Functional equations as a new tool for researching certain aspects of subject matter knowledge of functions in future mathematics teachers.

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The general tendencies, described in the introduction to the subtheme of CIEAEM 57: "Changes in People's Conceptions About Mathematics", suggest that mathematics is losing its status as a major subject. Also in Poland, both, general interest in mathematics and awareness of benefits resulting from mastering basic mental activities typical for mathematics (A. Z. Krygowska, 1986) are decreasing (R. J. Pawlak, A. Pfeiffer, 2003).

Mathematics teachers have found themselves in a difficult situation as they have to face those negative changes. Furthermore, while teaching mathematics "for everybody", they cannot overlook the exceptionally gifted students they occasionally encounter in the course of their work or neglect mathematically gifted students interested in developing their abilities.

Consequently, the need for a better, more comprehensive and thorough training of mathematics teachers becomes increasingly important (Z. Moszner, 2004; B. J. Nowecki, 2005). The teachers' professional training must be very thorough as the teaching process taking place at school should be based on teachers' profound knowledge. Students of mathematics at teacher training colleges are taught advanced abstract mathematics, its history and didactics of mathematics theory. Moreover, they study in depth school mathematical topics, participate in teaching practice in schools, and learn about various applications of mathematics (S. Turnau, 2003). Their competence in course subjects is evaluated in a detailed way by means of tests and exams. Unfortunately, these are not sufficient for monitoring the competence of the future teachers, as they only test knowledge and skills selectively and in the context of a given course (Z. Powązka, 2004; M. Przeniosło, 2004).

Therefore, there is a growing need for more detailed research on the future teachers' ***subject matter knowledge***. By this term I understand knowledge of abstract mathematics, its methods and history, which are indispensable for teaching. Undoubtedly, in order to reach that end, it is necessary to find tools which are appropriate not only for **carrying out the research** on the competence but also for **continuous assessment** of that competence.

R. Even (1990) put forward a general conception of examining teachers' subject matter knowledge concerning a given concept and illustrated it using the example of the function concept. I have attempted to apply R. Even's theoretical framework in order to analyse data from my research.. However, it turned out that many substantial modifications, extensions and specifications had to be made in that conception. After implementing these changes I distinguished the following components of teachers' subject matter knowledge of a given mathematical concept (M. Sajka, 2005):

1. **The essence of concept**, i.e. knowing the definition of the concept and its origin, understanding the key "idea" of the concept and its basic properties (Z. Dyrszlag, 1978; R. Even, 1990; H. Freudenthal, 1983; J. Konior, 2002a; Z. Semadeni, 2002a, 2002b; A. Sierpińska, 1992; A. Sfard, 1991).
2. **Representations and languages related to concept**, i.e. knowledge of representations of the concept, understanding different languages related to the concept and using them in an appropriate way (R. Even, 1990; E. Gray, D. Tall, 1994; M. Klakla, 2003b, M. Sajka, 2005; A. Sierpińska, 1992).
3. **Basic repertoire of concept designations,** i.e. having at one's disposal a set of concept designations, adjusted to the level of teaching and a thorough understanding of these (R. Even, 1990; Z. Dyrszlag, 1978).
4. **Analysing of concept designations,** i.e. ability to examine concept designations from many points of view and to construct concept designations fulfilling additional conditions (R. Even, 1990; Z. Dyrszlag, 1978, J. Konior, 2002b).
5. **The strength of concept,** i.e. knowledge about the power of concept in mathematics and ability to use that power in solving problems (R. Even, 1990; H. Freudenthal, 1983, J. Mioduszewski, 1996).
6. **Mathematical culture**, i.e.
   (a) knowledge of **elements of mathematical method** (R. Even, 1990; A. Z. Krygowska, 1977),
   (b) mastering basic approaches and behaviours unique to mathematics, i.e. **mathematical activities** - goals

from II level of education according to A. Z. Krygowska (1986) (for instance: generalising, specifying, defining, deducing, reducing, role of examples and counter-examples, ability to prove a theorem, creative activities (M. Klakla, 2002a), transfer of the method (M.Klakla, 2002b)),

(c) mastering approaches and **intellectual behaviours** which can be developed by mathematics and then transferred to everyday situations outside of the mathematical context - goals from III level of mathematics teaching (for instance: discipline and critical thinking (M. Klakla 2003a)),

(d) students' ability to **self-observation their mental activity** (J. Konior, 1993).

The aspects of mathematical culture enumerated above are considered here only in the context of solving the problems related to a given concept.

The aim of my research is to examine the subject matter knowledge of prospective teachers - students of mathematical teacher training colleges - concerning the concept of function.

The hypothesis being verified states that **problems related to functional equations can be used as new, multifunctional tools for revealing subject matter knowledge about the concept of function.** Such problems are unconventional in comparison with the ones the students - future mathematics teachers - work on in the course of their own school and university education. What is more, in order to solve them, one needs a more general approach to the concept of function based on seeing it as a finished, fully shaped object (J. Bergeron, N. Herskovics, 1982; Z. Dyrszlag, 1978; E.Gray & D.Tall, 1994; A. Sfard, 1991; A. Sierpińska, 1992), a part of a wider structure. I have used the term multifunctional research tools in reference to selected functional equations since the results of my research suggest that they can reveal simultaneously several aspects of the subject matter knowledge concerning functions. Furthermore, they can reveal both positive and negative aspects of the understanding of the concept by a given person (defined for example by M. Klakla, M. Klakla, J. Nawrocki, B. J. Nowecki, 1989) for the concept of function (M. Sajka, 2003).

The research subjects were students who were already qualified as primary and secondary school teachers and were undergoing training qualifying them as mathematics teachers in high schools with about 1,5 year of training still to be completed before obtaining their M.A. diploma. The main part of the research consisted in providing written solutions to functional equations by the students. The research subjects were to solve 4 problems on which they worked in two stages, with a month's break in between. Once the written part had been completed, individual interviews were conducted with selected students. Both written and spoken material provided by the students has been analysed. In my paper I will quote as an example one problem used in the research.

PROBLEM 1

A function f: $\mathbf{R}\rightarrow \mathbf{R}$ *is such that holds the requirement* $\mathbf{f(x+1)\ -\ f(x+3)=0}$ *for any arguments.*
Is it possible to give an example of a function that fulfils this requirement and, in addition:
(a) is linear (justify your answer)?
(b) is not linear but is continuous in the domain (justify your answer)?
(c) is discontinuous in the domain (justify your answer)?
(d) what can you say about all the functions fulfilling this requirement?

The terms and language used to formulate the above task are taken from Polish high school curriculum. Thus to solve them one does not need any additional university level knowledge. Students were informed about it at the beginning of their work. It should be mentioned that in Polish schools elements of the domain of a function are called "arguments of the function" and the term is commonly used already at secondary school level (at the age of 12 to 15 years). Moreover, by the term "linear function" we mean function $f\colon \mathbf{R}\rightarrow\mathbf{R}$, whose formula is: $f(x) = ax + b$ where a, b are real coefficients.

The text of the problem has a complex structure but there is only one right way of its interpretation. This problem consists of four sub-problems: (a), (b), (c), (d). In the first sentence of the problem we assumed the existence of a function which fulfils the functional equation. It eliminates any possible student's doubts and he/she can undertake an attempt at searching any examples of the functions having the required properties.

It should be mentioned that this task is not a typical functional equation problem because we want to get merely examples of functions fulfilling this equation. Only question (d) is an attempt to solve it. I call such kinds of tasks "problems related to functional equations".

The above task was undertaken by 62 future teachers but only 2 persons solved it successfully.

Having analysed all the solutions submitted, one might conclude that PROBLEM 1 may reveal the following elements of the student's subject matter knowledge related to functions:

Ad.1 THE ESSENCE OF FUNCTION CONCEPT

- understanding the function not only as a process but also as an object or understanding the function only as a process (through the way of searching for and then selecting or constructing an example). Understanding the function merely as calculating or mapping process prevents a student from solving this problem.

Ad.2. REPRESENTATIONS AND LANGUAGES RELATED TO FUNCTIONS

- ability to choose an appropriate representation (for example in the task (b) it is easier to draw the graph of a suitable function than to construct its formula),
- preferences concerning representation (providing examples),
- ability to use representations correctly (providing examples),
- potential equating graphs or function formulas with functions (epistemological obstacle observed),
- understanding symbols used in relation to functions (symbols $f(x+1)$, $f(x+3)$, requirement $f(x+1)-f(x+3)= 0$. Lack of understanding the functional symbolism prevents a student from solving this problem.),
- ability to use these symbols (for instance, if a student had not associated the requirement $f(x+1)=f(x+3)$ with periodicity of the function he/she should have started comparing the values of function f for arguments $x+1$ and $x+3$),
- ability to discover general meaning of the requirement $f(x+1)-f(x+3)=0$, namely realising that $f(x+3)=f((x+1)+2)$ and noticing in that periodicity of the function, fundamental period 2,
- ability to use the language related to the concept of function (by justifying that the examples given fulfil the requirements of the problem and providing answers to (d)),

Ad.3. BASIC REPERTOIRE OF FUNCTION DESIGNATIONS

- knowing the definition of linear function and its understanding,
- knowing the definition of periodic function and its understanding,
- some elements of a set of designations of the concept – especially among periodic, linear, continuous and discontinuous functions (an overview of these is necessary in order to provide examples),
- ability to select designations for the requirement given (selecting or constructing examples),
- understanding these designations (by selecting examples and justifying correctness of the choice)

Ad.4. THOROUGH ANALYSING OF CONCEPT DESIGNATIONS

- knowing and understanding general proprieties of functions such as periodicity of functions, continuity and discontinuity of the function in its domain
- different approaches to examining designations (point-oriented, range-oriented or examining general properties)

Ad.5. THE STRENGTH OF FUNCTION

(The task cannot reveal this element of the subject matter knowledge)

Ad.6. MATHEMATICAL CULTURE

- being able to understand elements of logic (conjunction and alternative, using general and specific quantifier. They are necessary in order to understand the problem.),
- knowledge of elements of mathematical method and knowing the rules of proving and invalidating a theorem,
- knowing the ways of substantiating a thesis unique to mathematics,
- searching for example or counter-example,
- justifying correctness of the example chosen,
- using deductive reasoning,
- generalising and specifying,
- reading a mathematical text (the problem to be solved)

Moreover, all the aspects of a given person's subject matter knowledge revealed by the task show both elements of his/her **concept image** (S. Vinner, 1983) or, according to another theory, a **procept** of function (E. Gray & D.Tall, 1994) and the persons beliefs - epistemological obstacles, which could possibly be described separately. The problem also allows us to observe occurrence of procedural and conceptual knowledge and the relation between them.

Below I present four randomly selected students' answers. Because of limitations resulting from the scope of this article I am not providing their analysis. In my research I make such detailed analyses of individual solutions in the light of the components of teachers' subject matter knowledge of a function concept I have singled out (see also: M. Sajka, *What subject matter knowledge about the concept of function should the teacher have?*, in this book). The exemplary students' answers provided in this text should be treated as an illustration of the general conclusions presented above. Spoken statements of the students participating in the research were translated from Polish and I include the original version of representations and statements in the mathematical language.

Answer Nr 1 written by Paulina:

A function f: $\mathbf{R} \to \mathbf{R}$ is such that holds the requirement $f(x+1) - f(x+3) = 0$ for any arguments.
Is it possible to give an example of a function that fulfils this requirement and, in addition:

(a) is linear (justify your answer)?

A constant function is such a function. E.g. f(x)=3 ($\forall x \in R$ f(x)=3). Thus, in particular f(x+1)=3, f(x+3)=3, f(x+1)-f(x+3)=0.

(b) is not linear but is continuous in the domain (justify your answer)?

(insert fig. 1 about here)

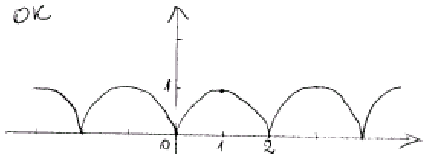

Figure 1

Continuous not linear function with a period of 2.

(c) is discontinuous in the domain (justify your answer)?

(insert fig. 2 about here)

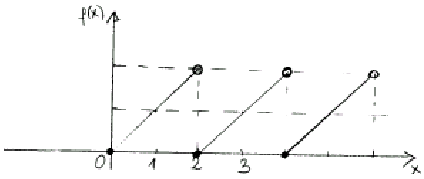

Figure 2

Discontinuous periodic f. with a period of 2.

(d) what can you say about all the functions fulfilling this requirement?

All the functions fulfilling this requirement f(x+1)-f(x+3)=0 are periodic functions with a period of 2.

Answer Nr 2 written by Joanna:

A function f: $\mathbf{R} \to \mathbf{R}$ is such that holds the requirement $f(x+1) - f(x+3) = 0$ for any arguments.
Is it possible to give an example of a function that fulfils this requirement and, in addition:

(a) is linear (justify your answer)?

NO, because f(x+1)=f(x+3). If x=2 then f(x+1)=2, f(x+3)=5.

(Insert fig. 3 about here)

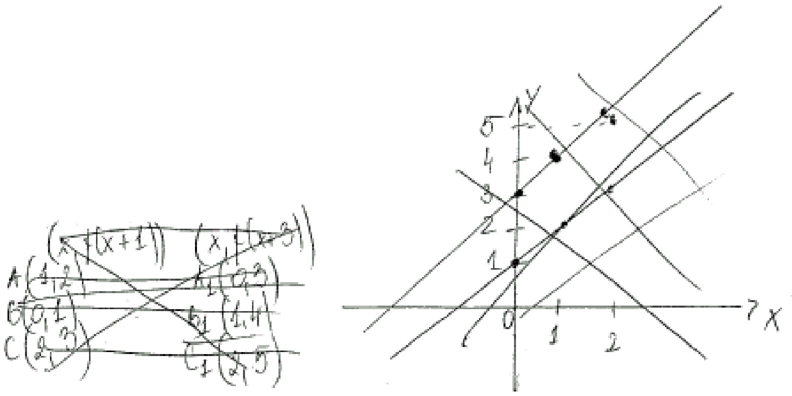

Figure 3

Thus two different values are assigned to one argument.

(b) is not linear but is continuous in the domain (justify your answer)?

NO,

Eliminato: *because it is not*

(c) is discontinuous in the domain (justify your answer)?

I DON'T KNOW

(d) what can you say about all the functions fulfilling this requirement?

I DON'T KNOW

Answer Nr 3 written by Agnieszka:

A function f: $\boldsymbol{R} \to \boldsymbol{R}$ is such that holds the requirement $f(x + 1) - f(x + 3) = 0$ for any arguments.
Is it possible to give an example of a function that fulfils this requirement and, in addition:

(a) is linear (justify your answer)?

Such a function doesn't exist because f(x+1)=f(x+3)
f(x+1) – translation of the function f by the vector [-1,0]
f(x+3) – translation of the function f by the vector [-3,0]
Thus these straight lines are parallel.

(b) is not linear but is continuous in the domain (justify your answer)?

Yes, such a function exists, e.g.

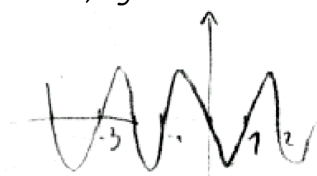

Figure 4

(insert fig. 4 about here)

(c) is discontinuous in the domain (justify your answer)?

Nothing comes to my mind. Probably it exists but I don't know what its graph would look like.

(d) what can you say about all the functions fulfilling this requirement?

I don't know.

Answer Nr 4 written by Paweł:

A function *f*: $\boldsymbol{R} \to \boldsymbol{R}$ is such that holds the requirement $f(x + 1) - f(x + 3) = 0$ for any arguments.
Is it possible to give an example of a function that fulfils this requirement and, in addition:
(a) is linear (justify your answer)?

Yes, it does exist e.g. if f(x) = 1, then f(x+1)-f(x+3)=1-1=0.

(b) is not linear but is continuous in the domain (justify your answer)?

Yes, it does exist but I don't have any example.

(c) is discontinuous in the domain (justify your answer)?

Yes, it does exist but I don't have any example. It seems to me that it would be the function:

$$f(x) = \begin{cases} 0 \text{ for } x \in R \setminus Q \\ 1 \text{ for } x \in Q \end{cases}$$

(d) what can you say about all the functions fulfilling this requirement?

These are all the periodic functions with a period of 2.

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