SALVADOR LLINARES* and ANA ISABEL ROIG

SECONDARY SCHOOL STUDENTS' CONSTRUCTION AND USE OF MATHEMATICAL MODELS IN SOLVING WORD PROBLEMS

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ABSTRACT. This study focussed on how secondary school students construct and use mathematical models as conceptual tools when solving word problems. The participants were 511 secondary-school students who were in the final year of compulsory education (15–16 years old). Four levels of the development of constructing and using mathematical models were identified using a constant-comparative methodology to analyse the student's problem-solving processes. Identifying the general in the particular and using the particular to endow the general with meaning were the key elements employed by students in the processes of construction and use of models in the different situations. In addition, attention was paid to the difficulties that students had in using their mathematical knowledge to solve these situations. Finally, implications are provided for drawing upon student's use of mathematical models as conceptual tools to support the development of mathematical competence from socio-cultural perspectives of learning.

KEY WORDS: abstraction, conceptual learning, mathematical competence, mathematical model, modelling, problem solving, stages of learning

INTRODUCTION

The way in which secondary students solve textual word problems provides information on their mathematical literacy (Silver, 1992; Verschaffel, Greer & De Corte, 2002). However, some researchers have revealed that many students merely carry out arithmetical operations on the quantities specified in the problem-question without taking into account the conditions of the situation. Greer (1993, p. 244) suggests that "students are liable to respond to word problems according to stereotyped procedures assuming that the modelling of the situation described is "clean" [...] Instead of being able to assume simple and unproblematic modelling, students should have to consider each textually represented situation on its merits, and the adequacy and precision of any mathematical model they propose."

*Author for correspondence.

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Research indicates that students use informal strategies for solving problems in their attempts to give meaning to stated situations. The strategies they use reveal their understanding of the relationships expressed in the situations and how they use this information to solve problems (Gravemeijer, 1994; Johanning, 2004; Nesher, Hershkovitz & Novotna 2003; Tzur, 2000). Johanning (2004) found evidence to show that students who had not learned algebra used systematic guess-and-check strategies in order to solve algebraic problems. Johanning (2004, p. 384) points out that "students will use the information obtained from previous guesses to make reasonable new guesses that will systematically move them toward the correct solution [...] In order to devise a system to systematically guess and check with, students had to understand the underlying structure of the problem and articulate it into a formal plan." The strategies used by these students can be interpreted as attempts to identify the structure of the situation by taking into account the unknown quantities and the totality. In other words, they constructed and used a model of the situation through a process of mathematical modelling.

From this point of view, the process of mathematical modelling involves the construction of a mathematical representation (a model) of the situation in order to think about it, explain it or formulate predictions based on the structural characteristics of the situation (elements, relationships, patterns or operations, etc.) (Blomhøj, 2004; Lange, 1996; Verschaffel et al., 2002). From this perspective, we assume that the mathematical conceptions play the role of models and/or tools in certain situations. In this sense, the student identifies the quantities and variables in the situation as well as the relationships between them, in order to come to a decision regarding the situation and to communicate that decision. Mathematical notions, such as linear function and multiple, can have a role as tools to handle a situation. Säljö (1999b, p.147) points out that "a fundamental assumption in a sociocultural understanding of human learning is precisely this: learning is always learning to do something with cultural tools (be they intellectual and/or theoretical)." Mathematical knowledge can be seen as a set of tools for thinking and acting. So, becoming mathematically competent has to do with how students appropriate and gain expertise in using tools for thinking and acting. From this, the mathematical knowledge (concepts, structures, ... that can be used to organise a situation can be considered as "mathematical models." When students use some aspects of this mathematical knowledge for thinking and acting in a situation, we can understand that they are using a mathematical model at some level of development as a tool for thinking (Izsák, 2004).

The modelling process is thus guided by a goal and its end-product is a model of greater or lesser sophistication. The analysis unit is the student in action using tools of some kind. The idea of a “degree of development” of the modelling capacity is therefore meaningful when the student is learning to construct and use models in order to become mathematically competent (Lesh & Harel, 2003). Therefore, learning can be identified as the extent to which learners are able to use the model constructed in different situations. Similarly, learning can be understood as the ability to use a particular set of tools in productive ways and for particular purposes (Säljö, 1999a). In the process of interpreting situations, constructed mathematical models can be used by students as tools which can help them to understand the situation and to formulate predictions as distinct aspects of the anticipatory phase in the mathematical conceptual learning (Greeno & Hall, 1997; Säljö, 1999a; Simon, Tzur, Heinz & Kinzel, 2004; Tzur & Simon, 2004). In this sense, Kastberg, D’Ambrosio, Mc Dermott & Saada (2005) suggest that students must also integrate personal and mathematical knowledge reasoning with knowledge of content areas other than mathematics as an essential factor in the continued development of mathematical power.

Simon et al. (2004) argue that a learner’s goal-directed activity and its effects (as noticed by the learners) serve as the basis for the formation of a new conception. This new conception can be a mathematical model for the situations. Simon et al. (2004) establish their characterization of the mechanism to explicate the conceptual learning, reflection on activity-effect relationships, from Piaget’s (2001) reflective abstraction. They identify the following components of the mechanism: “the learners’ goal, the activity sequence they employ to try to attain their goal, the results of each attempt (positive or negative), and the effect of each attempt (a conception-based adjustment). Each attempt to reach their goal is preserved as a mental record of experience” (Simon et al., 2004, p. 319). This process is supported by the capacity of the learner to compare the effects of his/her activity sequence (identifying invariant relationships in the situation) with his/her goal. So, the identification of invariants in each comparison is the product of an abstraction of the relationship between the activity and the effect. In this sense, “an abstracted activity-effect relationship is the first stage in the development of a new conception” (in this case a model of the situation). In this process Simon et al. (2004, p. 319) elaborate on the two phases of Piaget’s (2001) reflective abstraction: the projection phase and the reflection phase, and note that “the regularities abstracted by the learners are not inherent in the situation but rather a result of the learners’ structuring of their anticipation-based observations in relation to their goals and related (existing) assimilatory structures."

The research reported here was carried out to contribute to two lines of research. First, it helps to explain the process of modelling situations as evidence of the students' mathematical competence. Second, it obtains information that helps us to understand better the development of new conceptual entities. The specific goal of this research was to investigate how secondary school students construct and use mathematical models as conceptual tools when solving word problems. We ask: what are the characteristics of the modelling process generated by these students?

Method

Participants and Tasks

511 students participated in the survey (47.7% girls and 44.8% boys, 7.4% omitted to indicate their sex). Their ages ranged from 15 years and 5 months to 18 years and 3 months ($A = 16.15$ years old). A five-question test paper was first prepared. Three of the questions were word problems in which students had to make a decision and justify it. The other two problems were mathematical contexts for generalising (numerical patterns). In this paper we present an analysis of the problem-solving strategies used by the students in order to solve the three word problems (Figure 1).

The mathematical concepts and procedures that can be useful to solve these problems are included in the standard secondary-education curriculum, so it was reasonable to suppose that the students were acquainted with them. But, as some research work has shown that students learn to identify word problems as mathematical exercises requiring the identification of an algorithm (Greer, 1993; Silver, 1992; Verschaffel et al., 2002), an attempt was made in the word problems presented to avoid the stereotyped nature of school maths problems. In the tasks, the students were asked to make a decision regarding the situation and to justify it. They were also asked to express the reasons why they thought their decision was the right one. The idea behind using this type of problem is to avoid encouraging students to simply look for an operation they think suitable for solving the problem.

Our objective was to characterise the way in which students gave meaning to the situation and how they solved it using mathematical models. To this end the students were asked to justify their decision,

Q1: THE JOB OFFER
Job offers for pizza delivery workers have appeared in a local newspaper.
Pizza takeaway A pays each delivery worker 0.6 euros for each pizza delivered and a fixed sum of 60 euros a month. Pizza takeaway B pays 0.9 euros for each pizza delivered and a fixed sum of 24 euros a month.
Which do you think is the better-paid job?
Make a decision and explain why your choice is the better one.

Q2: THE FESTIVAL ENCLOSURE
The festival committee in your area wishes to prepare a rectangular festival enclosure, with a surface area of 400 m^2. The enclosure is to be fenced off with metal fencing costing 30 euros a linear metre.
What are the best dimensions for the site, if the cost of the fencing is to be reduced to a minimum?
Explain why the dimensions you have chosen are the best.

Q4: THE DANCE FLOOR
A floor-tile manufacturer has donated a quantity of floor tiles to the festival committee. Each tile is 33 centimetres long and 30 centimetres wide. The committee has decided to lay a square dance floor within the festival enclosure, but you have to tell them:

a) *the length of each side of the smallest square that can be made with this size of tile without cutting any of them*
b) *what other sizes of square dance-floors could be laid using only uncut tiles of this size, and why?*

In your reply to the committee, explain what you have done.

Figure 1. Word problems in which students must make a decision and provide an argument.

i.e., to explain the reasoning behind their decision rather than giving a mere number as the result of an operation. In this way the problem-solving process would enable the student to become involved in a "communicative process" with other people.

Finally, and in order to obtain further information on the ways in which students modelled the situations, 71 students, chosen at random, were interviewed. The aim was to get the students to verbalise the thought-processes they had used in solving the problems (Goldin, 2000), and to this end the interviewer discussed the students' solutions and elicited reasons for the decisions arrived at. The interviews were audio-taped and later transcribed. Analysis of the replies provided greater insight into the modelling process followed in each situation by the students (Clement, 2000).

Mathematical Models in the Problems

All three problems could be modelled in different ways, some of them requiring advanced mathematical content. However, the situations could also be interpreted and manipulated using the mathematical knowledge that final-year secondary-school students are assumed to possess. In this situation of problem solving we assume that the students' goal is to provide an answer and justify it. So, the activity of solving word problems can be considered as a goal-driven-activity.

P1: The Job Offers. A mathematical model that can be used in this problem is the comparison between two linear functions.

$$\textit{Monthly earnings in job A} = 0.6 \times \textit{pizzas delivered} + 60$$
$$\textit{Monthly earnings in job B} = 0.9 \times \textit{pizzas delivered} + 24$$

The number of pizzas delivered and the monthly earnings appear in the mathematical structure of the problem as variables. The relation between the two makes it possible to relate the money that can be earned with the number of pizzas delivered. In order to make a comparison, the number of pizzas delivered must be regarded as a variable common to pizza takeaway A and pizza takeaway B. The amount to be earned in each takeaway must be compared by assigning the same values to the "number of pizzas delivered" variable in order to calculate final earnings in A and B. This comparison makes it possible to identify the change produced in the wages offered by each of the two takeaways as the number of pizzas increases. It is essential to consider these characteristics in order to properly justify the decision taken. Although it would be appropriate to find out the average number of pizzas delivered from each establishment so that one could estimate earnings, the key issue in the model used by the students in this situation in order to generate an argument is to consider the number of pizzas delivered as a variable amount.

P2: The Festival Enclosure. One mathematical model for this situation is the multiplicative relation "$a \times b$ = constant" where the quantities "a" and "b" may vary. There are three relationships that have to be identified: the relation between the length of the sides of the rectangle and its area (a $\times$ b = area), the relation between the length of the sides of the rectangle and its perimeter ($2a + 2b$ = perimeter), and the relation between the price per linear metre of the fencing and the final cost of the fence (30 $\times$ perimeter = final cost) Two conditions affect these relationships. In the first place, the area must be equal to a constant quantity ($a \times b = 400$), and secondly, the cost of the fencing must be minimal ($30[2a + 2b]$ = a minimal quantity). This second condition may be reduced to the idea that the perimeter of the site must be minimal ($2a + 2b$ = a minimal quantity). In this situation two aspects must be borne in mind: the variability in the sides of the rectangle *versus* the non-variability in the area (possible pairs of values (a, b) such that $a \times b = 400$), and the variability of the perimeter as a function of the length of the sides. The conditions to be considered are interdependent, since the condition of "constant area" limits the set of data due to be dealt with under the second condition of "minimal perimeter." Although one can use calculus to find the minimum of some function like $f(a)=a+400/a$, to secondary school students the challenge was to manage the different conditions of situations and the influence of variability of some values in determining the smallest perimeter.

P4: The Dance Floor. One mathematical model for this situation that students can use is the idea of the lowest common multiple. The meaning of the common multiple comes into play due to the need to establish the relationship between the length of each side of the tiles and the length of each side of the finished dance-floor. In this situation, the student should realise that if rectangular tiles whose area is $a \times b$ are used, a square floor can be laid each side of which will be a common multiple of a and b. To come to this realisation, some students can use a way of "mathematising" diagrams. At some level of development the idea of the lowest common multiple will make it possible to determine the size of the smallest possible square floor that can be laid. Another multiplicative relation that applies to this situation is that starting from a square of side c other larger squares can be formed whose sides will be multiples of c ($2c$, $3c$, $4c$ ...) and, therefore, common multiples of "a" and "b". Although the lowest common multiple can be considered a mathematical model in this situation, students can use other tools in their way towards the "mathematization" of a situation.

Data Coding. Procedure and Analysis

The students' answers were analysed taking into account the way in which each student set up and used a model in order to interpret the situation and then made a decision using that model. The construction and use of the model set up by the student was inferred from the written text supplied by the student, and by the way in which he or she justified the final decision. Using a constant-comparative methodology (Strauss & Corbin, 1994), two researchers characterised from a small sample of the problems the way in which the students' answers indicated the construction and use of a model. We used a grounded theory approach in the analysis of students' answers, so we identified characteristics in the students' approach to the solving of the problem and these characteristics led to definition of levels of development in the process of constructing and using mathematical models as tools in word problem solving. The way in which students considered the variability of the quantities, the conditions that had to be fulfilled by these quantities in the situations given, and the way in which the students discerned generalities in the particular data determined the characteristics of the modelling processes generated by the students. We interpret these characteristics from the process involving students' goal-directed activity and the natural process of reflection.

We continued to review answers, observing how each one aligned or did not align with the initial descriptions. Thus, when we noticed an anomaly in the description of a characterisation, we modified this description or developed a new characteristic and re-examined several earlier answers for similarities and discrepancies. The discrepancies were discussed by the researchers, which led to modifications in the initial characterising of the answer or modifications in the characterisation of the development of the model. Following this process of analysing students' answers we generated a description of different levels of development. In this way, we were able to revise the coding scheme and to compare the new characterisations with previously coded protocols until the classification scheme became stable. We characterized four levels of development in the modelling process employed by the students, taking into account how they set up and used a model in order to arrive at a decision and the way in which they used it in order to justify this decision (Figure 2).

Results

The construction and use of models was revealed in the way in which students were able to think beyond the particular cases in order to incorporate in their model the influence of the variability of the quantities. The way in which students used the information provided by the particular cases was different in each situation, which suggested some influence of the structure of the situation on the modelling process. In problem 1 (comparison of linear functions) and in problem 2 (multiplicative structure with conditions) the students used the particular cases to generate a general model of the situation, but in problem 4 (common multiple) they relied on the general model to generate particular values when they could not remember a suitable algorithm.

In order to discover the importance of the students' use of particular cases in the modelling process, we propose to show the characteristics of the different problem-solving processes as generated by the students for each of the problems. This will enable us to see how each situation was

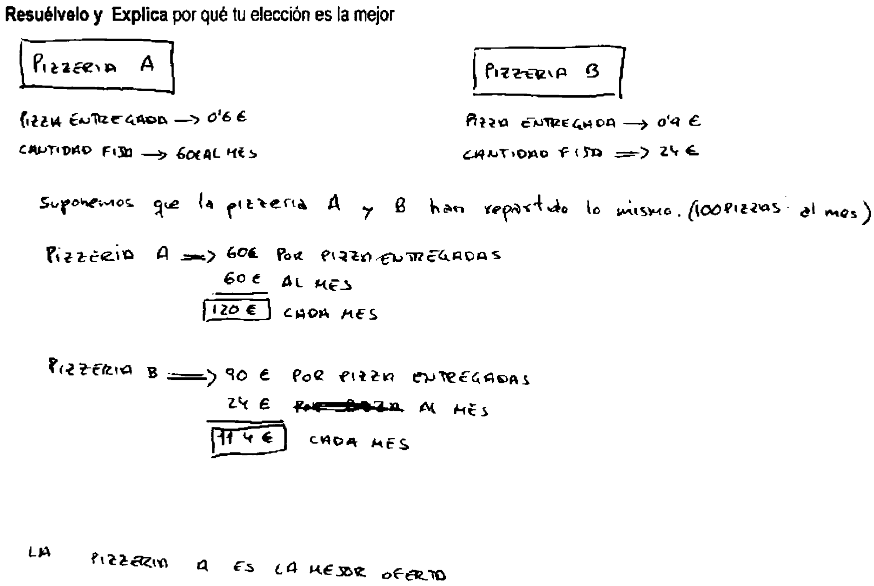
Level	*Characteristics*
LEVEL 0	*The student doesn't appear to be able to trigger activities to accomplish the goal.* *He/she paraphrases the description of the situation or gives confused explanations.* *Although the student sometimes appears to identify some of the variables which may be relevant in the situation, (s)he is unable to establish any meaningful relation between them.* *Apparently meaningless arithmetical operations are carried out, or false relationships are established.*
LEVEL 1	*The student aims at a certain goal and initiates activities to accomplish it.* *The quantities and relationships involved in the situation are identified, but "global" comprehension of the situation is incomplete, which prevents the student from developing an effective model with which to interpret the situation and justify the decision taken.*
LEVEL 2	*The student is able to anticipate what he/she needs to achieve the goal by making goal-directed adjustments and identifies activity-effect relationships.* *Some relevant aspects of the situation are identified and the relationships between them are established, thus revealing a structural understanding of the problem.* *An effective model is constructed in order to facilitate the search for an answer, but the model is not used appropriately for decisions to be made.*
LEVEL 3	*The model and its use are the result of the students' structuring of their anticipation-based observations in relation to their goals and related (existing) assimilatory structures.* *A model for the situation is constructed or identified and is used in an appropriate manner so that decisions can be made and justified.*

Figure 2. Levels of development of constructing and using mathematical models as tools in textual word problem solving.

interpreted mathematically and how the student based his or her decisions on the mathematical model constructed.

Problem 1: The Job Offer

This situation was the one the students modelled best. The characteristic of the mathematical model for this situation is the changing attractiveness of the monthly income according to the number of pizzas delivered. Students who realised how variations in the number of pizzas delivered would affect monthly income were able to construct an appropriate model for the situation and to justify their decision. However, it was not easy for them to handle the idea of variability, as many students relied only on the information provided by this particular case. For instance, JJ/31 (see Figure 3) based his decision on the information provided simply by supposing that 100 pizzas were delivered each month. JJ/31 calculated the amount earned at each of the pizza takeaways if 100 pizzas were delivered each month and then compared them ("let us suppose that in each of the two takeaways, A and B, the same number of pizzas are delivered (100 a month)"). JJ/31 identified the relationship between


Resuélvelo y **Explica** por qué tu elección es la mejor

PIZZERIA A

PIZZA ENTREGADA → 0'6 €

CANTIDAD FIJA → 60€ AL MES

PIZZERIA B

PIZZA ENTREGADA → 0'9 €

CANTIDAD FIJA ⇒ 24 €

Suponemos que la pizzeria A y B han repartido lo mismo. (100 pizzas al mes)

PIZZERIA A ⇒ 60€ POR PIZZA ENTREGADAS

60 € AL MES

$\boxed{120 €}$ CADA MES

PIZZERIA B ⇒ 90 € POR PIZZA ENTREGADAS

24 € ~~POR PIZZA~~ AL MES

$\boxed{114 €}$ CADA MES

LA PIZZERIA A ES LA MEJOR OFERTA

Figure 3. JJ/31's decision for word problem 1, based on the consideration of one particular case.

monthly earnings and the number of pizzas sold, the amount received for each pizza and the fixed sum paid every month, but the decision taken is justified only in one particular case ("the job offer at pizza takeaway B is the better of the two"). In this type of answer, students used one or more particular cases, but the information they obtained from their calculations could not go beyond the particular, and could not, therefore, enable them to perceive the influence of variability of the quantities on the decision they were expected to make. This manner of modelling the situation reveals difficulty in integrating the influence of variations in the number of pizzas on the model constructed. This was a typical behaviour pattern for level 1.

One important aspect in the model-construction process for this situation is the awareness of the influence of the variability in the number of pizzas on comparisons of possible earnings. In this situation, $n = 120$ pizzas is the key amount that tips the scales when considering the job offers made by pizza takeaway A and pizza takeaway B. The way in which a student might consider this aspect could vary.

One way of realising the influence of the number of pizzas delivered on the benefit obtained is when more than one specific case is used to compare the two offers. Answers that relied on several particular cases revealed that those students had begun to understand the variable behaviour of the final wage. Answers based on several different particular cases showed that the learners understood the variable behaviour of the wages. But the information obtained from particular cases made the difference between level 1 models and level 2 models. At level 2, the learner is able to see beyond the particular cases in order to focus his or her attention on the behaviour of the difference between the two wages and on how the variability affects this behaviour (finding a pattern) (only 15.7% of the students were at this level).

For example, JJ/60 (see Figure 4) used two particular cases: “if 100 pizzas are delivered” and “if 110 pizzas are delivered.” The information obtained from the calculations performed in these two cases appeared to suggest to the student that the difference between the two job offers gradually diminished, which led him or her to affirm that “job offer B is the better one, so long as more than 110 pizzas are delivered.” JJ/60 seems to have constructed a model for the situation in which the influence of the variability in the number of pizzas delivered is seen to affect the decision that has to be made. Thus, the general information obtained from the two particular cases led the student to discover the influence of the variability and therefore to integrate this variability into

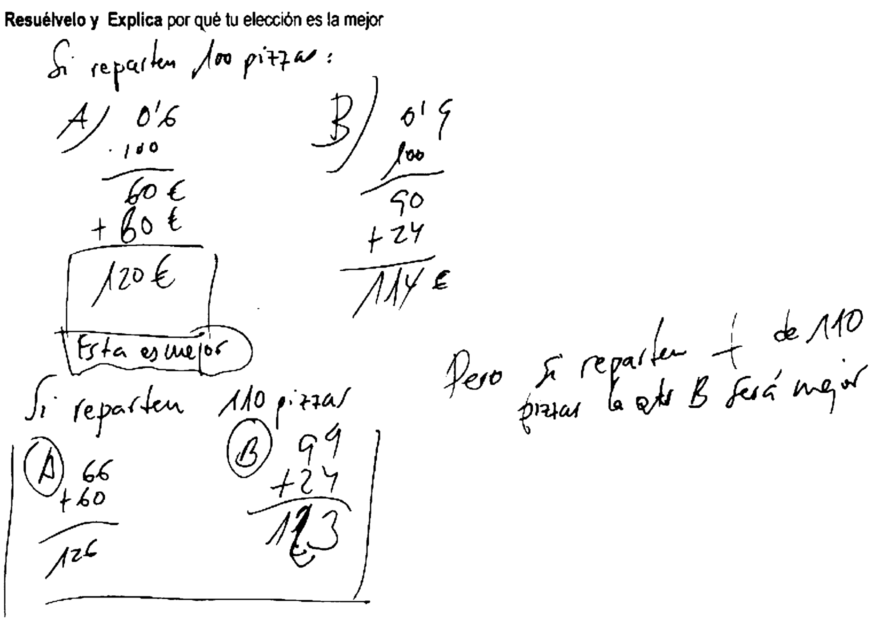

Resuélvelo y Explica por qué tu elección es la mejor
Si reparten 100 pizzas:
A) 0'6 · 100 = 60 €; + 60 € = 120 €
Esta es mejor
B) 0'9 · 100 = 90; + 24 = 114 €
Si reparten 110 pizzas
A) 66 + 60 = 126
B) 99 + 24 = 123
Pero si reparten + de 110 pizzas la oferta B será mejor

Figure 4. JJ/60’s answer for word problem 1 in which he/she sets the goal and initiates activities that can achieve the goal.

the model for the situation. JJ/60's decision is not based only on the results of the two particular cases, but does recognise the change that characterises this situation when (s)he states that "if more than ... are delivered, (then) ...." However, although JJ/60 makes a decision based on the influence of the variability of the number of pizzas, the student's handling of this variability is not entirely appropriate since (s)he states that "job offer B is better so long as more than 110 pizzas are delivered," which is not really correct (for example, if 115 pizzas are delivered, pizza takeaway A's offer is still better). This kind of behaviour was a characteristic of the answers we judged to be at level 2 in the process of constructing and using a mathematical model.

Those students who properly handled the influence of the variability of the quantities on their interpretation of the situation in order to make a decision made proper use of the model constructed. For example, FP/13 (see Figure 5) identified the relevant information in the situation and

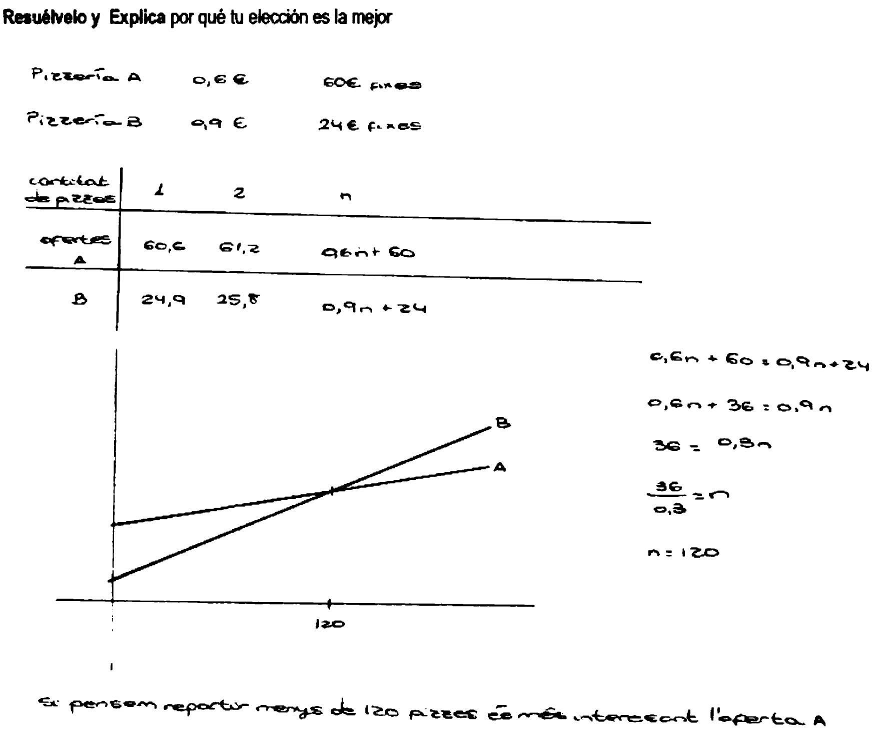


Figure 5. FP/13's answer for word problem 1 in which he/she is able to anticipate what is needed to accomplish the goal and makes goal-directed adjustments.

used a table to represent the variability in possible earnings depending on the number of pizzas delivered. This student used the letter "n" to represent the number of pizzas (as a way of representing the general nature of the variable quantity "number of pizzas") and indicated the relationship between what one could earn at pizza takeaway A and pizza takeaway B by equalising the two functions in order to arrive at an equation. The algebraic manipulation of the equation $0.6n + 60 = 0.9n + 24$ enabled him or her to obtain $n = 120$, which was interpreted as the number of pizzas delivered in order to earn the same at the two takeaways. The graphic representation of what could be earned at each of the takeaways as a function of the number of pizzas delivered made it possible to show the influence of variations in the number of pizzas delivered on the final wage offered by the two pizza suppliers. The model is thus used to make a correct decision "If we intend to deliver fewer than 120 pizzas job offer A is more attractive." In this case, the graphic representation of the situation has been used to communicate the influence of the number of pizzas delivered when determining the wage paid by each of the takeaways.

In this answer, by considering the influence of the variability in the number of pizzas delivered, FP/13 was able to take into account elements which made it possible to make an appropriate decision. On the other hand, the use of graphic representation of the functions and the equation equalising the functions made it possible to communicate the reasoning behind that decision. It is therefore the student's argumentative process in communicating the decision that reveals the efficiency of the model constructed.

On the whole, students were able to construct a model for the situation based on the influence of variations in the number of pizzas delivered if they realised that the behaviour patterns of their possible earnings would vary according to the number of pizzas delivered. A model for the situation constructed on these lines enabled students to perceive changes in the possible earnings offered by the two pizza takeaways. Finding the number of pizzas delivered that would mark the change in tendency was determined by the way in which each student handled the variability, either by expressing the equation $0.6n + 60 = 0.9n+24$ (where n is the number of pizzas) or by repeated use of particular cases successively approaching $n = 120$. We can understand this behaviour as the development of an activity sequence. In either case, it is by appropriately integrating the influence of variations in the number of pizzas delivered on the interpretation of the situation after observing general tendencies in particular cases that students are able to develop an appropriate model bringing together all the quantities and relationships, and to use this model effectively in order to make decisions.

Problem 2: The Festival Enclosure

The situation given here relies on the idea that for a constant area, the rectangle of shortest perimeter is a square. The way in which students handled this property in the problem-solving process gave an indication of how they constructed a representation of the situation. However, it was not easy for students to identify and use this property because they had to manage simultaneously the two conditions imposed by the situation, namely, the shortest possible perimeter and the constant area. Almost two thirds of the students gave confused explanations or simply paraphrased the description of the situation. The difficulties they encountered in constructing a model for this situation can be seen in the answer given by JJ/4 (see Figure 6).

JJ/4 identified the quantities and the relationships between them for one of the conditions of the situation, but was unable to handle both conditions simultaneously. JJ/4 identified the condition of constant area and expressed it in two particular cases, as dimensions of 50×8 and 400×1. Using information obtained form these two cases, JJ/4 decided that “in A, the fencing works out at 2,220€ and in B at 24,060€. A is

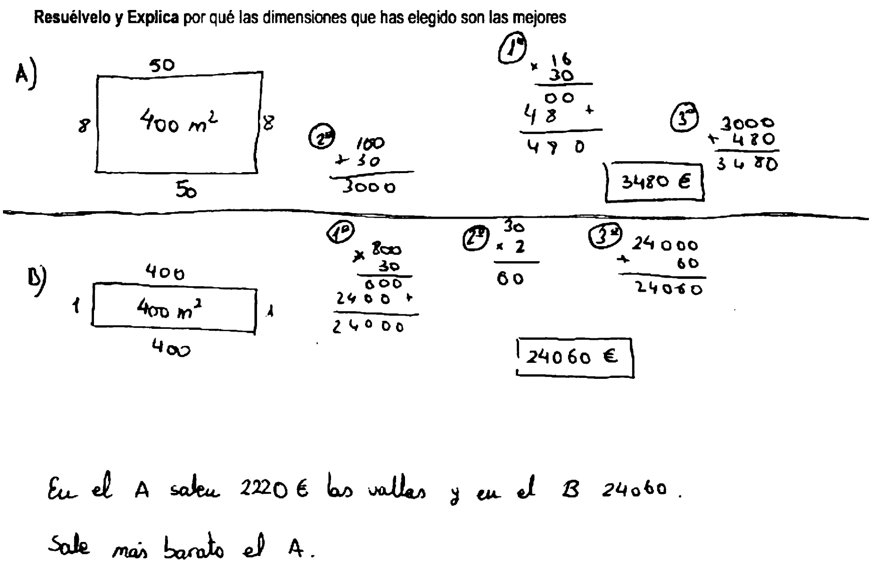


Figure 6. JJ/4’s answer. Decision based on information from two cases.

cheaper.” The calculations on which his or her decision is based reveal that JJ/4 recognised the quantities involved and the relationships between them, but could not see the influence of the variability in the two quantities (length of the sides) on the cost of the fencing. Being able to recognise this influence involves realising that the perimeter diminishes as the difference between the length of the sides of the rectangle becomes smaller.

It is precisely when students identify this influence and incorporate it into the model they construct that they are able to make a well-founded decision. For example, JJ/39 (see Figure 7) also used two particular cases (dimensions of 10×40 and 16×25), but was aware of the changes that occur when the length of the sides varies “the more similar the dimensions of the sides, the shorter the perimeter,” referring to the fact that as the difference between the length of the sides diminishes, so the perimeter also gets shorter. The information obtained from the two cases enabled this student to go beyond the particular and to appreciate the global behaviour of the situation by considering the influence of the variability in the length of the sides on the length of the perimeter. This characteristic is not found in level 1 answers (for example JJ/4’s answer, Figure 6). However, although JJ/39 integrated the influence of the

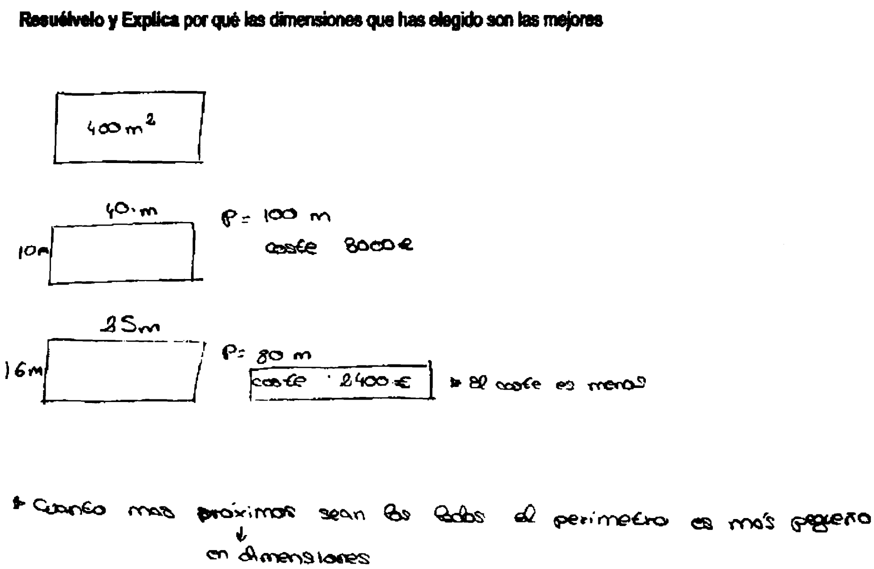
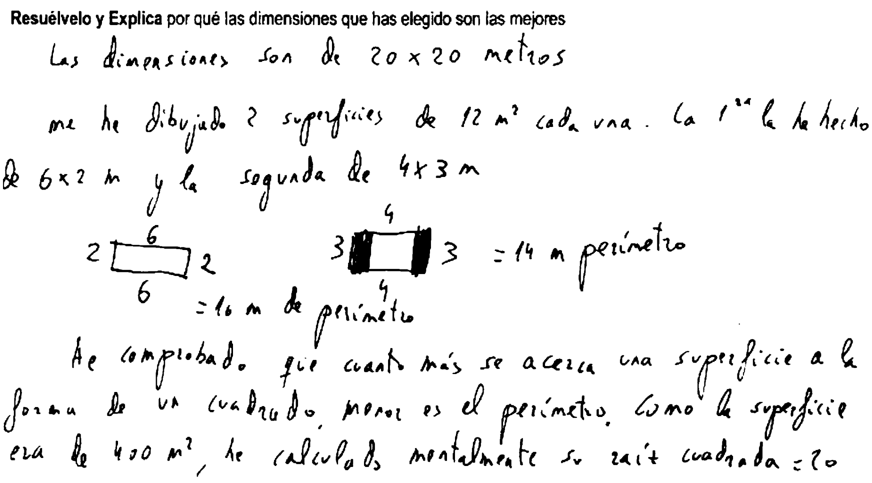


Figure 7. JJ/39’s answer, in which variability is recognised from two particular cases but is not used in making a decision.

variability in his or her answer, (s)he did not make use of it in making a decision.

On the other hand, there were answers that we interpreted as showing that the student was able to recognise the influence of variability in the difference of the length of the sides on the length of the perimeter of the festival enclosure and then be able to use it in making a decision, which is a characteristic aspect of useful modelling in this situation. An example for this was JJ/23's answer (see Figure 8). This student used two particular cases for a rectangle with an area of 12 m^2 (dimensions of 6×2 and 3×4). From these cases the student was able to identify the influence of variability in the sides on the length of the perimeter. JJ/23 wrote "the closer the surface gets to the shape of a square, the shorter the perimeter. As the area is 400 m^2, I have mentally calculated its square root, 20." This enabled the student to conclude that for the situation as given, "the dimensions are 20×20 metres." We interpret this behaviour such that the student has constructed a model that was the result of his/her structuring of their anticipation-based observations in relation to their goals and related (existing) assimilatory structures.

Globally, the three answers rely on the use of two particular cases but there are differences in the information thus obtained by each student. While JJ/4 did not go beyond the particular cases in order to make a decision, JJ/39 and JJ/23 both identified, from the calculations carried out, the influence of the variability in the length of the sides on the cost


Resuélvelo y Explica por qué las dimensiones que has elegido son las mejores

Las dimensiones son de 20 x 20 metros

me he dibujado 2 superficies de 12 m² cada una. La 1ª la he hecho de 6x2 m y la segunda de 4x3 m

6, 2, 2, 6 =16 m de perímetro

4, 3, 3, 4 =14 m perímetro

He comprobado que cuanto más se acerca una superficie a la forma de un cuadrado, menor es el perímetro. Como la superficie era de 400 m², he calculado mentalmente su raíz cuadrada =20

Figure 8. JJ/23's answer for word problem 2, in which the influence of variability is recognised.

of the fencing (length of the perimeter). The information obtained from the two cases led them to realise that the greater the similarity in the length of the sides, the shorter the perimeter. JJ/23 handles this influence appropriately in order to make a decision. A model for the situation based on this influence makes it possible to carry out a meaningful search for the shortest-perimeter rectangle under the condition of constant area. This search was carried out using repeated particular cases—different lengths of the sides—gradually approaching $a = b = 20$. This qualitative difference in the use of particular cases in order to make sense of the variability in quantities was interpreted as a difference between level 2 and level 3 in the development of modelling processes.

In this situation, in contrast to what happened in problem 1 (the pizza takeaway job offers), students were able to handle the variability without having to represent it with a "letter". This may be interpreted as the students' ability to handle variability in quantities without having to represent the variability itself. In any case, this characteristic of the modelling process may be linked to the problem-type (underlying structure of the problem).

Problem 4: The Dance Floor

This problem proved to be the most difficult one for the students to solve. More than 80% of them were incapable of making sense of it. The idea of the common multiple is the key to the modelling of the situation, linking it to the problem's two conditions of "square floor" and "without cutting any of the tiles." The student has to realise that by using rectangular tiles measuring 30×33 cms different-sized square floors can be laid so long as the sides are multiples of 30 and 33. But students can handle these relations in different ways, some of which more efficient than others. It was not easy for students to see this principle. Furthermore, the way in which the students used the idea of the common multiple to try to solve the problem depended on whether or not they remembered the procedure for obtaining lowest common multiples. Students who did not know a procedure relied on particular cases to carry out successive approximations until they discovered two numbers which, when multiplied by 30 and 33, gave the same result (a common multiple). Once these numbers had been found they were able to obtain the lowest common multiple. The process, however, was far from simple.

For example, the difficulty in handling the two conditions in the problem which involved the idea of the common multiple is revealed in the answer provided by MH/9 (see Figure 9). MH/9 used the idea of a multiple in order to look for an interpretation of the situation by carrying out a number of calculations. MH/9 sought different multiples of 33 by multiplication and identified 33×10 as 330 and 33×20 as 660 as relevant cases which would provide information. Then the student tried to relate these numbers to some multiple of 30 via the expression $30x = 330$; x being therefore 11. (S)he then looked for multiples of 11 without apparently making sense of the numbers obtained. The procedure used seems to have been guided by the goal of finding a common multiple of 30 and 33 (the learner's goal), but the difficulty of handling the particular cases led the student to declare "I think it can't be done because if the tiles are not square, how can they form a square figure?" This reply reveals the difficulty encountered when a meaning is

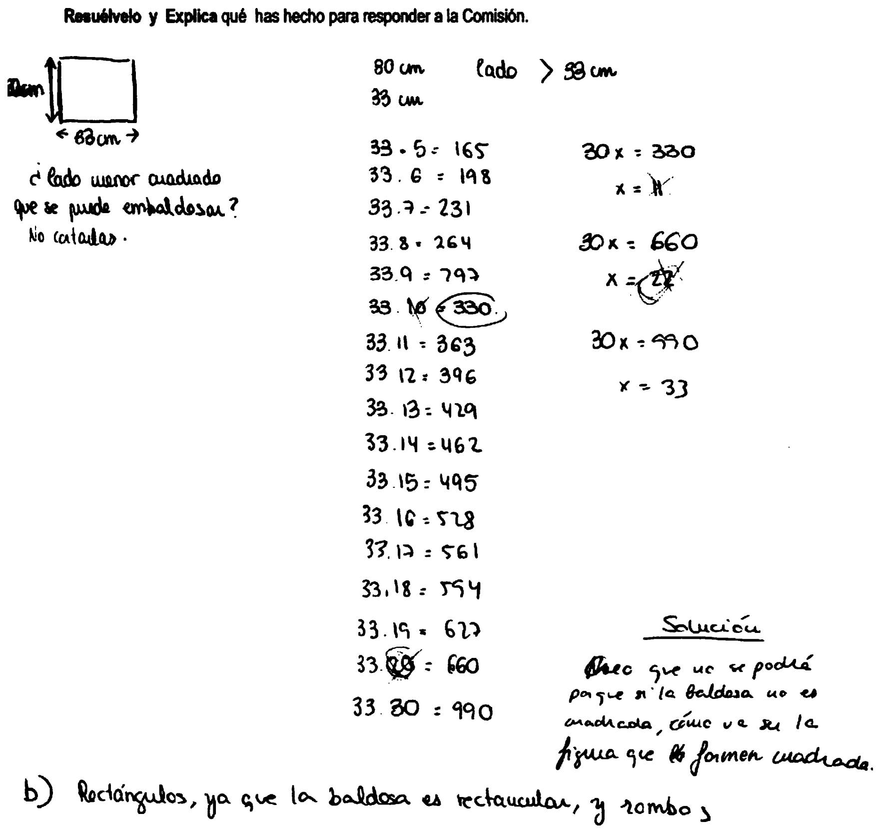


Figure 9. MH/9's answer for word problem 4: the student initiates activities which can achieve the goal but has difficulty in handling the information from particular cases.

not assigned to the letter “x” in relation to a situation that has to be solved. The way in which the student crossed out the expressions $x = 11$ linked to $30x = 330$, and $x = 22$ linked to $30x = 660$ seems to indicate an attempt to find a sole value for “x” which is a multiple of both 30 and 33.

However, in some instances the use of the idea of a common multiple as a general model in this situation made it possible to offer a correct answer, though the procedures followed by the students were different depending on whether they remembered an algorithm for the calculation of the lowest common multiple. MH/35 (see Figure 10), for instance,

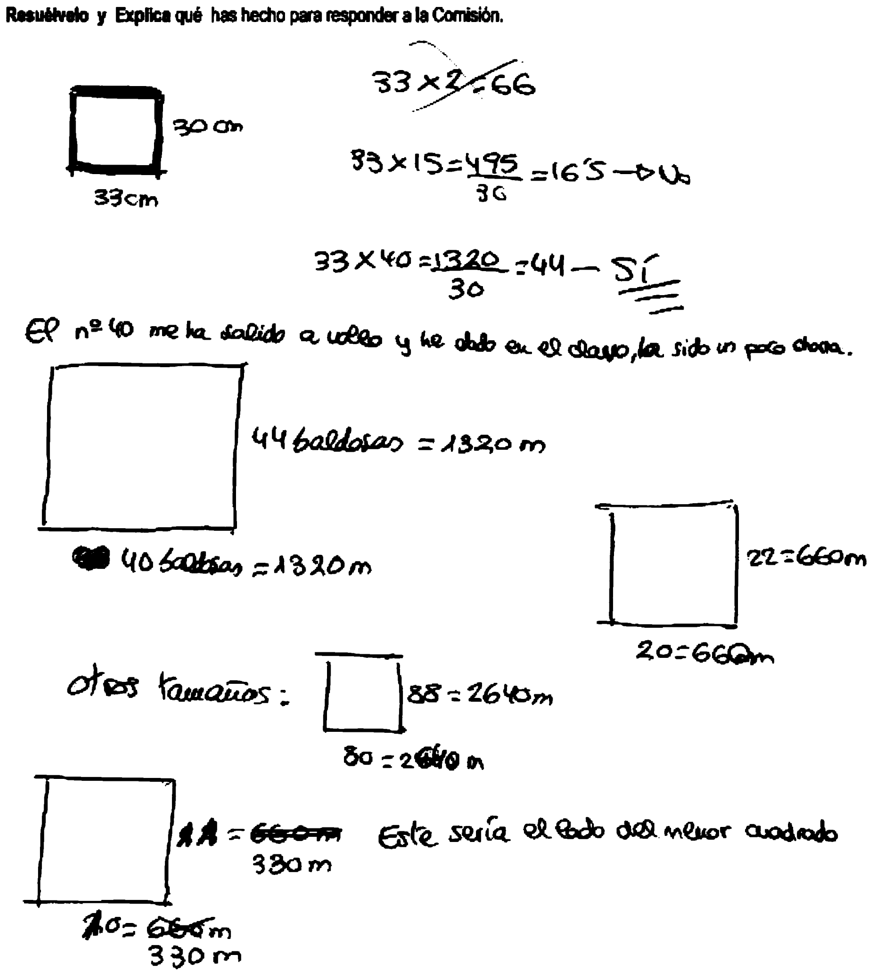
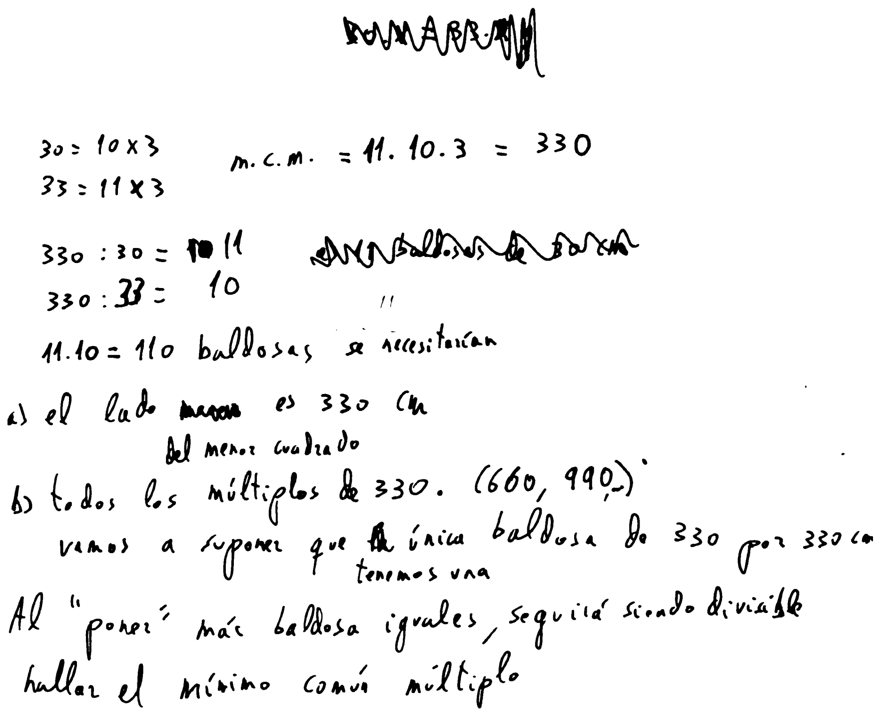


Figure 10. MH/35’s answer for word problem 4 from a general model through particular cases to make sense of the situation.

recognised the role played by the idea of the lowest common multiple, but could not remember the algorithm required to calculate it. These students then resorted to trial-and-error procedures. MH/35 looked for multiples of 33 ($33 \times 2 = 66$, $33 \times 15 = 495$ and $33 \times 40 = 1320$) in order to see if they were also multiples of 30 (66 is out of the question, it is obvious that 495 is not a multiple of 30 because the division does not give a whole number, while 1320 is a multiple). The student then searches among multiples of 33 for one which is also a multiple of 30. This manner of proceeding seems to show that MH/35 was seeking by trial and error a common multiple of 30 and 33, as can be seen in the written answer "the number 40 has come up purely by chance [by trial and error] and I have hit the nail on the head. It was a bit of a fluke [i.e., good luck]." Although the way in which the student writes the operations are not arithmetically correct, they do reveal the mental process that (s)he appeared to follow. MH/35 then suggests as a possible solution laying 40 tiles along one edge (the 33-cm ones) and 44 30-cm tiles along the other. From this starting-point other particular cases were discovered, for which the number of tiles was doubled (80 by 88) or halved (20 by 22). Being aware that the different sizes of floor could be found by multiplying or dividing the number of tiles laid seemed to show the student the path to follow in order to discover the dimensions of the smallest possible square floor. This number was obtained by dividing the number of tiles laid by two in the case of a rectangle measuring 20 tiles by 22.

On the other hand, students who remembered a method for calculating a lowest common multiple came to a much more direct conclusion. JJ/23 (see Figure 11), for example, calculated the lowest common multiple directly and gave as other possible sizes "all multiples of 330," justifying this decision by stating that "we have in effect one large tile measuring 330 by 330 cms" with which to construct further dance floors, and asserting that the floors thus constructed "would still be divisible." The model constructed, apart from being the appropriate one, was used correctly in order to justify the decision made.

What these answers seem to have in common is the fact that the students identified the idea of the lowest common multiple as a model for the situation, the difference being that some knew, and others didn't, a procedure for calculating the lowest common multiple. Students who could not remember the procedure resorted to a trial-and-error method using particular cases. In this situation, students who were aware of the idea of the lowest common multiple as the general idea that could organise the situation, resorted to particular cases in order to organise their search when they could not remember an algorithm. From this point



Resuélvelo y Explica qué has hecho para responder a la Comisión.

30 = 10 x 3 m.c.m. = 11. 10. 3 = 330
33 = 11 x 3

330 : 30 = 11
330 : 33 = 10

11.10 = 110 baldosas se necesitarían

a) el lado es 330 cm
del menor cuadrado

b) todos los múltiplos de 330. (660, 990,...)
vamos a suponer que tenemos una única baldosa de 330 por 330 cm

Al "poner" más baldosa iguales, seguirá siendo divisible

hallar el mínimo común múltiplo

Figure 11. JJ/23's answer for word problem 4 shows how the mathematical knowledge of multiples is used as a tool to think about the problem in order to justify the decision made.

of view, we must conclude that in this situation it was the model—the general thing—which governed the use of particular cases, giving rise to a "meaningful search," even though the procedure employed was one of trial and error.

Discussion

The aim of this study was to find out how secondary school students constructed and used mathematical models in order to make decisions in solving word problems and thus obtain information about how the modelling process is developed as an example of the development of new conceptual entities. The students involved in our research solved word problems and wrote down their solving processes and a justification of the solutions. This evidence enabled us to observe how they constructed and used their mathematical knowledge at some level of development as a model of the situation in order to make and justify a decision.

When faced with a word problem, a student should somehow represent its structure by identifying the quantities and the relationships between them in order to make a decision and justify it, whereby involving a process of mathematical modelling. This process is understood as a cyclical process composed of a number of sub-processes (Blomhøj, 2004; Simon et al., 2004; Verschaffel et al., 2002). This process of mathematical modelling is heavily influenced by the learner's goal. Modelling should therefore be conceived as a process determined by the context in which "he or she (student) is able to perceive both the situation or phenomenon modelled and the mathematics in play as two separated but interrelated objects. This is in fact the core of the matter both in relation to the potential and the difficulties connected to the learning of mathematical modelling" (Blomhøj, 2004; p. 146). A fundamental aspect of the mathematical modelling process is thus "recognizing the underlying structure of the situation," which implies recognition of what quantities and variables are involved in the situation, the relationships between them, the prevailing conditions and the role played by these conditions in achieving the required goal. The students in this study had considerable difficulty with a modelling process of this nature. Only one fifth of the answers analysed revealed that the student had identified relevant aspects of the different situations and was able to translate these into mathematical objects and relationships. The way students handled the different mathematical concepts as models in the situations given—comparison of linear functions, the multiplicative relation $a \times b$ = constant with the condition that $a + b$ = minimum, and the lowest common multiple—enabled us to observe the different approaches employed, which in turn made it possible to characterise different levels of development in the modelling process. The processes enacted by students show us different aspects of the mathematical concept learning and the role of mathematical knowledge in the modelling process.

Two ideas were shown to be important in this process: firstly, the relationship between the general and the particular revealed in the different ways in which students used particular cases; and secondly, the difficulty students encountered in using mathematical knowledge as a model at some level of development to solve the problems. These two ideas come from different roles played by the projection and reflection phases of reflective abstraction (Piaget, 2001/1977; Simon et al., 2004) in the development of new conceptual entities.

Recognizing the Underlying Structure of the Situations: The Relationship Between the General and the Particular

In the process of recognising the structure of the situations, the students used particular cases for different purposes, thus revealing a dialectic relationship between the general and the specific determined by their goal (the learner's goal). In some situations, the use of particular cases provided information which facilitated the construction of appropriate models. In the first two situations, for instance, the use of particular cases enabled the students to identify the influence of the variability in the quantities. On the other hand, in situation 3 (the dance-floor problem) students made use of particular cases for a different purpose. In this situation, once the idea of the lowest common multiple had been identified as a useful general model in order to interpret the situation, the use of particular cases helped in particularising the model when no appropriate algorithm could be remembered. Therefore, the different ways in which students made use of particular cases underlines the importance of the relationship between the general and the particular, upon which recognition of the structure of the situation is founded.

This difference in students' behaviour may be caused by differences in the levels of conceptual development of the mathematical concepts involved. In the first two situations the students seem to be in the projection phase in which they sort out records based on the results (Simon et al., 2004). In situation 3, the students' conceptualisation of the mathematical concept (the model) is at a level which enables the learner to use this model to generate other particular cases.

From this standpoint, the information obtained by the students from calculations carried out with particular cases is seen as an activity sequence that seems to be a determining factor in the characteristics of the model constructed. That is to say, the mathematical concepts seen as a model of the situation is the product of reflection on an activity-effect relationship. This information enabled the student to go beyond the particular and to perceive the influence of variable quantities on the decision to be taken. This can be interpreted as an abstraction of the relationship between activity and effect. Failure to perceive this influence leads to a particular-cases-based level 1 model for the situation, leading to decisions that relied exclusively on a few specific cases. From the point of view of learning mechanisms as reflections on activity-effect relationships

(Simon et al., 2004), this situation can be seen as one in which the student is at the participatory stage but not yet at the anticipatory stage. A modelling process of this kind is devoid of a complete structural understanding of the situation. Students who considered the influence of the variability of quantities (level 2) had achieved this complete structural understanding of the situation. In these cases, students anticipate what he/she needs to accomplish the goal making goal-directed adjustments. At the same time, appropriate use of the model constructed was determined by the correct handling of the variability as a result of the students' structuring of their anticipation-based observations in relation to their goals and related (existing) structures (level 3).

In situation 3, on the other hand, students relied on a general realisation (they identified the idea of the lowest common multiple as an appropriate model) and resorted to particular cases as a process by which to calculate the lowest common multiple when they could not remember an algorithm. The use of the model identified in this situation enabled the student to carry out a meaningful search among the particular cases. This search for the lowest common multiple was sometimes carried out by a process of trial and error and on other occasions by a more systematic method. In this situation the use of particular cases made it possible to see the particular from the general (the model), which generated a meaningful search for the answer.

We think that the characteristics of the modelling processes identified in this study coincide with the two aspects of modelling defined by Gravemeijer (1997) as "modelling as a form of organizing" and "modelling as a form of translating." Once the different quantities involved in the situation and the relationships between them have been recognised, the student is able to find a mathematical content or procedure which models the revealed structure ("modelling as a form of translation"). In this kind of modelling, the mathematical concepts provide potential models for the situations, as has been seen to occur in situation 3. The model may also emerge as the result of an organising activity ("modelling as a form of organizing") as has happened in situations 1 and 2. The models thus created reflect the different relationships which underlie the situation. "If a student does not have fitting ready-made solution procedures available, an organizing/modelling approach may enable the student to find an adequate informal solution" (Gravemeijer, 1997, p. 396).

What has become evident in our study is that the way in which students used particular cases and the information thus obtained determined their modelling capacity (both as translating and as a form of organising). This result suggests that the development of the two processes, perceiving the general in the particular and perceiving the particular from the general (making the general meaningful by proposing particular instances) is a characteristic of students' modelling processes and as such is an aspect of their abstraction process, and so of their mathematical competence.

On the other hand, the way in which students handled the particular cases in the modelling process may be interpreted by the application of the theoretical construct proposed by Simon et al. (2004), i.e., that of reflection on activity-effect relationships. From this point of view, the learner sets a goal and enacts an activity sequence. The reflection on the activity-effects relationships (the particular cases) enables the learner to identify invariant relationships. In accordance with Simon et al. (2004) these invariant relationships constitute a new level of anticipation (abstraction) which we have called a new level in the modelling process. In this sense, we considered that the use of reflection on the activity-effect relationship (Simon et al., 2004) and the use of aspects of modelling (Gravemeijer, 1997) provide complementary information to explain the progressions from one level of competence to the next.

Mathematical Knowledge in the Modelling Process

The situations used in this study required students to make a decision and communicate it. Solving this type of problem revealed the severe difficulty encountered by students in representing the structure of the situation in order to make a decision by using their knowledge of mathematics. The results show that they were incapable of recognising the internal structure of the situations (variables and relationships) in order to develop or identify an appropriate mathematical concept as a model for the situation. [Problem 1: comparison of two linear functions ($y = 60 + 0.6x$, $y = 24 + 0.9x$), expressing the equation $60 + 0.6x = 24 + 0.9x$, solving the equations and interpreting the results; Problem 2: considering the rational function $f(x) = x + 400/x$; Problem 4: Calculating the lowest common multiple]. According to Greeno (1991, p. 175), "Knowing a set of concepts is not equivalent to having representations of the concepts but rather involves abilities to find and use the concepts in constructive processes of reasoning [...] the person's knowledge [...] is in his or her ability to find and use resources, not in having a mental version of maps and instructions as the bases for all reasoning and actions."

The difficulties that students were seen to experience in using their mathematical knowledge as a conceptual tool for problem-solving may be due to educational factors. The problems presented in mathematics classrooms are usually given within the framework of a particular topic, which provides all the necessary tools required to find the mathematical model that fits the situation. These problems become mere directed tasks rather than modelling tasks. This makes it difficult for students to develop habits of enquiry regarding the quantities and relationships in a given situation and to find the mathematical tools that can help to solve the problem. The situation is beginning to change however, with projects that teach students to design models as an instrument in the learning of mathematics (Blomhøj, 2004; van Dijk et al., 2003).

On the other hand, Säljö (1999b, p. 149) argues that learning has to do with how people appropriate and gain expertise in using tools for thinking and acting that exist in a given culture. Our study has shown the difficulties encountered by students in the use of mathematical concepts as tools for thinking and acting in problem-solving situations. From a socio-cultural perspective of learning, meaning is created via interactions between people when we convert our experiences into language and express them. The results of this study shed light on the difficulties encountered by students in recognising the mathematical structure in word problems. The results also show that many of the students who do achieve a certain understanding of this structure fail to use the mathematical tools at their disposal. It is then possible for students to make sense of situations by relying on the identification of units of information in the text rather than by searching for mathematical tools. One consequence of this implication is that problem-solving should not be treated as the application of previously established procedures but rather as situation modelling processes in which the main objective is to make sense of the different aspects of a given situation (Blomhøj, 2004; Gravemeijer, 1997).

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Departamento de Innovación y Formación Didáctica, Facultad de Educación
Campus de San Vicente del Raspeig, Universidad de Alicante
Alicante, 03080, Spain
E-mail: sllinares@ua.es