DIFFICULTY AND OBSTACLES WITH THE CONCEPT OF VARIABLE

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INTRODUCTION

There are a lot of studies on the obstacles that the pupils meet in the passage from the arithmetic thought to the algebraic thought. Some of them reveal that the introduction of the concept of variable represents the critical point of transition (Matz, 1982; Wagner, 1981, 1983).

This concept is complex because it is used with different meanings in different situations. Hismanagement depends precisely on the particular way of using it in the activity of problem-solving.

The notion of variable could take on a plurality of conceptions: *generalized number* (it appearsin the generalizations and in the general methods); *unknown* (its value could be calculated considering the restrictions of the problem); *"in functional relation"* (relation of variation with other variables); *totally arbitrary sign* (it appears in the study of the structures); *register of memory* (in informatics) (Usiskin, 1988).

It is possible that many difficulties in the study of algebra derive from the inadequate construction of the concept of variable (Cfr. Chiarugi, I. *et alii*, 1995). An opportune approach to this concept should consider its principal conceptions, the existing inter-relationships between them and the possibility to pass from one to the other with flexibility, in relation to the exigencies of the problem to solve.

The difficulties met by the subject that learns can be very close to those experimented by generations of mathematicians. Some experimental studies (Harper, 1987; Sfard 1992) seem to confirm the thesis of Piaget on the convergence between historical development and individual development (Garcia and Piaget, 1989). From the historical analysis carried out in Malisani (1996, 1999), we drew some important observations on the development of the algebraic language, pointing out evidence the slow and progressive evolution of the symbolic language and the passages over the linguistic levels of the different phases: rhetoric, syncopated and symbolic. In the syncopated phase we observed the necessary to have recourse to other languages: natural, arithmetical or geometric, in lack of an adequate symbolic language. Beginning from the reflections effected on the epistemological and historical-epistemological representations, we planned the study of the obstacles that the pupils meet in building up and assimilating certain concepts, in the passage from the arithmetical thought to the algebraic thought.

The aim of the present work is to study some characteristics of the period of transition between the arithmetical language and the algebraic language. We want to analyze if the different conceptions of variable are evoked by the pupils in the resolution of problems and if the procedures in natural language and/or in arithmetical language prevail as resolutive strategies, in absence of an adequate mastery of the algebraic language.

The experimental research supposes the preparation of a-didactical situations (Brousseau, 1986, 1998). Therefore we want to examine particularly the role of the phases of formulation and validation in the communication and the socialization of the knowledge. In these phases precisely the pupil tries to convince the others of his own strategy, he must reason, discuss the situation and share or look over his opinions. In other words, the pupil is involved emotionally by the situation, the knowledge becomes a social knowledge. Thus we put in evidence the role of the emotion in the learning (Ferreri & Spagnolo, 1994)(1).

This experimentation was effected, thanks to the collaboration of a group of teachers coordinated by the Prof. Teresa Marino and Elsa Malisani, in a few classes of middle school (11-12 years of age) and high school (14-15 years of age) of Piazza Armerina (a provincial town in the province of Enna - Italy). This work is included in a project of experimentation on the teaching-learning of Mathematics entitled *"Inferring, conjecturing and demonstrating in the school of all"*, coordinated by Prof. Filippo Spagnolo. This paper belongs to the doctoral thesis of next publication: "The concept of variable in the passage from the arithmetical language to the algebraic language in different semiotic contexts"(2).

HYPOTHESIS

H_1: The pupils evoke the different conception of variable (constant, unknown, "thing that varies", etc.) also in the absence of an adequate mastery of the algebraic language.

H_2: The procedures in natural language and/or in arithmetical language prevail as resolutive strategies in absence of an appropriate mastery of the algebraic language.

METHODOLOGY OF RESEARCH

The experimental work was divided into three phases: in the first one, the teachers prepared the a-didactical situations and they carried out a-priori analysis of the problem (Cfr. Brousseau, 1986; Brousseau, 1998). In the second and the third one, the experimental data was analyzed qualitatively and quantitatively(3). In this paper the quantitative analysis of the protocols of the phases of formulation and validation is introduced fundamentally.

Twenty seven pupils attending the first year of middle school in two different classes (11-12 years of age) and thirty nine pupils belonging at two first classes of the Psycho-Pedagogical High school (14-15 years of age) participated in the experimentation.

The teachers performed the didactic experimentation on the resolution of the magic square: "complete the square inserting the lacking numbers, so that the sum of the numbers of every line, column or diagonal is always the same".

The use of the magic square has different motivations: it is a problem that can be adapted well enough to the experimentation in the two scholastic levels, because it can be introduced with different modalities and with different degrees of difficulty. But mainly, the magic square allows to study some aspects of the period of transition between the arithmetical language and the algebraic language.

THE A-DIDACTIC SITUATION

- **Phase 1: Delivery** (time 30')

  The teacher simulates the game with the pupil and she explain the procedure for the compilation of the square 3×3. Then two pupils continue the game at the blackboard. Successively, the other students complete other magic squares 3×3, playing in groups of two. They decide of common accord the numbers to insert, choosing therefore a suitable strategy.

- **Phase 2: Action (individual work)** (time 50')

  The students must compile individually a square 4×4, writing on the sheet the various phases of the adopted strategy. They must decide between the possible individualized strategies, which is the more convenient and motivate it. At the end of this phase, the pupils hands over his own work that will come successively assessed quantitatively through a special grid.

- **Phase 3: Formulation (game of team)** (time 20'-30')
  All students are divided into groups. The teacher gives every group the following magic square 4 × 4: *"Complete the magic square in way that the greatest number to insert is equal to 92"*.

Sum 26 + a

14		1	
	9	12	
11		a	10
	16	13	

  Every group has a spokesman. Inside the group the pupils discuss, every student tries to convince the others of his own strategy. Therefore the formulation of knowledge comes about. This phase and the following one are recorded and assessed qualitatively.
- **Phase 4: Validation (time 20').**
  In this phase the pupil must make the feed-back, he must reason, discuss and share or look over his opinions. The students take conscience of the definite strategy of common accord and then they write the demonstration on a sheet. The game is won by the team that succeeds in completing the square first and convincing the whole class of the strategy used by them and thus formulating a valid demonstration.

DISCUSSION AND CONCLUSIONS

Inside the group the pupils discuss during the phase of formulation, every student tries to convince the others of his own strategy. Thus, he must communicate, deducing and conjecturing. The aim of the initial discussions is to decode *"what does it represent the symbol a"* present in the magic square. From the qualitative analysis of the protocols it emerges that the symbol *"a"* represents by the pupils of the middle school: *"…a number that had to be added to all the columns, to all the lines and the two diagonals"* (intuition of the concept of variable), *"a symbol that could be replaced by a number"*, *"a constant equal to 0"*. Although these pupils had not approached the study of algebra yet, they considered the symbol *"a"* under different aspects: constant, numerical value, 0, "thing that varies".
From the qualitative analysis of the protocols of high school we observe that the symbol *"a"* assumes different aspects for the pupils, for example: "…we *must not give any value to a*", "…*a must be a value, because a is a constant, it is not variable*", "*a is an unknown therefore it must be replaced with a number that does not overcome 92…*", *"a must be considered like a kind of variable and we must subtract…"*. The expression of Felicia is eloquent when she talks of the value that the sum of the square assumes: " *… it depends on the meaning that we give a"*. Precisely, a difficult characteristic of symbolic values is that their precise nature changes, they can assume different aspects that have one characteristic in common: the fact that they are abstract.
Therefore the symbol *"a"* can to assume different aspects for the pupils of the high school: constant, not negative constant, numerical value, variable, unknown and symbol without any value. These conceptions depend on the particular way of using the symbols in the activity of problem-solving and from the individual development of the algebraic thought. Therefore these results allow to falsify the first hypothesis: "The pupils evoke the different conceptions of variable, also in the absence of an adequate mastery of the algebraic language".
The phase of formulation considers the research of a resolutive strategy. In the phase of validation the students must make the feed-back, they must reason, share or look over his opinions. They are motivated to discuss a situation, because they must choose a winning strategy of common accord and formulate a valid demonstration.

From the analysis of protocols we observe that some students of the middle school complete the magic square using the negative numbers as "numbers to subtract", but they did not succeed in operating with the symbolic value *"a"*. They used essentially resolutive strategies in arithmetical language and/or natural language, but they did not find a winning strategy.
Most pupils of high school effected algebraic calculations, instead, but in certain cases they made some errors ($26 + a$ like $26a$).
In one class of high school the winning group used the arithmetical procedure, that consists to considered that at least a box must contain the number *92* and that the sum must be equal to $26+a$. They took in examination the column in which the numbers *1*, *12* and *13* are inserted, their sum is *26*, therefore they substituted to *"a"* the maximum number that is *92*. The other group began with an arithmetical-algebraic procedure, instead, inserting some values: *5*, $-4+a$, $9+3a$, etc.; but they did not succeed in completing the square because they did not understand to what box they were to attribute the value 92. Therefore, the students abandoned this strategy and they used also the arithmetical method, too.
In the other class of high school, both groups used the arithmetical-algebraic procedure, considering that a box of every line and every column must contain the symbol *"a"*; thus, they completed the square. In the boxes the pupils noticed the presence of: a, $a – 2$, $a – 4$, $a – 6$; stimulated by the teacher to complete a feed-back and therefore, they deduced that the value 92, the greatest, was to be attributed to *"a"*.
It is interesting to underline that, the pupils did not adopt any algebraic strategies anticipated in the a-priori analysis.
Therefore these results falsify the second hypothesis: "The procedures in natural language and/or in arithmetical language prevail as resolutive strategies in absence of an appropriate mastery of the algebraic language".
We want to put in evidence the importance of the a-didactical environment in this experimentation. The pupils have been involved in their discussions emotionally and they have been able to express their own ideas spontaneously and to choose between the numerous solutions.

NOTES

(1) Two systems of learning have been hypothesized in neurophysiology. A system uses the hippocampus and the amygdale as structures and it bases his operation on the association emotion-learning. The other one uses the striated bodies and it is based on the repetition of the stimulus. The first kind of learning is at the base of the cognitive memory, whereas the second one is a learning of type repetitive (Ferreri & Spagnolo, 1994).
(2) The thesis: "The concept of variable in the passage from the arithmetical language to the algebraic language in different semiotic contexts" was effected by Elsa Malisani to get the degree of Philosophy Doctor, under the supervision of the Prof. Filippo Spagnolo. This thesis will be published shortly on the site Internet http://dipmat.math.unipa.it/~grim.
(3) For the quantitative analysis we used the software of inferential statistic Chic 2000 (Classification Hiérarchique Implicative et Cohésitive) and the factorial analysis survey S.P.S.S. (Statistical Package for Social Sciences).

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