DIFFICULTY AND OBSTACLES WITH THE CONCEPT OF VARIABLE

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INTRODUCTION

There are a lot of studies on the obstacles that the pupils meet in the passage from the arithmetic thought to the algebraic thought. Some of them reveal that the introduction of the concept of variable represents the critical point of transition (Matz, 1982; Wagner, 1981, 1983).

This concept is complex because it is used with different meanings in different situations. His management depends precisely on the particular way of using it in the activity of problem-solving.

The notion of variable could take on a plurality of conceptions: *generalized number* (it appears in the generalizations and in the general methods); *unknown* (its value could be calculated considering the restrictions of the problem); *"in functional relation"* (relation of variation with other variables); *totally arbitrary sign* (it appears in the study of the structures); *register of memory* (in informatics) (Usiskin, 1988).

It is possible that many difficulties in the study of algebra derive from the inadequate construction of the concept of variable (Cfr. Chiarugi, I. *et alii*, 1995). An opportune approach to this concept should consider its principal conceptions, the existing inter-relationships between them and the possibility to pass from one to the other with flexibility, in relation to the exigencies of the problem to solve.

Kücheman (1981) has shown that most of the pupils between 13 and 15 years of age treat the letters in expressions or in equations like specifics unknowns before as generalized numbers or variables in a functional relation. Trigueros, M. *et alii* (1996) have demonstrated that the beginners university students have a fairly poor conception of variable in its aspects of generalized number and functional relation. Panizza *et alii* (1999) have shown that the linear equation in two variables is not recognized from the pupils like an object that defines a set infinite pairs of numbers. The notion of unknown would not be effective to interpret the role of the letters in this type of equations.

Malisani (2002, 2005) shows that in the context of a problematic situation, if the conception of variable as unknown prevails, then its relational-functional aspect is not evoked. The pupil understands the relational-functional aspect more easily than the variable in presence of visual representative registers, evoking the mental model of the equation of the straight line. Thus, the equation becomes "perceivable" through the graph and the student can "visualize" more easily that it is satisfy by the plurality of solutions. The student is more inclined to consider the variable under the unknown aspect, searching the unicity of the solution of the linear equation, in absence of the graphic representation.

From our study we have (Malisani 2002, 2005) we have demonstrated that the resolutive procedures are supported predominantly by the natural language and/or by the arithmetical language as symbolic systems, when the pupils do not have an adequate command of the algebraic language.

We have also seen that the translation from the algebraic language to the natural language is difficult for the pupils. Some of them succeed in producing the text of a problem that does not result meaningful for the given equation. Others limit themselves, instead, to carry out a purely syntactic manipulation of the formula showing a particular solution that verifies the equation.

The present article intends to examine carefully these conclusions, studying the relational-functional aspect of the variable in the activity of the problem-solving and of the formulation of a problem. We want to analyse how are the conceptions of the unknown and of functional relation set going in the context of the problematic situation and if the natural language and/or the arithmetical language

prevail as the symbolic systems in absence of an adequate mastery of the algebraic language. We also want to investigate the difficulties that the students meet in interpreting the concept of variable, in the process of translation from the algebraic language to the natural one.

METHODOLOGY OF THE RESEARCH

The experimentation was carried out with four students of 16-17 years of age of the Scientific Experimental High School of Ribera (AG) Italy.

The questionnaire presents four questions, but in this paper we will introduce the resolution of the first two queries (Appendix). In the first of them, the variable takes on the relational- functional aspect in the context of a problematic concrete situation. We also ask to think over the quantity of solutions. With this question we want to analyze the resolution strategy used and if the unknown notion interferes with the interpretation of the functional point of view.

The second question asks the formulation of a problem. This must be resolved by means of a given equation, that is, the student must carry out the translation from the algebraic language to the natural language. We consider that this activity represents a fundamental point. It allows to reveal the difficulties that the pupils meet in interpreting the variable under the relational-functional aspect.

The pupils have worked to pairs. They must agree in their discussion before they could write. The whole interview was recorded on audio-cassette and successively it was transcribed.

DISCUSSIONS AND CONCLUSIONS

From the analysis of the protocols of the first problem we notice that the resolutive procedures are based on the natural language and they follow the pace of spoken thought in which the semantic control of the situation is developed and takes place.

On the one hand, the second pair exploits the semantic control of the quantities in relation to the problem in determining the bonds of the numerical universe. On the other hand, they are mixed up in a not very clear and redundant discussion that only brings them to develop a long and twisted procedure with a labyrinth of hypothesis and against-hypothesis. So the ambivalence of the natural language in expressing certain relations between the elements in game becomes evident.

The symbolic language is completely absent in the first protocol. In the second one, instead, it appears in the final part of the resolution when the pupils perform the translation of the problem to an equation. They immediately point out the necessity to give the variable x and y a meaning in relation to the context of the problem, that is they manifest the need to connect the "original story of the problem" (word problem) with the "story reported in symbols" or the *symbolic narrative* using the terms of Radford (2002a)[1].

However in the second protocol the pupils use the symbols in a superficial way, only to communicate, but not to resolve the problem. Therefore the control that the formula can operate on the flow of the verbal reasoning is missing.

In the first protocol the predominant conception of variable necessary to resolve the first problem is that of unknown. The students calculate particular solutions by resolving two linear systems. Since they do not know the criterion by which the sums of money can be divided, they conclude that the solutions are infinite. The impossibility to find this criterion is equivalent to the impossibility to form a single system. Therefore the passage from the single solutions to infinite solutions is produced through the systems of equations. In other words, for this pair of students the infinite solutions constitute a set of single solutions coming from the resolution of different linear systems that contain the given equation. Accordingly they do not state the problem of the bonds imposed by the context in which the expression is considered.

The relational-functional aspect of the variable prevails, instead, in the second protocol. So the infinite solutions constitute a set of pairs of values that are obtained by varying one of them and calculating the other, beginning from the linear dependence between the variables.

The two couples of students begin the second question by effecting a purely syntactic manipulation of the equation to find some solutions. In the second protocol we observe a game of signs without

sense in some passages. That is, it would seem that the students see the equation like a string of arbitrary symbols, a string governed by arbitrary rules (Linchevski and Sfard, 1991).

From the study carried out we deduce that the pupils actually confuse the activity of solving an equation with that of inventing a problem which originates from an equation. We think that this difficulty is due to a matter of didactic contract: at school usually the students resolve problems, they do not invent problems.

The formulation of a problem from an equation implicates fundamentally three activities:

➢ Choosing an adequate context to give meaning to the equation
➢ Identifying the objects of the context that represent the variables
➢ Individualising the properties of the objects that are pointed out by the relation expressed in the equation

We believe that the critical stage is precisely: "to individualize the elements of the context to be associated to the variables". In the second protocol we assist at the attempt to choose a context of "market and apples", but the students do not succeed in identifying x and y with the quantities of apples of two different subject-objects: two shopkeepers, two different varieties, two different cassettes, etc. Thus they formulate the text of a classical arithmetical problem with specific numerical values (the coefficients of the equation); in the attempt of bettering the statement, they succeed only in inserting a variable and therefore they abandon this context.

The two couples of students resolve the query producing a text similar to the first problem. This means to deal with the context "money and bets" and the elements "two persons that play". They must only adapt the properties of the objects to the new relation that the equation expresses. We thought that this activity would have brought about the paraphrasing of the text of the first problem, but it was not so obvious, especially in the second protocol. One couple felt the need to make the variable emerge in the text of the problem and to interpret the minus sign; their final formulation is the consequence of a gradual elaboration.

According to Radford (2002b), in some occasions the symbols produced by the pupils (in this case the minus sign) constitute simplified writings (scripts) that tell important parts of the original story[2]. Therefore the exigency to effect the interpretation is associated to the possibility of conferring the correct algebraic sense to the expression and not that of the scripts. On the other hand, the phrase: *"The betted money is this x. x and y are the unknowns of what they have betted"* (Line 451) represents the connection between the equation and the text of the problem, that is between the symbolic narrative and the story of the problem. After having individualised the objects of the context, it is necessary to work on the expressed relation in the equation.

In the two protocols we clearly observe an important loosening between the symbolic language and the possibility of finding a different context from "money and bets", to give meaning to the equation. We think that this is not the consequence of the lack of a certain amount of creativeness, but the result of an insufficient control on the symbols. This is revealed in the impossibility to associate the variables to some elements of the context.

From the study carried out it results evident that an equation alone does not activate forms of productive thought, it is not considered absolutely like the interpretative model of a problem or better still as a class of problems.

To study these conclusions in depth it would be interesting to analyze the existing relation between the variables of an equation and the objects of the context that represent them, from a semiotic perspective of the discourse. It would be important to study how the construction of the sense of a symbolic expression takes place in the space in which the dominion of the symbolic narrative still has not been achieved completely and the story of the problem is just outlined.

NOTES

(1) Radford (2002a) prefers to speak of symbolic narrative to point out the translation of a given problem into an equation. According to the author this term allows to indicate that a story is still told, but in mathematical symbols.

(2) Radford (2002b) considers that, for some pupils, the minus sign in the expression x - 2, does not always indicate a subtraction on the unknown, sometimes it represents the sign of a simplified writing in relation with the original story.

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APPENDIX: QUESTIONNAIRE

1- Charles and Lucy win the lottery the total sum of euro 300. We know that Charles wins the triple of the betted money, while Lucy wins the quadruple of her own.
   a) Determine the sums of money that Charles and Lucy have betted. Comment the procedure that you have followed.
   b) How many are the possible solutions? Motivate your answer.

2- Invent a possible situation-problem that could solve using the following relation of equality: *6 x – 3 y = 18*. Comment the procedure that you have followed.