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Functional equations as a new tool for researching certain aspects of subject matter knowledge of functions in future mathematics teachers.

The description of the second sub-theme, included in the introduction to the theme of CIEAEM 57 "*Changes in Society: A Challenge for Mathematics Education*": "Changes in People's Conceptions About Mathematics", states the following:

In high schools, more and more frequently mathematics teachers are left alone in their professional engagement to teach relevant mathematical content to their students. At that level, particularly in some States, in the USA, mathematics tends to become an optional subject for students. At university level in some countries (e.g. Italy and France) students tend to avoid curricula that demand a high level of competence in mathematics (particularly in advanced mathematics), and the number of mathematical courses offered in technical faculties is falling.

Can the teaching of mathematics allow students to access the "hidden value" of the subject as a crucial component of scientific knowledge, of technology, etc.? What changes are needed in the teaching of mathematics at high school and university levels, in order to cope with this aim?

The above tendencies show that mathematics seems to be losing its status as a major subject. Also in Poland, both, general interest in mathematics and awareness of benefits resulting from mastering basic mental activities typical for mathematics are decreasing (A. Z. Krygowska, 1986; R. J. Pawlak, A. Pfeiffer, 2003).

Mathematics teachers have found themselves in a difficult situation as they have to face those negative changes. Furthermore, while teaching mathematics "for everybody", they cannot overlook the exceptionally gifted students they occasionally encounter in the course of their work or neglect the mathematically gifted students who are interested in developing their abilities.

Consequently, the need for a better, more comprehensive and thorough training of mathematics teachers becomes increasingly important (Z. Moszner, 2004; B. J. Nowecki, 2005). The teachers' professional training must be very thorough since the teaching process taking place at school should be based on teachers' profound knowledge. Students of mathematics at teacher training colleges are taught advanced abstract mathematics, its history and didactics of mathematics theory. Moreover, they study in depth school mathematical topics, participate in teaching practice in schools, and learn about various applications of mathematics (S. Turnau, 2003). Their competence in course subjects is evaluated in a detailed way by means of tests and exams. Unfortunately, these are not sufficient for monitoring the competence of the future teachers, as they only test knowledge and skills selectively and in the context of a given course (Z. Powązka, 2004; M. Przeniosło, 2004).

Therefore, there is a growing need for more detailed research on the future teachers' ***subject matter knowledge***. By this term I understand knowledge of abstract mathematics, its methods and history, which are indispensable for teaching. Undoubtedly, in order to reach that end, it is necessary to find tools appropriate not only for **carrying out the research** on the competence but also for **continuous assessment** of that competence.

R. Even (1990) put forward a general conception of examining teachers' subject matter knowledge concerning a given concept and illustrated it using the example of the function concept. This conception could possibly be adjusted to teaching conditions in other countries (M. Sajka, 2005a).

I have attempted to apply R. Even's theory to the reality of teaching mathematics in Poland. However, it turned out that some substantial modifications, extensions and specifications had to be made. After implementing these changes I distinguished the following components of teachers' subject matter knowledge of a given mathematical concept:

1. **The essence of concept**, i.e. knowing the definition of the concept and its origin, understanding the key "idea" of the concept and its basic properties (Z. Dyrszlag, 1978; R. Even, 1990; H. Freudenthal, 1983; J. Konior, 2002a; Z. Semadeni, 2002; A. Sierpińska, 1992; A. Sfard, 1991).
2. **Representations and languages related to concept**, i.e. knowledge of representations of the

concept, understanding different languages related to the concept and using them in an appropriate way (R. Even, 1990; E. Gray, D. Tall, 1994; M. Klakla, 2003b, M. Sajka, 2005; A. Sierpińska, 1992).

3. **Basic repertoire of concept designations,** i.e. having at one's disposal a set of concept designations, adjusted to the level of teaching and a thorough understanding of these (R. Even, 1990; Z. Dyrszlag, 1978).
4. **Analysing of concept designations,** i.e. ability to examine concept designations from many points of view and to construct concept designations fulfilling additional conditions (R. Even, 1990; Z. Dyrszlag, 1978, J. Konior, 2002b).
5. **The strength of concept,** i.e. knowledge about the power of concept in mathematics and ability to use that power in solving problems (R. Even, 1990; H. Freudenthal, 1983, J. Mioduszewski, 1996).
6. **Mathematical culture**, i.e.
   (a) knowledge of **elements of mathematical method** (R. Even, 1990; A. Z. Krygowska, 1977),
   (b) mastering basic approaches and behaviours unique to mathematics, i.e. **mathematical activities** - goals from II level of education according to A. Z. Krygowska (1986) (for instance: generalising, specifying, defining, deducing, reducing, role of examples and counter-examples, ability to prove a theorem, creative activities (M. Klakla, 2002a), transfer of the method (M.Klakla, 2002b)),
   (c) mastering approaches and **intellectual behaviours** which can be developed by mathematics and then transferred to everyday situations outside of the mathematical context - goals from III level of mathematics teaching (for instance: discipline and critical thinking (M. Klakla 2003a)),
   (d) students' ability to **self-observation their mental activity** (J. Konior, 1993).
   The aspects of mathematical culture enumerated above are considered here only in the context of solving the problems related to a given concept.

The aim of my research is to examine the subject matter knowledge of prospective teachers - students of mathematical teacher training colleges - concerning the concept of function.

The hypothesis being verified states that **problems related to functional equations can be used as new, multifunctional tools for revealing subject matter knowledge about the concept of function.** Such problems are unconventional in comparison with the ones the students - future mathematics teachers - work on in the course of their own school and university education. What is more, in order to solve them, one needs a more general approach to the concept of function based on seeing it as a finished, fully shaped object (J. Bergeron, N. Herskovics, 1982; Z. Dyrszlag, 1978; E.Gray & D.Tall, 1994; A. Sfard, 1991; A. Sierpińska, 1992), a part of a wider structure. I have used the term multifunctional research tools in reference to selected functional equations since the results of my research suggest that they can reveal simultaneously several aspects of the subject matter knowledge concerning functions Furthermore, they can reveal both positive and negative aspects of the understanding of the concept by a given person (defined for example by M. Klakla, M. Klakla, J. Nawrocki, B. J. Nowecki, 1989) for the concept of function (M. Sajka, 2003).

The research subjects were students who were already qualified as primary and secondary school teachers and were undergoing training qualifying them as mathematics teachers in high schools with about 1,5 year of training still to be completed before obtaining their M.A. diploma. The main part of the research consisted in providing written solutions to functional equations by the students. The research subjects were to solve 4 problems on which they worked in two stages, with a month's break in between. Once the written part had been completed, individual interviews were conducted with selected students. Both written and spoken material provided by the students has been analysed. In my paper I will quote as an example one problem used in the research.

PROBLEM 1

A function f: $\boldsymbol{R} \to \boldsymbol{R}$ *is such that holds the requirement* $\boldsymbol{f(x + 1) - f(x + 3) = 0}$ *for any arguments.*
Is it possible to give an example of a function that fulfils this requirement and, in addition:
(a) is linear (justify your answer)?
(b) is not linear but is continuous in the domain (justify your answer)?
(c) is discontinuous in the domain (justify your answer)?
(d) what can you say about all the functions fulfilling this requirement?

The above task was solved by 62 students of a mathematical teacher training college. In the paper I will present several chosen solutions and discuss two of them in more details in the light of the conception presented above. I will concentrate on the elements of teacher subject matter knowledge revealed in the course of solving the problem. These two solutions reveal respectively poor and thorough student's subject matter knowledge of the concept of function. Having analysed all the solutions submitted, one might conclude that PROBLEM 1 may reveal the following elements of the student's subject matter knowledge related to functions:

Ad.1 THE ESSENCE OF FUNCTION CONCEPT

- understanding the function not only as a process but also as an object or understanding the function only as a process (through the way of searching for and then selecting or constructing an example),

Ad.2. REPRESENTATIONS AND LANGUAGES RELATED TO FUNCTIONS

- ability to choose an appropriate representation (looking for examples),
- preferences concerning representation (providing examples),
- ability to use representations (providing examples),
- potential equating graphs or function formulas with functions,
- understanding symbols used in relation to functions (symbols $f(x+1)$, $f(x+3)$, requirement $f(x+1)-f(x+3)=0$),
- ability to use these symbols (for instance comparing value of function f for arguments x+1 and x+3),
- ability to discover general meaning of the requirement $f(x+1)-f(x+3)=0$), namely realising that f(x+3)=f(x+1+2) and noticing in that periodicity of the function, fundamental period 2,

- ability to use the language related to the concept of function (by justifying that the examples given fulfil the requirements of the problem and providing answers to (d)),

Ad.3. BASIC REPERTOIRE OF FUNCTION DESIGNATIONS

- knowing the definition of linear function and its understanding,
- some elements of a set of designations of the concept (an overview of these is necessary in order to provide examples),
- ability to select designations for the requirement given (selecting or constructing examples),
- understanding these designations (by selecting examples and justifying correctness of the choice)

Ad.4. THOROUGH ANALYSING OF CONCEPT DESIGNATIONS

- knowing and understanding general proprieties of functions such as periodicity of functions, continuity and discontinuity of the function in its domain
- different approaches to examining designations (point-oriented, range-oriented or examining general properties)

Ad.5. THE STRENGTH OF FUNCTION

(The task cannot reveal this element of the subject matter knowledge)

Ad.6. MATHEMATICAL CULTURE

- being able to understand elements of logic (conjunction and alternative, using general and specific quantifier),
- knowledge of elements of mathematical method and knowing the rules of proving and invalidating a theorem,
- knowing the ways of substantiating a thesis unique to mathematics,
- searching for example or counter-example,
- justifying correctness of the example chosen,
- using deductive reasoning,
- generalising and specifying,
- reading a mathematical text (the problem to be solved)

Moreover, all the aspects of a given person's subject matter knowledge revealed by the task show both elements of his/her **concept image** (S. Vinner, 1983) or, according to another theory, a **procept** of function (E. Gray & D.Tall, 1994) and the persons beliefs - epistemological obstacles, which could possibly be described separately. The problem also allows us to observe occurrence of procedural and conceptual knowledge and the relation between them.

References:

Bergeron, J., Herscovics, N.: 1982, Levels in the Understanding of the Function Concept, in: van Barneveld G., Verstappen P. (ed.) *Proceedings of the Conference on Functions, Report 1,* Foundation for Curriculum Development, Enschede, 39-46.

Dyrszlag, Z.: 1978, *O poziomach i kontroli rozumienia pojęć matematycznych w procesie dydaktycznym*, WSP w Opolu, seria B: Studia i Monografie nr 65.

Eisenberg, T.: 1991, Functions and associated learning difficulties, in: Tall, D. (ed.), *Advanced Mathematical Thinking*, Mathematics Education Library, Kluwer Academic Publishers, Dordrecht, 140-152.

Freudenthal, H.: 1983, *Didactical Phenomenology of Mathematical Structures*, D. Reidel Publishing Company, Dordrecht.

Even, R.: 1990, Subject matter knowledge for teaching and the case of functions, *Educational Studies in Mathematics* **21**(6), 521-554.

Gray, E., Tall, D.: 1994, Duality, Ambiguity, and Flexibility: a "Proceptual" View of Simple Arithmetic, *Journal for Research in Mathematics Education* 25/2, 116-140.

Klakla, M., Klakla, M., Nawrocki, J., Nowecki, B.: 1989, Pewna koncepcja badania rozumienia pojęć matematycznych i jej weryfikacja na przykładzie kwantyfikatorów, *Dydaktyka Matematyki* **13**, 181-221.

Klakla, M.: 2002a, Kształcenie aktywności matematycznej o charakterze twórczym na poziomie szkoły średniej, in Żabowski, J. (ed.), *Materiały do studiowania dydaktyki matematyki*, t. 3, Wydawnictwo Naukowe NOVUM, Płock, 263-274.

Klakla, M.: 2002b, Transfer metody, in Żabowski, J. (ed.), *Materiały do studiowania dydaktyki matematyki*, t. 3, Wydawnictwo Naukowe NOVUM, Płock, 275-298.

Klakla, M.: 2003a, Dyscyplina i krytycyzm myślenia jako specyficzny rodzaj aktywności matematycznej, *Studia Matematyczne Akademii Świętokrzyskiej* **10**, 89-106.

Klakla, M.: 2003b, Формирование творческой математической деятельности учащихся классов с углубленным изучением математики в школах Польши, Wydawnictwo Naukowe NOVUM, Płock, 223 str.

Konior, J.: 1993, Samoobserwacja aktywności myślowej studentów na zajęciach z przedmiotów kierunkowych jako element przygotowania do zawodu nauczyciela matematyki (raport na podstawie doświadczeń z pracy ze studentami kolegium nauczycielskiego), *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki* **15**, 37-55.

Konior, J.: 2002a, Czym jest pojęcie matematyczne (szkic z perspektywy nauczania i uczenia się) in: Żabowski, J. (ed.), *Materiały do studiowania dydaktyki matematyki*, t. 4, Wydawnictwo Naukowe NOVUM, Płock, 11-30.

Konior, J.: 2002b, Opracowywanie definicji pojęć matematycznych w nauczaniu szkolnym, in: Żabowski, J. (ed.), *Materiały do studiowania dydaktyki matematyki*, t. 4, Wydawnictwo Naukowe NOVUM, Płock, 31-46.

Krygowska, Z.: 1986, Elementy aktywności matematycznej, które powinny odgrywać znaczącą rolę w matematyce dla wszystkich, *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki* **6**, 25-41.

Mioduszewski, J.: 2004, , Ciągłość. Szkice z historii matematyki. WSiP, Warszawa, 1996.

Moszner, Z.: 2004, Refleksje na temat kształcenia nauczycieli matematyki, *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki* **26**, 255-265.

Nowecki, B.: 2005, Koncepcja kształcenia nauczycieli, *Annales Academiae Paedagogicae Cracoviensis*, Wydawnictwo Naukowe Akademii Pedagogicznej w Krakowie.

Pawlak, R. J., Pfeiffer, A.: 2003, Matematyka w szkole. Do czego zmierzamy?, *Studia Matematyczne Akademii Świętokrzyskiej* **10**.

Powązka, Z.: 2004, Z badań nad rozumieniem treści zadania matematycznego, *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki* **26**, 165-178.

Przeniosło, M.: 2004, Images of the limit of function formed in the course of mathematical studies at the university, *Educational Studies in Mathematics* **55**, 103-132.

Sajka, M.: 2003, A secondary school student's understanding of the concept of function – a case study, *Educational Studies in Mathematics* **53**, 229-254.

Sajka, M.: 2005, Refleksje na temat pewnej koncepcji określania wiedzy przedmiotowej nauczycieli matematyki, *Annales Academiae Paedagogicae Cracoviensis*, Wydawnictwo Naukowe Akademii Pedagogicznej w Krakowie.

Semadeni, Z.: 2002, Trojaka natura matematyki: idee głębokie, formy powierzchniowe, modele formalne, *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki* **24**, 41-92.

Sfard, A.: 1991, 'On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin', *Educational Studies in Mathematics* **22**, 1-36.

Sierpińska, A.: 1992, 'On understanding the notion of function', in Dubinsky E., Harel G. (ed.), *The Concept of Function. Aspects of Epistemology and Pedagogy*, MAA Notes, vol. 25, 25-58.

Turnau, S.: 2003, Kształcenie nauczycieli matematyki – u nas i gdzie indziej, *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki* **25**, 231-240.

Vinner, S.: 1983, Concept definition, concept image and the notion of function, *International Journal of Mathematical Education in Science and Technology* 14(3), 293-305.