont trouvé 2 réponses et 7,2% ont trouvé 3 réponses.

Flexibilité : Les stratégies employées étaient surtout essai et erreurs. Aucune solution ne contenait plus d'une stratégie. Dans quelques cas, les élèves se sont servis des propriétés des nombres (pair et impair) et d'une recherche systématique de toutes les façons d'obtenir des sommes requises.

Originalité : Le tableau ci bas représente la fréquence des réponses obtenues. Nous pouvons remarquer que deux réponses (R = 3 et R = 4) étaient les plus rares, donc, elles sont considérées comme originales, selon notre définition.

Réponse	Pourcentage (%)
trouvée	
3	3,1
4	3,1
7	21,2
8	39,3
9	33,3

En somme, le potentiel créatif pour ce problème était surtout au niveau de la fluence et de l'originalité, car un total de 5 différentes réponses ont été trouvées pour ce problème et que 2 parmi les 5 étaient originales puisque très peu d'élèves les ont trouvées. Pour ce qui est de la flexibilité, celle-ci n'est pas exploitée du tout. Les élèves semblent se contenter d'une seule stratégie pour résoudre le problème. Nous pensons que le potentiel créatif serait plus élevé pour des problèmes plus riches.

Compte tenu la nature exploratoire de notre étude et du nombre limité de solutions analysées, nous ne pouvons pas tirer de conclusions plus profondes. Par contre, en faisant une analyse avec plusieurs problèmes et avec un plus grand échantillon de solutions, on pourra mieux répondre à nos questions de recherche. Toutefois, nos analyses démontrent que les enseignants voulant développer la créativité mathématique chez leurs élèves peuvent se servir de ce problème et de l'espace collectif de solutions pour organiser des discussions à l'intérieur de la communauté virtuelle ou en salle de classe afin de permettre aux élèves d'exploiter leurs différentes solutions, les inciter à chercher ensemble différentes stratégies et de chercher d'autre solutions originales ou de prouver qu'ils en existent plus.



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PATTERNS OF CHANGE IN SOLVING DYNAMIC AND STATIC PROBLEMS

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Subject: Problem solving and institutionalization of knowledge.

Dans cet article, nous analysons l'évolution dans le temps du nombre de réponses correctes et incorrectes ainsi que du pourcentage de l'absence de réponses à des problèmes provenant d'un test international à choix.. Le principal intérêt à ce travail est de relier certaines caractéristiques de problèmes à certains changements. Pour catégoriser les problèmes, nous avons retenu les catégories « dynamique » et « statique ». Les problèmes dynamiques impliquent un changement dans leur configuration, une transformation du contexte ou une analyse des variantes ce que n'impliquent pas les problèmes statiques. Ces deux catégories de problèmes semblent définir différents patterns de réponses et ce, à différents niveaux scolaires. La connaissance de tels comportements pourrait aider les enseignants à mieux structurer les activités de résolution de problèmes tout au long de la scolarité.

INTRODUCTION

We have chosen for the classification of math problems a kinematical point of view. A kinematical view is significant for a couple of reasons, which go from the curriculum to the individual. In general, the curriculum provides a succession of topics and activities across ages from explorations to formal knowledge, which supposes an evolution from dynamic searches to more static descriptions (Singer, 2007). This is relevant for many countries because the structure of contemporary curriculum in mathematics throughout the world reflects unifying tendencies that show its globalization (e.g. Atweh & Clarkson, 2001). When referring to students, while in the first years of schooling children perception seems to be mostly dynamic-rhythmic, while progressing to high school, students' explanations involve more formal arguments of a static nature, induced by the new acquired knowledge (Singer & Voica, 2008). If we look at the individual's development through learning, the dynamic infrastructure of mind (Singer, 2008) allows successive phases in the process of cognitive growth. Therefore, an analysis of the dynamics of problems could offer new data on the problem solving process to both: the teaching practice, and the didactical research.

We further define two general categories of problems. A *dynamic* problem supposes a movement of a configuration, or a transformation of the context, or some degrees of freedom which lead to more possible results. Conversely, a *static* problem does not suppose motion or variability of data. These definitions equally refers to the text of the problem as well as to the cognitive processes involved in problem solving activities and allow classifying problems in two categories that do not overlap: dynamic or static. Because the classification emphasizes some complex characteristics of problems, sometimes it is not obvious to which category a specific problem belongs to. For instance, problems with similar text can be situated in different classes, since the essential criteria is the dynamics of the content (Singer & Voica, 2006). It has to be mentioned that, for some problems, different persons might have different opinions about



to which category a problem belong. This can happen because each one has its own way to formalize the problem and to interpret it. The importance of a characteristic depends on the possibility to highlight significant differences in some descriptors. In the present paper, we are interested to see how the dynamic-static characteristic affects *no-answers* and the ratio of correct answers per distracter with highest percentage (descriptors).

To exemplify our method, we next look at two problems in detail. The two problems (one dynamic and one static) were chosen such to belong to the same domain (geometry) and to make reference to the same type of configuration.

Problem 1 (static)

In a rectangular triangle, a and b are the legs. If d is the diameter of the inscribed circle and D of the circumscribed one, then d + D is equal to:

A) a+b **B)** 2(a+b) **C)** 0.5(a+b) **D)** \sqrt{ab} **E)** $\sqrt{a^2+b^2}$

Problem 2 (dynamic)

In the figure there are two half circles of radius 2 cm and centers *E* and *F* and a circle of diameter *AB* (*A* and *B* are on the two half circles so that $AE \perp EF$ and $BF \perp EF$). The area of the grey region is:

A) $2\pi + 1 \text{ cm}^2$ **B)** 7 cm^2 **C)** $2\pi \text{ cm}^2$ **D)** 8 cm^2 **E)** $2\pi + 2 \text{ cm}^2$

The first problem is a static one because, in order to solve it, we need to visualize the inscribed circle, as determined by the tangent points. The finishing of the drawing with these segments doesn't change the initial configuration, but rather puts in evidence the data given in the problem. For solving it, it is needed to identify equal segments, but this process will not induce changes in the configuration. On the contrary, the second problem is dynamic, because a fast solution (it has to be reminded that these problems are given at a competition) supposes to transform the figure by translating parts of the grey area (two quarters of the disc of diameter AB) over two white quarters (circles with radius AE and BF). By this translation, the area asked in the problem can be easily computed as the area of the rectangle ABFE.

We consider as given that problems can be categorized in a disjoint way into static and dynamic, and there are several types of static and dynamic problems (Singer & Voica, 2006). However, we have degrees of dynamic and static characteristics, especially for problems that allow a dynamic and static approach. In this article we argue in favor of the following claims: 1) no-answers are in general higher for static problems; 2) dynamic problems have a lower rate of growth when we look at the ratio between correct answers and the percentage of the maximum distracter than static ones; 3) the rate of change in case of the dynamic problems depends on the type of the dynamic problem; 4) static problems increase the ratio within the advances of the grades due to increase in handling mathematical knowledge or due to alternatives in solving the problem; 5) there is a change in the ratio between static and dynamic problems used in school in favor of the static ones as progressing in schooling; 6) these analyses inform both classroom practice and didactics on better approaches on problem solving.

METHODOLOGY

Sample and tools. We applied the dynamic-static classification to the problems given at the international applied mathematics contest Kangaroo in order to allow the analysis of the relation



between problems and students' responses. Kangaroo is a multiple choice contest where the students have to answer, in 75 minutes, 30 questions of 3, 4 or 5 points worth. For each question five answers are given, but only one is correct. Wrong answers are penalized by decreasing the accumulated points by the quarter of the problem's worth, meanwhile no-answer or multiple answers are not taken into account (but are still stored). The tests are common for two consecutive grades (from grade 3 up). In Romania, each year, there are approximately 250,000 participants at the contest. The number of participants decreases between grades, from an initial 40,000 in grade 2 (7-8 years old) to around 2,200 in grade 12 (18-19 years old). Given the high number of participants, the analysis of responses can give statistically significant results.

Method. In order to sustain the claims, we analyzed several aspects by looking at the dynamic and static problems given across several grades at the Romanian edition of the Kangaroo competition. Problems were selected in pairs (dynamic-static), such to show certain similarities in content or in concept and to have been given at several grades. We examined the: evolution of average non-answers percentage, variation in the ratio of correct and maximum valued distracter or, in some cases, the sum of distracters with highest percentage of selection.

RESULTS AND DISCUSSION

In the general analysis we looked at the evolution of the average of non-answers (table1). It can be observed that there is a continuous increase for both types of problems, but up to grade 6 dynamic problems have a higher rate of non-answers average than static ones. From grade 7, there is an inversion that remains stable in time. We argue that this is due to the relation that static problems have with formal knowledge. As participants advance in their studies, problems are more focused on the use of formal knowledge, but also to handle them becomes heavier for students, so they avoid answering to those they consider too difficult at the first sight.

Grade	3	4	5	6	7	8	9	10	11	12
Static	4.34	2.81	5.37	3.50	6.56	5.53	8.61	8.78	8.54	8.22
Dynamic	5.05	3.44	5.27	4.77	5.37	4.76	5.58	5.50	7.62	7.30
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Table 1. Average of non-answers (in %) per problem type and grades

The differences between static and dynamic problems are maintained at particular analysis. In order to see whether the difference in these characteristics of the problems triggers different behaviors from participants, we analyzed the ratio between the percentage of correct answers and the maximum percentage from the distracters. Table 2 contains the results.

Problem	Classification	Grade 7	8	9	10	11	12
1	Static	0.90	0.85	0.91	1	1.11	1.16
2	Dynamic			0.82	0.88	0.87	1.01

Table 2. Pattern in time of the correct/maximum distracter ratio for a dynamic and a static problem

It can be observed directly that the ratio is growing faster for the static problem than for the dynamic one. In the case of the static problem, this can be due to the fact that students can reason more effectively about the segments as they advance in their studies. The increase in case of

the dynamic problem is slower, but still gets to one at the highest grade (that is, only in grade 12 the correct answers overcome the wrong ones).

We consider that this particular result is not suggesting that in grade 12 students solved the problem dynamically; on the contrary, we consider it as a confirmation of the fact that static thinking becomes dominant in time. The reason for such interpretation comes from considering the answers to distracters, not only the ratio. In case of problem 1, in grade 7 and 8, the distracter with maximum percentage was B, that is $2 \cdot (a+b)$; meanwhile from grade 9 it is the distracter E, that is $\sqrt{a^2 + b^2}$. But, distracter E is an answer that should be eliminated at once, since it represent the diagonal of the triangle (D) and it is obviously not the correct one. Still, choosing this answer reflects that students "retrieved", almost in an automatic way, the knowledge that was linked to a right angle triangle with the given legs (Pythagoras theorem) and do not estimate the plausibility of the answer. The same argument goes for the dynamic problem. The most frequently chosen distracter (from grade 10 up) was C, which expresses the area of a half circle (so, obviously less than the grey area). Only in grade 9, the distracter with maximum percentage was E, at least not so clearly wrong as C. Based on what kind of thinking is required by the most frequent error, we suppose that the same way of thinking was applied in case of correct solutions. This is why we argue that, in case of the dynamic problem, the 1.01 ratio at grade 12 could be still the expression of a static thinking. It has to be reminded that this is also possible because the problem allows a static way of solving (by drawing the triangles BFO, AEO – where O is the intersection of the three circles), even if that one is not the most efficient (at least as the needed time considered). In conclusion, beside being an argument in favor of hypotheses 2 and 3, the above situation illustrates that the ratios for dynamic problems that also have an alternative static way of solving increase slower than for "completely" static ones (hypothesis 4), but probably faster than for "pure" dynamic problems.

CONCLUSIONS

We looked at two categories of problems: dynamic and static, and argued that they define different patterns of behaviors. This hypothesis was verified by a general overview of a multiple choice contest problems and particular analysis of a pair of static-dynamic problems. The first analysis focused on the evolution of average percent of non-answers from grade 3 to grade 12 and concluded that after grade 7 (when in most of the curricula there is a complete turn into teaching formal mathematics: proving, formal reasoning, various formulas, etc.) static problems have constantly higher percent of non-answers, meanwhile during all grades there is an increasing tendency for both types of problems. The particular analysis underlined that static and dynamic problems show different patterns of ratio (correct answers per distracter) change between grades: with static ones growing faster. The rate of growing depends on the type of static or dynamic problem, concretely whether there are alternative solutions to the problems. We consider these results as highly relevant to teaching and assessment, since they treat complementary aspects of problem solving.



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RELATIONSHIP BETWEEN ITEM DIFFICULTY RATIO AND SUBJECTIVE PERCEPTIONS OF DIFFICULTY IN MATHEMATICAL PROBLEM SOLVING

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Subject: Problem solving and institutionalization of knowledge.

Dans cet article, nous présentons les résultats d'une evaluation préliminaire de la relation entre la performance en résolution de problèmes mathématiques, la perception des solutionneurs quant à la difficulté des probèmes et les caractéristiques de ces problèmes. Les problèmes provenant de l'environnement virtuel CASMI sont caractérisés par leur contexte familier et par le processusanticipé de résolution. Une analyse exploratoire nous amène aux conclusions préliminaires suivantes : il existe des patterns que l'on peut identifier relatifs à la difficulté perçue des problèmes et chacun de ces patterns est relié aux caractéristiques du problème ainsi qu'à la difficulté relative de l'item. Nous pointons comment ces résultats peuvent être utiles aux enseignants.

1. AIMS OF THE STUDY

In their lesson preparation, mathematics teachers have to evaluate the difficulty of problems chosen for students. There are benefits of doing so: it allows selecting the problem that is appropriate to the student's knowledge and, lets increase gradually difficulty levels in order to keep student motivated (Newman *et al.*, 1998).

Research related to difficulty of mathematical problems can be grouped as follows. First, we have studies that search for domain-independent measures of difficulty. Grondlund (1981) defined difficulty as the ratio between the number of correct and total number of solutions. He introduced the term *item difficulty ratio* (ITR) for this concept and, today, is still one of the most used measures. Another common measure from the same category is response time: time passed between the presentation and resolution of an item (Mason et al., 1992). Second, there are the studies that search for factors that affect success in solving a problem and thus influence its difficulty level. For example, Lane (1991) found that difficulty of algebraic word problems depends on such factors as number of intermediate results to be obtained, a necessity to reformulate result in one short sentence and familiarity of problem context. Hornke & Habon (1986) define these factors as *cognitive*. On the other hand, Jerman (1983) used the term *complexity factor* to denote elements that relate to the number of steps required to achieve results, number of partial results and computational complexity. In these cases, difficulty was measured as ITR. A third line of studies focuses on problem categories and their difficulty. Researchers in this line, define frameworks to classify problems and then try to associate difficulty with categories. The difficulty in question can be the subjectively perceived difficulty (Craig, 2002), ITR (Caldwell & Goldwin, 1987) or a researcher established scale based on the degree of solution correctness (Galbraith & Haines, 2000). In this approach, the number of difficulty categories is another aspect. In case of ITR we have a continuous measure; in the other cases we have crisp values. In Craig (2002), difficulty measures were established by comparison of problems from different categories trying to set up a one to one correspondence between problem category and diffi-

culty. The same is done in Galbraith & Haines (2000) where the authors propose a three-level taxonomy according to the increasing mathematical demands.

Our exploratory study belongs to the latest line of research. However, we analyze the problems in a different way. First, we aim to establish a set of problem characteristics that would reflect several stages of the solving process. Rather then working with a priori defined problem categories (as in the above mentioned studies) we search for combinations of these characteristics related to ITR. On the other hand, we look at subjective perceptions of difficulty on a threelevel scale (from easy to difficult). Instead of using these levels of individually perceived difficulty, we look for patterns of perceived difficulty (PPD) in order to relate them to sets of problem characteristics. There are some benefits in doing so. First, relating problem characteristics to ITR, we obtain a more general way to assess difficulty. Second, by using PPD, we allow to include variations of perceived difficulty (it is important since the appreciation of difficulty will vary from student to student) and redefine difficulty categories based on these patterns (instead of a priori defining them). Third, relating PPD with a set of problem characteristics, we have, again, the advantage to identify aspects that make the problem more difficult. Last, studying the relation between ITR and PPD we can have an insight as from where differences arise and which are the probable error sources (since, ITR is related to correct solution). To know these characteristics may be useful for classroom teaching since it helps to do problem selection by teacher by balancing between difficulty perceptions and maintaining students' interest in solving problems. Therefore, we are trying to find an answer to the following questions:

- how can we properly characterize a problem;

- how PPD can be identified and what they mean;

- what is the relation between PPD and characteristics; ITR and characteristics; ITR and PPD.

2. METHODOLOGY

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The problems were selected from the CASMI website (<u>www.umoncton.ca/casmi</u>), an internet community providing students with challenging mathematical and science problems at various levels of difficulty (Freiman & Lirett-Pitre, 2009). New problems are posted every two weeks and participation is open to everybody. After submitting a solution electronically, participants, who are schoolchildren and pre-service teachers, have been asked to assess the problem's difficulty on a three level scale: *not difficult at all, a bit difficult* and *very difficult*. For each problem, the total number of submitted solutions and the number of correct solutions are also available.

In order to characterize these problems, we focused on the context of the problem (*familiarity*) and on the way to solve it. The *familiarity* means that the situation described in the problem can be associated to some kind of everyday life experience. The second aspect refers to the *solution process*, namely to the possibility to find the solution by a step-by-step approach. This means that the solution can be reached by trying out a limited number of configurations, combinations and the elements involved in these variants are all given in the problem. The contrary way is to identify a general procedure to treat the problem and then apply this to the particular case of the problem. Examples of such type of problems are combinatorial problems.

Each problem - for the purpose of our analysis - is described as a vector of 6 components: familiarity (1- present, 0 - not), solution process, ITR, percents of *not at all*, *a bit* and *very diffi*-

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cult. For the exploratory study, we selected 5 problems in the way to have different combinations of characteristics (table 1 and appendix). Maximum percent is in bold.

Pb.	Fami-	Solution	ITR	Not at	Somehow	Very dif-	PPD	Interpretation
	liarity	process	(%)	all difficult (%)	difficult	ficult"		of PPD
1	<u>1</u>	<u>1</u>	0.72	0.797	0.145	0.058	<u>1</u>	Very easy
2	<u>1</u>	<u>1</u>	0.83	0.825	0.124	0.052	1	very easy
3	0	1	0.69	0.526	0.385	0.087	2	Facy
4	1	0	0.37	0.585	0.367	0.047	2	Lasy
5	0	0	0.42	0.410	0.474	0.121	3	Intermediate

Table 1. Characteristics and different indicators for the analyzed problems

3. PRELIMINARY CONCLUSIONS

Patterns of perceived difficulty: A pattern is defined by the distribution of percentages on the three difficulty levels; in such way we can identify three patterns (PPD column). The interpretation of a pattern is based on the percent of the *not at all difficult* level (last column). For example, for values higher than 75 we have *very easy* problems, for values between 50 and 75 we have *easy* ones.

Relation between ITR and PPD: For the patterns 1 and 3 (PPD column) we found a correlation between ITR and maximum values of difficulty levels (relation highlighted in italic in table 1). However, for the pattern 2, ITR values are split. We explain this case thru problem 4 (line 4, in table1). Meanwhile ITR is low, the percent on *not difficult at all* is in average range (50-75) which suggests that students know how to solve the problem (thus, they consider it easy), but make mistakes in the solving it (so, ITR is low). Further research is needed to confirm this finding.

Relation between problem characteristics and patterns of difficulty: As it can be observed from the table (values in italic and underlined), *very easy* problems are familiar and allow a step by step approach. As mentioned, familiarity makes easier the understanding of the problem since there is no need for interpretation of mathematical terms. The step by step approach, allows easy solution: one tries out combinations given in the problem's text or interprets data. On the other hand, *easy* problems have either familiarity either solving process, but not both. It might be that participants perceive problem as difficult due to the lack of familiar context (and that lack manifests itself in difficulties to understand / interpret the problem) or because the problem needs a general procedure in order to be solved. *Intermediate* problems are neither familiar nor step by step. In this case, difficulties can arise from interpretation /understanding of the problem and solving it. This seems to be the reason why the value on the *very difficult* level is above 10%.

More generally, it seems that the lack of familiar context characterize *easy* and *intermediate* problems (lines 3, 5 in table 1). Still, this doesn't mean that problems in familiar setting will be necessarily *easy* ones (see the example of problem 4, table1). Further analysis is required to properly asses the role of each characteristic over PPD.



Relation between problem characteristics and ITR: We found that ITR was higher for stepby-step problems (grey cells in table 1). Again, these problems can be solved by a combination of given data and do not (necessarily) require the use of formula or complex operations that could lead to mistakes and perhaps, to incorrect solutions (so, lower item difficulty ratio). However, the analyzed data are not enough to draw conclusions on the role of the familiar characteristic over ITR.

4. SUMMARY

In the paper, we treated the problem of ITR, PPD and their relation to problem characteristics. Five problems, from the CASMI site, were analyzed, each with a high number of solvers. We report three preliminary conclusions. First, three patterns of subjective difficulty have been identified. These patterns show different percent distributions on the difficulty evaluation levels.

Second, there is a relationship between the two problem characteristics, familiarity and step by step approach, and the identified patterns. Familiar and step by step problems are in the *very easy* category; meanwhile, when only one of the characteristics is present, problems are *easy*. Problems with none of the characteristics present are *intermediate*, with more than 10% of the participants judging them as *very difficult*. Third, ITR is directly related to the step by step characteristic: its presence defines high ITR. On other hand, the role of familiarity in ITR is still not clear.

The experiment suggested several lines for further research. First, these preliminary conclusions should be confirmed by a larger scale evaluation. Second, the link between PPD and problem characteristics should be further analyzed by using sequences of problems in which the problem characteristics are modified and each problem is assessed separately. Third, it is necessary to think of special settings that would allow to properly assessing the role of the familiarity characteristic in getting children into mathematics: inciting to solve problems, but also to pass between problems with and without this characteristic.



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Appendix: – problems used in the analysis

1. Paul goes to the shop, where they sell videogames and sweets. Paul goes to the sales section in order to check for new discounts. After a moment, his friend, Marc arrives. Since he has no money, Paul decides to share his money with him. Paul has 60\$ and gives the half to Marc. Marc decides to buy a videogame "Guerre civile" (11\$), a copy of "Le chevalier" (17\$) and 12 chocolate bars (50c per bar). Paul, who can't eat sweet because of his diabetes, only buys videogames: a copy of "Supercourse" (13\$) and one of "Le chevalier" (17\$). At that moment, Paul's father arrives and offers to pay the rest of money if needed. How much Paul's father should pay?

2. Marc puts five apples on the table. He cuts three into halves. How many apples are still on the table?

3. Here is a number game: a. think of a number; b. multiply by 2; c. add 8 to the result; d. divide the new result by 2; e. subtract the number you initially thought of; f. your answer is 4.

Try out several numbers. What do you get as final answer? Can you explain in mathematical terms how you got the answer?

4. The Doucette family invites their friends to celebrate Thanksgiving in a restaurant. All tables there are square and only one person can be seated on one side. We can arrange tables as we wish, but tables must touch at east on one side each other. Beside the 4 family members, there are 12 guests. What is the minimum number of tables needed?

5. The administration of the park Hiberville decided to transform their 8 km side triangular track into a rectangular with keeping the same length. Which are the different possibilities for the lengths of the sides when all of them are integer numbers? Could you find all solutions?

DIFFERENT WAYS TO SOLVING OF ONE PROBLEM AS A MEANS FOR STIMULATING THE INTEREST AND CREATIVE THINKING OF COLLEGE STUDENTS IN MATHEMATICS

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Résumé

L'objet de cet article réside dans de nouvelles approches dans l'enseignement des preuves mathématiques, en ce qui concerne les séries. Ces méthodes sont basées sur deux vues alternatives. Soit, en considérant une formule donnée comme le cas particulier d'un résultat plus global, soit en la considérant elle même comme la règle propre à une variété de cas particuliers, à priori sans liens communs.

As it was written in the Second Announcement of CIEAEM 61 the one of the main teacher's tasks is in "motivating students by putting mathematical concepts into context and linking them to other academic subjects and everyday life". Mathematics teachers look for attractive topics for their lessons, which truly draw attention to studies of mathematics and provoke student's creative thinking, rather than merely provide them technical skills. Such topics may be directed to different mathematical problems, but they may also integrate different mathematical themes by applying different methods to solving of a single problem. The latter is in the spirit of the Poincare's definition of mathematics as "the art of giving the same name to different things" (in Rose, N., 1988). By solving a single problem in various ways students are likely to gain a deeper understanding of the problem. Moreover, by being exposed to different kinds of mathematical ideas, students are provided with a range of effective patterns for their own mathematical thinking. Thus, our presentation takes as it motto and guiding principle Polya's advice that "It is better to investigate one problem from many points of view, than to solve many problems from one point of view" (Polya, G., 1962). We will present an extended example, based on this principle, related to the formula for computation of the sum of powers of two:

$1+2+2^2+...+2^n$

In our presentation, we shall give seven different approaches to discovery and proof of the formula for this sum based, as a rule, on finding connections of it with some practical processes. We are sure that diversity of approaches to solution of a single problem stimulate creative thinking and arose interest of the students in mathematics.

Non-formal Interest-Evoking Introduction for students:

According to one well-known ancient legend the emperor of China so loved the game of Chess that he offered the inventor of the game anything he wished as a gift. The inventor asked that a grain of rice be placed on the first square of a chess board, two grains on the second square, four on the next, then eight, 16, 32, 64, 128, 256 and so on until the entire chess board was covered. After the inventor had left, members of the court were calculating a long time the number of grains of rice to pay the inventor:



$$1 + 2 + 2^2 + \dots + 2^{63}$$

They were astonished to find that this was more quantity of rice then existed in the entire world. This much rice would cover the entire earth several inches deep. (Pedoe, D., 1973).

Here we present some different approaches to the formula for this sum.

1. Multiplication by 2

$$S_n = 1 + 2 + 2^2 + 2^3 + \dots + 2^n$$

 $2S_n = 2 + 2^2 + 2^3 + \dots + 2^n + 2^{n+1} \Longrightarrow 2S_n = S_n + 2^{n+1} - 1 \Longrightarrow S_n = 2^{n+1} - 1.$

(This summation is demonstrated and used as far back as Euclid's *Elements* (300 BCE). It is the most widespread school approach to this sum because in this way the common formula for computation of the sum $S_n(q) = 1 + q + q^2 + d^3 + ... + q^n$ may be easily proved. But from the affective point of view it is formal and not very impressive method. We check many times the first year students (of engineering college) competence about this sum and it was found that only quarter of them were able to give the precise formula for its computation and no more than 5% of the asked students were able to prove it. This state says about the little attention of the secondary school teachers to this sum and giving it as some formal mathematical equation without understanding its intrigue philosophical essence which my be done as you can see in some of the next approaches.

2. Decay Integer Model [Expansion approach]

 $2^{n+1} = 2^{n} + 2^{n-1} + 2^{n-1} + 2^{n-2} + 2^{n-2} + 2^{n-2} + 2^{n-1} + 2^{n-2} + 2^{n-2$

This solution may be described as mathematical model of nuclear decay of some material consists from 2^n atoms (with use of half-decay period). Here frame used for number of atoms left at the proper stage of decay.

3. Fraction decay model

$$S_{n} = 1 + 2 + 2^{2} + 2^{3} + \dots + 2^{n}; \qquad F_{n} = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n}}$$

$$2^{n} F_{n} = 2^{n} + 2^{n-1} + 2^{n-2} + \dots + 2 + 1 = S_{n}$$

$$1 = \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \dots \Rightarrow$$

$$1 = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n}} + \frac{1}{2^{n}}$$

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n}} = 1 - \frac{1}{2^{n}} \Rightarrow F_{n} = 2 - \frac{1}{2^{n}} \Rightarrow S_{n} = 2^{n} F_{n} = 2^{n+1} - 1$$

4. Tournament Model

K players play hand-to-hand and the loser leaves the field. How many games will be played till the final game?

Logical Solution:

After one game, one player leaves. In the end K - 1 of K players leave the field and so K - 1 games were played and it is the answer.

<u>Special case</u> $K = 2^{n+1}$ <u>direct solution:</u>

If there were $K = 2^{n+1}$ players in the beginning as in the first round 2^n games will be played; in the second round 2^{n-1} games will be played, ..., in the last round 1 game will be played. So the total number of the games is: $2^n + 2^{n-1} + ... + 2 + 1$.

So, combining two mention above solutions, we have:

 $1 + 2 + 2^{2} + \dots + 2^{n} = K - 1 = 2^{n+1} - 1$.

5. Probability Model

What is the probability that in a family with n children at least one is a daughter [assumed that $p_{male} = p_{female} = 0.5$]

Divide all elementary events into n subsets according to the number of births before the first girl was born:

$$p(A) = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n}} \quad P(\overline{A}) = \frac{1}{2^{n}}; \quad p(A) = 1 - p(\overline{A}) = 1 - \frac{1}{2^{n}}$$
$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n}} = 1 - \frac{1}{2^{n}} \text{ and see the final stage of the solution } 3.$$

6. Distance computation Model



7. Binary Arithmetic Method

In binary notation: $1 = 1; 2 = 10; 2^2 = 100; 2^3 = 1000, \dots$ and so on $S_n = 1 + 10 + 100 + 1000 + \dots + 100\dots 0 = 111\dots 1 (n + 1 \text{ times } 1)$ $S_n + 1 = \underset{n+1 \text{ times } 1}{111\dots 1} + 1 = \underset{n+1 \text{ times } 0}{1000\dots 0} = 2^{n+1}$

All or part of the approaches given in this paper was presented, at various times, to lecturers, teachers and college students. Consistently, we were told that the lesson was a very interesting one, that it revealed new ways of thinking, and that it included moments of surprise. It was evident that the approach demonstrated to the students the strength and the beauty of mathematical thinking. For some of the students it initiated or increased their interest in mathematics.

Conclusions

• The approach presented in this paper promotes a positive attitude towards mathematics in which mathematics is regarded as a powerful and a beautiful instrument for problem solving.

• It combines notions and methods from different fields of science and real life and highlights the universality of mathematics - how one simple formula may describe different problems from different fields of knowledge.

• Considered in light of research into the value of multiple representations, for example, it provides an opportunity for two-way communication between theory and practice in mathematics teaching.

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Patterns in the city: a mathematics project

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One aspect of mathematics education, in this century, is the lack of scientific culture and curiosity of young people as well as the low mathematics knowledge of students. These students' low mathematics skills translate into lack of motivation for learning this subject, traditionally identified as a hard task. It is our belief that it is important to show the invisible face of mathematics around us to get students and teachers more motivated. In this presentation we introduce an overview of this project whose first goal is to promote the mathematics culture of elementary pre-service teachers and students through the observation and exploration of the urban environment while designing mathematics materials for elementary education. We will present the description of some of the tasks, in particular those connected with patterns.

Un aspect de l'enseignement des mathématiques, dans ce siècle, c'est le manque de culture scientifique et de curiosité des jeunes et leur faible connaissance des mathématiques. Les faibles compétences des élèves en mathématiques se traduisent par l'absence de motivation pour l'apprentissage de cette discipline, traditionnellement reconnue comme difficile. C'est notre forte conviction qu'il est important de montrer la face invisible des mathématiques autour de nous pour que plus d'étudiants et d'enseignants en soient motivés. Dans cette communication, nous présenterons brièvement ce projet dont le premier objectif c'est de devélopper la culture des mathématiques élémentaires chez les enseignants et les étudiants par le biais de l'observation et l'exploration de l'environnement urbain tant bien que la conception de matériaux pour l'enseignement des mathématiques. Nous présenterons aussi la description de certaines tâches, en particulier celles liées aux padrons.

Between 2006-2008 we developed a project¹ that was oriented by the following reasons: As teachers educators we have been noticing the lack of scientific culture and curiosity of young people that is one of the most important features in mathematics education in this century. This happens in short anywhere around the world, but in particular in Portugal, since the recent international studies (e.g. TIMSS, PISA) have shown that our students have weak performance in Mathematics tests. These students' low mathematics skills translate into a lack of motivation for learning this subject, traditionally identified as a hard task.

When students enroll in school they carry, with them, a negative attitude towards mathematics, subject about which they have already heard, most of the times in an inexact way, but about which they have a limited knowledge yet. This situation provokes in the teachers despondency, lack of motivation and of alternative strategies to break this vicious cycle in which mathematics teaching have been transformed. This happens not only at schools but it is a socially installed attitude towards mathematics.

It is our strong belief that mathematics is accessible to everybody and is present everywhere around us. Only actions that show these features of mathematics and conduct to a great aware-

¹ This project, MatCid, ref^a CV/33-2006, was supported by Ciência Viva Agency

ness of people in general and students in particular, will allow us to recover the delay and invert general unmotivation.

With this project, we intended to promote the contact with a contextualized mathematics, starting from the daily life features, walking through and analyzing the city where we live in, connecting some of its details with exploration and investigation tasks in school mathematics. Its aim is to promote the mathematics culture of elementary (pre-service and in-service) teachers and students through the observation and exploration of the urban environment, while designing mathematics curriculum for elementary education. At the same time they learn social and historical events related with what they saw, contributing for a more civic, historical and cultural knowledge of the place where they live.

Background

Teaching a contextualized, applied, experienced, visual, intuitive mathematics to all students, promoting peer and teacher/students interaction, using didactical materials, and communicating their own ideas can contribute to a more significant learning of mathematics. In this perspective, learning requires an active and reflexive student engagement in significant and diversified tasks. In addition, most of mathematics students' failures derive from the affective classroom environment, once it can seriously compromise their initial expectations and motivations. After all we must develop and stimulate creative thinking in the mathematics classroom.

Teachers have a determinant role in the teaching process, so according to that perspective, teacher education should promote a new vision about mathematics knowledge and its teaching. Teacher education must create opportunities either for pre-service or for in-service teachers to explore their world and discover that mathematics is everywhere, connecting mathematical ideas to real world interests, experiences, and empowerment. Teacher education must as well develop teachers' competences such us to be aware, critic and more confident in their mathematical abilities, but most of all teachers have to be alert to know and to discover the "invisible" mathematics that surrounds us.

Among the different mathematical tasks that we use in mathematics classes, those involving problem solving play an important role in the learners' lifes. To look for a pattern is a powerful problem solving strategy and mathematics may be defined as the *science of patterns* (e.g. Devlin, 1999; Orton, 1999). Pattern study has a growing importance in particular on algebra, since "algebraic thinking" has become a catch-all phrase for the mathematics teaching and learning that will prepare students with the critical thinking skills needed to fully participate in society and for successful experiences in algebra. Algebra as the study and generalization of patterns means that students should be able to observe a pattern, form a general algebraic rule and then be able to justify that rule (e.g. Lannin et al., 2006; Mason, 1996; Rivera & Becker, 2005). Moreover patterns are everywhere; we see them in nature, in architecture and in art. In this presentation we will give a special attention to different types of patterns: those related to numbers and algebra and those related to tessellations and friezes, where students can generalize in numerical or geometric contexts.

The project – methodology and procedures

The aim of this project is to awake elementary pre-service teachers, students and population in general to the beauty and utility of mathematics in daily life, discovering it, unveiling it, and exploring the multiples features of the city where we live. Our main goals are: to promote mathematics culture of population; to contribute to the architectural, natural and historic knowledge of the city; to contribute to a more positive view of mathematics; to promote the mathematics culture of elementary teachers and students through the observation and exploration of the urban environment while designing mathematics curriculum for elementary education; to promote mathematic and transversal competences of students; and to contribute to the professional development of teachers.

According to the aims of this project we adopted an exploratory methodology where the participants were pre-service elementary teachers of mathematics; in-service teachers; and elementary students of grades 1-6. The project had different phases, but it started in the classes of teacher education at a School of Education. During the classes of Didactics of Mathematics some themes were selected of different features of the city, such as: documents; traffic and signs; gardens; monuments; buildings; windows; forged iron; tiles; tessellations; and regional embroideries, clothes, *palmitos*, pottery, gold. Then those students, future teachers, had to walk around the city of Viana do Castelo looking for different elements, according to a previous theme and work it mathematically at an elementary level proposing adequate tasks for children (grades 1-6). These tasks were used, with some adaptations, in the posters presented in the exhibition.

The different data was collected and analyzed by the project team according to the previous objectives and the expected products, supported by literature on teaching and learning mathematics education for elementary levels and teacher training.

The products

The main products/tasks created within this project were:

(1) One booklet of tasks exploring mathematic details of particular itineraries of the city of Viana do Castelo. This booklet is for using by common citizens that enjoy mathematics while visiting our city, in teacher training (pre-service and in-service) and in elementary school mathematics;

(2) One CD-ROM/DVD and a Website with information, materials and selected resources for students and teachers of elementary education;

(3) An exhibition – ExpoMatCid. This exhibition was and can be presented in schools and places of general public access. It includes posters with mathematical tasks grounded on several aspects of the city of Viana do Castelo. These tasks are to be solved by people with elementary mathematics knowledge and also intend to motivate young students to mathematics;





and

(4) A mathematical walk through the city, where students of grades 1-6 solve mathematics tasks during a narrative path through some streets of Viana do Castelo and was integrated in the Summer Courses of our institution.

This task was replicated with in-service teachers and after with their own students.

Some conclusions

At the end of this project we hope to have contributed to: the promotion of science; a more positive attitude towards mathematics; and to broaden the vision of the possible connections that can be established between mathematics and the world around us. To design all the tasks in an adequate way to general population with basic mathematics knowledge wasn't always easy. A project that involved students till the secondary level would more and more provide to deepen mathematics subjects and would allow the exploration of different tasks.

Patterns are indeed a powerful resource to develop mathematics concepts and to establish several connections among different subjects and school levels. In general the intervenient in this project find the more obvious connections of patterns those related to geometry (geometric motions - flips, slides and turns – friezes, tessellations) because they are more visual and familiar. The new connection for these intervenient was that related with the development of the algebraic thinking through generalization of patterns. This approach it was less familiar because it wasn't in the Portuguese mathematics curriculum of school. To conclude we can say that the implementation of a project such as MatCid-*Mathematics in the City* has endless chances of exploration.



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