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FOUR STUDENT TEACHERS' PEDAGOGICAL REASONING ON FUNCTIONS

ABSTRACT. This study attempts to identify the influence of student teachers' subject matter knowledge for teaching on the process of pedagogical reasoning. This influence is studied through the way in which the concept of function is presented to pupils in teaching through the textbook problems. Our findings show that the four student teachers in our study differed in their subject-matter knowledge for teaching both in the different aspects of concepts they emphasised and in the use of a representation repertoire to structure learning activities. All of this conditioned the use of graphical and algebraic modes in their planning of subject matter to be presented to pupils. We explored also the influence of *images* of mathematics, teaching and learning on student teachers' organisation of the subject matter for teaching, but found this only slight. Finally, regarding the relationship between subject matter knowledge and pedagogical content knowledge in student-teachers' ways of knowing the subject matter, we offer some implications of these findings for mathematics teacher education programmes.

KEY WORDS: function concept, images, learning to teach, pedagogical reasoning, pedagogical content knowledge, subject matter knowledge

INTRODUCTION

Learning to Teach

Learning to teach is a complex process that is influenced by a range of factors. Over the last few years, research efforts have provided insight into this process and the factors conditioning it (Carter, 1990; Borko & Putman, 1996). Interest in the role of subject matter knowledge for teaching and in the process of transformation of subject matter for the purpose of teaching has grown, particularly since the work of Shulman and his colleagues (Shulman, 1986; Wilson, Shulman & Richert, 1987). These researchers have introduced the construct *Pedagogical Content Knowledge* to note "the ways of representing and formulating the subject that make it comprehensible to others . . . [it] also includes an understanding of what makes the learning of specific topics easy or difficult" (Shulman, 1986, p. 9). In this context, we use *pedagogical reasoning* as a theoretical construct to portray the transformation of content knowledge for the purposes of teaching.

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When we consider learning to teach as an active constructive process, one factor that influences the student teachers' learning is what they bring to the teacher education programme (Ball, 1990; Even, 1993; Simon, 1993). Further, the complexity of the interrelationships between the different components of student teachers' knowledge is seen in the different impact of mathematics education courses on their mathematical and pedagogical understanding (Wilson, 1994). This complexity shows the existence of several variables which influence student teachers' construction of the knowledge needed to teach and their development of pedagogical reasoning. For example, Even & Tirosh (1995) studied the influence of subject-matter knowledge and knowledge of learners on student teachers' instructional decisions. They found that the student teachers' organisation of the content for teaching was influenced by their subject matter knowledge and not so much by their knowledge of pupils' ways of thinking.

Likewise, the relationship between subject matter knowledge and pedagogical content knowledge has been considered as a key aspect in the development of student teachers' pedagogical reasoning (Even, 1998). Wilson (1994) described how a student teacher understood a function as a part of the mathematics that she would teach. He reported that the course integrating mathematical content and pedagogy had influenced her understanding of function (subject-matter knowledge) but did not affect her approach to teaching (beliefs). These studies help us to understand better the role played by the comprehension of a specific topic of subject matter in teacher learning (Cooney & Wilson, 1993), and point out the necessity for deeper research into the relations between the components of teacher knowledge.

Ways of Knowing the Subject Matter and Images

In this study, we use the term "teacher's ways of knowing the subject matter" to take into account the aspects of the mathematical content emphasised, their explicit connections and the different uses of modes of representation that the teacher emphasises in teaching.

For example, a teacher who emphasises the concept of function as an action can underline the meaning of function as a chain of operations. So, this teacher might put greater emphasis on algebraic modes and computational activities in teaching. On the other hand, a teacher can emphasise the function as a model for a real situation, using graphs. From these approaches, the activities of translating among different representation modes (e.g., models of real-world situations using functions) can play a different role in teaching. In turn, the different aspects of the concept emphasised, its explicit connections and the way in which the representation modes are used will define the teacher's goals. In the context of learning to teach, the student teacher's ways of knowing the subject-matter helps us to understand how a student teacher might make sense of the activities of teacher education programmes.

From other perspectives, other researchers have also emphasised the role that beliefs play in the process of learning to teach. Cooney et al. (1998) suggested that "the various ways in which the teachers structured their beliefs helped account for the fact that some beliefs were permeable whereas others were not" (p. 306). Grossman (1990) considered as one of the components of pedagogical content knowledge the overarching conception of what teaching a particular subject means. The overarching conception reflects aspects of the student teachers' beliefs that are more specifically related to how they think about the mathematical content for the pupils (what pupils should learn about mathematics) and the nature of mathematics. Since affective issues seem to be integrated with the beliefs, we think that the term "image" could be appropriate to describe the student teachers' beliefs and attitudes towards mathematics, learning and teaching. Calderhead & Robson (1991) argue that "Images ... represent knowledge about teaching but might also act as models for action and, in addition, they frequently contain an affective component, being associated with particular feelings and attitudes" (p. 3). For example, student teachers can hold an image of mathematics as abstract, unreal or as a set of systematic procedures. Likewise, the mathematical activity can be seen as a game or a mechanism that works properly. Johnston (1992) used the construct of "image" to identify the ways in which student teachers think about themselves as teachers and how this relates to their teaching practice. In this sense, the images can influence the perspective and models for action taken by student teachers and permeate aspects of their experience in teacher education programmes.

Pedagogical Reasoning and Teacher Learning

We have considered two aspects of teacher's subject-matter knowledge for teaching that seem to have some importance in the characterisation of teacher learning:

- a teacher's ways of knowing the subject matter and
- his/her images

From the perspective of teacher learning, it seems relevant to analyse the influence of these aspects on the ways in which student teachers structure learning activities (Grossman et al., 1989; McDiarmind et al., 1989). Influence on the organisation of mathematical content in teaching may be shown in the process of pedagogical reasoning, i.e., when student teachers transform the subject matter for the purposes of teaching and give arguments about it. Wilson et al. (1987) consider that during this transformation the 'critical interpretation' of subject matter becomes apparent, which involves 'reviewing instructional materials in the light of one's own understanding of the subject matter' (p. 119). For us, this critical interpretation includes the characteristics of the concept which are identified, the type of problem chosen and the order in which the different aspects of the concept are presented by the student teachers.

Likewise, in the transformation, the teacher's 'representational repertoire' becomes evident in the sense of the different activities, assignments, examples, and so on, that 'teachers use to transform the content for instruction' (Wilson et al., 1987, p. 120). We include here the modes of representation of subject matter that a student teacher emphasises and uses to convey something about the subject matter to the learner. Wilson et al. (1987) also consider 'the adaptation' of the subject matter, which involves fitting the transformation to the characteristics of the students in general. Globally considered, the interpretation, representation and adaptation contribute to generate an action plan for teaching specific subject matter. Within this framework, we pose the following question:

How do the student teacher's ways of knowing and their images about school mathematics and mathematics learning/teaching influence the ways in which they think about presenting the subject matter to pupils?

We focus on a specific topic, the concept of function, at secondary school level (pupils aged 14–16). We have chosen this particular domain because it is one of the most important curricular topics at this level. It is playing an increasingly important role in the secondary curriculum and is related to other subjects (e.g., Physics).

METHODOLOGY

Participants and Context

The study involved four university graduates who volunteered to participate: Juan, Rafael, Alberto and José (all names are pseudonyms). Each had obtained a degree in mathematics or in another branch of science. Their ages ranged from 22 to 25. They presented no special characteristics other than their interest in collaborating in anything that might help to improve the design of mathematics teacher education programmes.

In order to obtain the credits necessary to teach mathematics at secondary school level (pupils aged 12–18), they had to enroll on a post-graduate course. This course was focused on pedagogical, psychological and mathematics education issues (with only 30, out of 180 hours, focused on mathematics education issues). There was also a practical component (student teaching) in which each student teacher carried out his/her student teaching in a secondary classroom with a mathematics teacher as a tutor for four weeks. Data for the study were collected at the beginning of this post-graduate course. At this stage, the student teachers' knowledge about functions and their images resulted from their previous experiences in school and university mathematics courses and not from the educational theory of the postgraduate course.

The specific context of secondary mathematics teacher education in Spain has some particular characteristics. Student teachers receive wide training in mathematics (and sometimes in other branches of science such as physics, chemistry) for five years to obtain a degree in mathematics (or science). So, when the student teachers take the post-graduate course they have already a solid background in mathematics. This situation is different from those countries in which teacher training programmes have more or less fully integrated courses with mathematical subject content and mathematics education content. Specific training in mathematics for such a long time (five years) may influence a student teacher's images of mathematics and its teaching/learning. We believe it to be worthwhile to study the relationships between the ways of knowing and the images as references in the development of pedagogical reasoning processes. On the other hand, nowadays the secondary school curriculum in our country (Junta de Andalucía, Diseño Curricular Base, 1993) emphasises a modelling approach to functions. This curricular approach points out the reading and interpretation of graphs (linked to a real-situation) as specific objectives. This view, which gives less prominence to formal definitions and algebraic expressions, can be affected by the approach to the concept adopted by the teacher.

Data Collection and Instruments

We designed four interviews to obtain information about the student teachers' ways of knowing the concept of function and their images as well as to study the relationships between the aspects of function elicited and the mode of representation selected. We asked each student teacher to complete all the interviews. The first interview, which was semi-structured, was a general interview aimed at obtaining information concerning his/her biographical background related to mathematics and eliciting data regarding his/her images about mathematics, teaching and learning.

For the second interview, we asked students to engage in practical tasks with associated textbook problems. In consideration of the literature on students' understanding of functions (Vinner & Dreyfus, 1989; Leinhardt et al., 1990; Dubinsky & Harel, 1992) and on mathematical representations (Janvier, 1987; Kaput, 1991), we chose a set of 22 textbook problems that differed in two dimensions:

(i) the mode of representation in which the function concept was shown (real-world, situation, algebraic mode, graphs and tables);
(ii) the hypothetical activity that the problem demanded from the problem solver (Garcia & Llinares, 1995).

Each of these problems was written on a card to make the handling easier The two practical tasks designed with these problems were:

- 'Classification' task: Sort the 22 textbook problems relating to the concept of function, written on the cards, and provide arguments to explain the sets made. [Here, problems were named Textbook Problem 1, Textbook Problem 2, and so on.]
- 'Textbook Problem Analysis' task: Analyse the 10 textbook problems (chosen from the 22 textbook problems written on the cards) from different perspectives. [Here, problems were named Textbook Problem A, Textbook Problem B, and so on.]

In this interview, we posed questions like:

❖ Describe this problem in your own words.
❖ Do you think this task is necessary to teach functions?
❖ What mathematical content might be learnt with this problem?
❖ What objectives will you try to achieve?

We wanted to obtain information about the student teachers' reasons for using a specific problem in their teaching, and how they thought that a learner would solve it.

In the third interview we asked student teachers to use the textbook problems in the planning of a hypothetical teaching sequence for the concept of function and provide arguments that might justify their decisions. The aim was to identify what was behind the presentation of the mathematical content in the planning prepared by each student teacher.

We considered that making them think about the presentation of the subject matter to pupils using textbook problems might generate mental activity from which we could identify the source of their choices and decisions.

For the fourth interview, we designed four hypothetical situations (Cases 1 to 4). The cases were constructed bearing in mind the results of the review of research into the learning of functions (Leinhardt et al., 1990). The content of the cases was

(i) misconceptions in the overall interpretation of graphs;
(ii) the role of images that the pupils construct as a result of the type of task that they usually carry out;
(iii) the separation between visual and analytical processing of the information;
(iv) the difficulties created by the notion of variable.

Each case described a pupil's response to a problem with functions and posed several questions. Some of these questions focused on diagnosing pupils' thinking and asked student teachers to identify the causes for the pupil's response; some others asked about the way in which the teacher could help the pupil.

Data Analysis

The four interviews were recorded and transcribed. Using the transcriptions, different analyses were performed. In the first step, from each interview we identified the arguments used by the student teachers. The information obtained was categorised in relation to the two dimensions of subject matter knowledge for teaching that we had considered: *ways of knowing the subject matter* and *images*. In each of the categories obtained, we identified data relevant for the transformation of subject matter for teaching from the different interpretations of subject matter, the modes of representation emphasised, and information about the process of adaptation of the subject matter for the learner.

The results obtained have been organised into two subsections;

- ❖ Subject matter knowledge for teaching (a pre-service secondary teacher's ways of knowing and images);
- ❖ Processes of transformation of the subject matter described through critical interpretation, repertoire of representational modes and adaptation to pupils' mathematical thinking.

We continue with a discussion of the results of our analysis and their implications for mathematics teacher education programmes.

Texbook problem A. Reading a graph

Let us consider the graph of the function

$f(x)= 1/(1 + x^2)$

Could you answer the following questions?

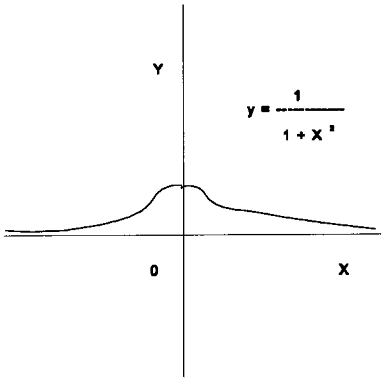
I. What is the maximun, or supremun?

II. What is the minimun, or infimum?

III. In which interval is the function increasing? And decreasing?

IV. Is the function banded?

V. Is the function symmetric? If it is, with regard to what?



Textbook problem 7

In the hall of a high school there is a machine with cans of soft drinks. One day, the machine's owner made a study of how many cans there were at each moment in the machine from 8 am to 8 pm.The results of the study are represented in the following graph. By looking at the graph carefully, try to answer these questions:

I. How many cans there were in the machine at 8 o'clock in the morning?

II. From which periods of time was no can consumed?

III. How many cans were consumed in the morning break, between 11am and 11:30 am?

IV. At what time was the machine filled?

V. From the graph, could you say at what time the afternoon school classes finish?

VI. When were more cans per hour sold, during the morning break or during the lunch break?

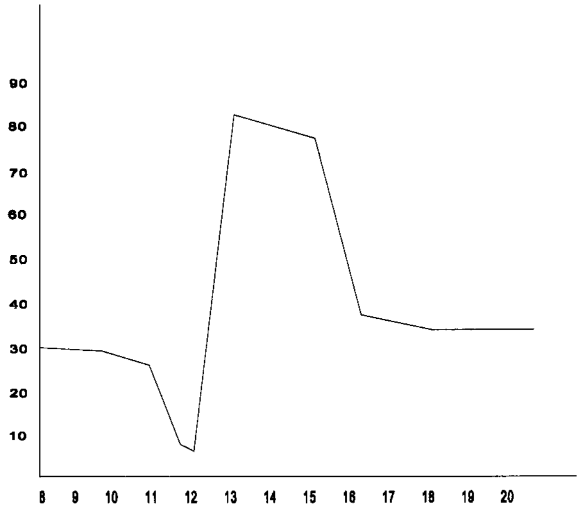


Figure 1. Three textbook problems and one of the hypothetical cases used in the interviews posed for the student teachers.

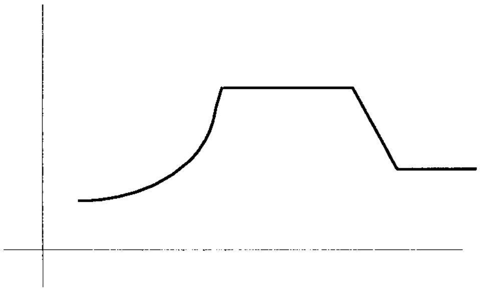
Texbook problem 11

Juan wants to buy a car. He can choose between a petrol car and a diesel car. The first car costs 1,300,000 pesetas to buy, and for every 100 km it consume 8 litres of petrol. The second car costs 1,500,000 pesetas to buy, and for every 100 km it consumes 5 litres of diesel. The price of petrol is 90 pts/l and the price of diesel is 60 pts/l. How many kilometres do you have to drive before the second car become more economical? For this case, fill the following table with the total cost of each car plus combustibles, and draw the respective graphic representations.

Km	5000	10000	15000	20000	30000
Gasoline car					
Kerosene car					

Case 1

When introducing functions and graphs in a class of 14–15 years-old pupils, tasks were used which consisted of drawing graphs based on a set of data contextualized in a situation and from equations. One day, when starting the class, the following graph was drawn on the blackboard by the teacher, and the pupils were asked to find a situation to which it might possibly correspond.



One pupil answered

"It may be the path of an excursion during which we had to climb up a hillside, then walk along a flat stretch and then climb down a slope and finally go across another flat stretch before finishing".

How could you respond to this pupil's comment? What do you think may be the reasons for this comment?

Figure 1. Continued.

RESULTS

Subject Matter Knowledge for Teaching

Student teachers' ways of knowing the subject matter. In order to describe the student teachers' ways of knowing the concept of function we identified the different aspects that they emphasised (see Figure 2) and the connections between them, and the roles of different modes of representation used.

	Emphasised aspects	**Representative protocols**
Juan	Relation between elements from sets Algebraic manipulation Function shows the existence of a rule or law that may be symbolised with an algebraic expression Function as a succession of operations	". . . mainly that a function leads me . . . elements from one set into other. Then to begin putting down a number of points and leading them on to others . . . (Juan, 682–683, Case 1). ". . . instead of putting all these numbers and to each number we do that, well we put . . . a generic number, an *x*, since *x* means all the elements in this set . . ." (Juan, 690–697, Case 1).
Rafael	Function as a factory Relationship of dependence and causality between input/output	"In my opinion, for the case of a function there has always been the typical example of a factory where . . . raw material goes in, undergoes a modification and another one comes out . . . there is a relationship between one object and another . . ." (Rafael, 133–144; General Interview).
Alberto	Function as a model of a real situation Study of the functional behaviour	". . . in practice, when there is a real problem I'll have to convert it into a mathematical model and look at the behaviour of the function mathematically . . . this (Textbook A) is perhaps the "skeleton" for a real problem. The real problem does not appear, but it is the "skeleton" of a possible real problem" (Alberto, 1001–1004; Textbook Problem A analysis.)
José	Relationship between variables Emphasis on the mathematical language and notation Graph displaying the functional behaviour	"They (the pupils) would see that . . . well, it is a correspondence of R into R, or which is the abscissa axis and which is the ordinate axis. The "*y*" can be represented . . . the functions can be represented graphically and . . . by means of tables of values" (José, 880–890; Classification) ". . . one can draw conclusions from a graphical representation of a function as regards its behaviour" (José, 887–889; Classification)

Figure 2. Student teachers' ways of knowing the concept of function.

All four student teachers in our study saw the concept of function as a correspondence between sets, but they emphasised different aspects of the mathematical concept for the purpose of teaching. Two of them, Juan and Rafael emphasised algebraic aspects and a view of functions as actions. For instance, for these students, the algebraic formulae $y = 4x + 3$ was viewed as "four times a number plus three". What they considered important was for pupils to learn formal aspects of the algebraic manipulation. Another student teacher, Alberto, emphasised the meaning of the relationship between the variables in real-world situation problems. For Alberto, the consideration of the function concept as a real situation model was an important aspect in teaching. Finally, the last student teacher, José, considered from a balanced perspective both function as a model in which graphs allow one to see how a phenomenon behaves and function as an correspondence of R into R. The possibility of considering simultaneously the aspects of concept that they had emphasised, various types of functions and specific properties of functions in the problems turned out to be quite difficult for these student teachers. For instance, only Rafael used different characteristics jointly in his classification of the textbook problems: linear versus non-linear, relating to reality versus formally mathematical, geometric versus. non-geometric and continuous versus discrete domains. However, as is shown in the following quotation from the classification of the textbook problems, even Rafael found it difficult to consider these characteristics simultaneously:

Although at first I thought that there could be many organisations [of the problems], the truth is that, as I have been looking at the problems, . . . I thought that it was difficult to change this [what I have written]. (Rafael, 1253–55; Classification).
. . . the thread . . . is the linear or non linear behaviour of the functions as compared to the behaviour . . . a geometric approach. (Rafael, 1623–64; Classification).

Finally, the four student teachers thought about the modes of representation of functions (graphic, algebraic, and real situation) in a different way, linked to the different aspects of the concept that they had emphasised. Juan and Rafael considered the activities of reading and interpreting graphs and translation between different modes of representation as a complement of the activities with algebraic mode (for them, the key activities with functions were the ones where pupils had to handle algebraic symbols). On the other hand, Alberto and José extended the range of uses of the graphs as "instruments" for solving situations (that is an idea linked to the notion of function as a model). For example, in the classification of the problems, Alberto mentioned three complementary perspectives of the graphs: forming part of the situation and transmitting information, an instrument for solving a problem, and representing relations.

Student teachers' images. As mentioned in our theoretical framework, the student teachers' images can mediate in the transformation process of concept of function from mathematical knowledge to its consideration as a teaching-learning object. Juan and Rafael viewed mathematics as a set of systematic procedures that may be used to solve problems and as knowledge organised in an accumulated way. For example, as they stated in the general interview:

So I have some information, which I have to use. Then it is clear that I have to go that way. Or rather, that I am supposed to do what the problem tells me . . . in a more systematic and reasoned way . . . (Juan, 531–535; General Interview).

In so far as it [mathematics] has been organised in an accumulated way, backing each other, perhaps it is a good way for dealing with the mathematical content that is being explained . . . (Rafael, 242–245; General Interview).

For Juan and Rafael, the logic that organises the mathematical knowledge constituted a referent for the introduction of teaching content. Furthermore, these student teachers thought that mathematical content in teaching was organised to solve the usual problems arising at the end of the chapters in mathematical textbooks as application problems. That is why they said: "I am going to do what the problems tells me . . .", and "So I have that information, which I have to use". On the other hand, José's image about the mathematics included two ideas: mathematics as an abstraction "mathematics are somewhat unreal, in the sense that when you are dealing with what is abstract, you move away from reality" (José, 1211–1214; textbook problem analysis), and mathematics as a set of useful instruments, "the pupil must realise that a mathematical instrument can help him/her to solve a real case" (José, 1206–1208; textbook problem analysis). This image allowed him consider two approaches to the concept of function (i) putting the emphasis on the formal aspects in the algebraic context and (ii) using the graphs as an instrument for studying the functional behaviour of any real phenomenon. He considered these two approaches independently.

Alberto associated several terms with school mathematics: mathematics as a game, that conveys the idea of something in which you participate and produces satisfaction; and mathematics as a unit composed by related parts. He said:

. . . the pupil recognises mathematics as a game, as a mechanism that is amusing and that always functions well (Alberto, 112–114; General Interview);
. . . I like the fact that they [the pupils] see mathematics rather as a unit, and that everything is related (Alberto, 940–942; Classification).

In addition, Alberto viewed the mathematics as a theory that helped to explain real situations,

. . . the mathematical theory (linked to a problem that you are going to explain) probably comes from real events . . . (Alberto, 350–352; General Interview).

For example, in his analysis of textbook problems, with respect to the Textbook Problem D,

Textbook Problem D
Draw the graphs for the following parabolas by locating the vertext first
$y = x^2 + 2$; $y = x^2 - 8x + 16$; $y = x^2 - 4x$; $y = -x^2 - 5$; $y = x^2 + 4x + 5$; $y = 2x^2 + 2$

Alberto stated:

. . . because it is probably the 'skeleton' of a real, practical problem, of a physical problem, of a problem of another type (Alberto, 1342; Textbook Problem D analysis).

Finally, student teachers' images about learning were very similar. For them, learning is a question of knowing the information previously provided by the teacher. From this perspective, the difficulties and mistakes of the pupils were associated with

. . . a lack of knowing the concept (José, 1301; Case 1);
. . . they just have a mistaken a priori idea of a function . . . They haven't understood the concept of function (Alberto, 1870–1873; Case 2).

This image about learning was related to the image of teaching as a process for transmitting information. All four student teachers viewed teaching as telling and learning as remembering. From this position, these student teachers held the image that mathematics teachers had the knowledge and the responsibility for transmitting it, and their pupils would assimilate it without any difficulty, as is shown in the following assertion about the most important thing for teaching mathematics:

The first thing is having something to teach. To know why it is important for me to teach that. And then, to know how to transmit to the pupil all those things I think necessary for him/her to learn, and to be able to transmit it so that he/she understands the idea adequately (Alberto, 59–62; general interview).

For these student teachers, the teacher is the one who decides what it is good or bad and the one who decides on the correctness of the answers to the problems.

Transformation of the Subject Matter

The hypothetical plans for teaching the concept of function that the student teachers prepared and the reasons they provided were influenced by their ways of knowing the concept of function. During their transformation of subject matter for planning a hypothetical teaching sequence, three aspects became apparent:

- their critical interpretation of the subject matter;
- the representational repertoire they used; and
- their adaptation of the subject matter.

Critical interpretation. In this section, we describe how the student teachers' ways of knowing, and their images, influenced the aspects of the concept of function identified, the type of problems chosen and the order in which the different aspects of the concept were presented.

Textbook Problem 1. Given the function f(x) = x^2 – 4, calculate f(0), f(1), f(3).
For which values of *x* is f(*x*) = 0?

Textbook Problem 16. Given the function f(*x*) = 2*x* + 3.
a) Make a table and draw a graph.
b) Indicate the points cutting the graph in the *x*-axis.
Indication the point cuttign the graph in the *y*-axis.
c) For which values of *x* is f(x) = –9?
d) For which values of *x* is f(*x*) > 0?

Figure 3. Textbook problems 1 and 16.

Juan and Rafael considered it important for the pupils to know this concept as a relationship between variables, emphasising the meaning of functions as a succession of operations. For example, Juan used the two problems shown in Figure 3 to start his hypothetical teaching sequence, which reflected the idea of a 1-1 correspondence and functions as a succession of operations.

He justified the use of these problems in his teaching sequence saying: "They are useful problems to introduce what a function is, the points, how the images are calculated ... they are easier because the function is seen directly". For Juan, these were the problems that reflected the idea of function most clearly.

Likewise, the starting point for the teaching sequence established by Rafael is captured in the following statement: "what should be clearly explained is that a function is a univalent correspondence between two sets". However, in order to achieve this objective, two characteristics that determined how he structured learning activities were evident in his decisions and curricular choices. The first characteristic was the identification of the functional relationships as linear or non-linear (content knowledge). The other characteristic was assigning a low difficulty level to the problems related to linear functions. Rafael selected Textbook Problem 11 for starting his hypothetical teaching sequence (see Figure 1). He said,

Initially, problems related to real-world situations, allowing the students' acquisition of intuitive concepts about what is going to be explained, are set out (Rafael, 1116–1118; Interview 3).

Rafael talked about the use of these problems in his teaching sequence:

To start off, with respect to what the most profound content is, I would go on explaining functions. Firstly, what a function is, then I would go on towards linear functions .... Within linear problems, the first things that I would use would be those that are related to the construction of graphs from the "evaluation" of algebraic expressions that they already know about ... (Rafael, 1080–1084; Interview 3).

What is relevant in this situation is that Juan and Rafael put their emphasis on the relationship between variables in an algebraic context and considered this relationship important for pupils' learning about functions. Although Rafael included problems describing real-world situations, they were used exclusively to provide an intuitive view of the idea of relationship between variables and input-output pairs. These problems were not used to promote graph reading and interpretation activities, which could show the function concept as a real-world situation model.

The other student teachers, José and Alberto, used a dual approach based on their ways of knowing the mathematical content and their images. José emphasised the idea of function as an abstract mathematical object and its implications (the meanings of the 'x' and of the 'y' in the system of Cartesian axes; calculation of images, domains, slopes. In the case of parabola, the vertex, relationship between concavity and the sign of the coefficient of x^2). In addition, and related to his images about mathematics, José also indicated the need to show that mathematics can be useful to solve a real situation, which was what justified the introduction of problems linked to real life for him. However, he made no mention of the possible relationship between the 'formal' aspect of the mathematical content (with the emphasis placed on the algebraic mode) and the use of graphs in the interpretation of situations of the teaching sequence. It is as if there were two 'worlds' for the function concept that the teacher must use in the teaching sequence, but at no time are the relationships made between the textbook problems for interpreting a situation using a graph and those based on the algebraic mode.

Like José, Alberto used a double criterion when interweaving "problems of a technical nature with real problems". For him, "technical problems" are problems without any real context that, in accord with his ways of knowing the concept of function, can be the "skeleton" of a possible real problem. The problems with a real context have a role for motivation, related to his image that considering the mathematics can help to explain real situations, "so that the individual can see the relationship that exists between theoretical mathematics and reality" (Alberto, 854–855; Classification). Alberto pointed out that this could occur after motivating the pupils through connecting the mathematical topic with real life:

> I would begin with this type of technical problems [but] always maintain the graphic representation as the most important aspect, because with the latter you can display all the study of increase, decrease, symmetry . . . (Alberto, 1102–1105; Classification).

That is to say, Alberto emphasised the study of the overall functional properties through the visualisation allowed by the graphic representation of a real life situation.

Representational repertoire. Here we consider the alternative ways for representing the concept of function that the student teachers used and the reasons provided.

The hypothetical teaching sequence that Juan and Rafael prepared highlighted the relational view of the concept from the 'point-image' perspective. These student teachers gave a priority role to algebraic representation. What it is important to emphasise here is not just the types of problems that the student teacher used, but also the objective that was pursued. Juan and Rafael considered the problems of reading and interpreting graphs as application problems. On the other hand, for José and Alberto the graphs and the real situations played a key role in their teaching sequence. For example, José's one objective was that pupils should become familiar with a formal definition of the function concept as a univalent correspondence of R into R, and with some skills needed to perform the graphs of the linear functions. So, he started his hypothetical teaching sequence with problems of graphic representations of linear functions. Likewise, Alberto complemented the use of the function as a model with the algebraic mode. He gave greater significance to the graphs in the teaching sequence, taking into account the different uses of the graphs: for instance, the graphs playing a complementary role to the algebraic expressions in order to study the overall properties of the functional relationships and as a means to understand a real situation.

Adaptation. Here we consider what characteristics of the pupils' learning of the concept of function (the pupils' prior knowledge and the most frequent mistakes) seem to be taken into account by student teachers.

Juan and Rafael considered a hypothetical level of difficulty of the problems while José and Alberto took into account the idea of 'motivation', but these ideas were always used in a general manner and without any more specification. Another idea that influenced the adaptation of mathematical content to pupils was the meaning given by the student teachers to the pupils' 'prior knowledge'. The four student teachers saw the prerequisite knowledge needed to solve the problems as the prior content that the teacher should have provided earlier. The textbook problems were seen as an application of mathematical content that had been explained in advance. The problems were seen as a means for the pupils to 'practice' the procedures provided beforehand by the teacher. None of the student teachers provided information regarding the pupils' mathematical understanding. The significance given to the pupils' prior knowledge by the student teachers was compatible with their images about teaching and learning as 'telling and remembering'.

DISCUSSION AND IMPLICATIONS

This study is an attempt to identify the influence of student teachers' ways of knowing the subject matter and images of mathematics, teaching and learning on their hypothetical presentation of subject matter for teaching in the context of functions. Along the same lines as other studies (Wilson, 1994; Even & Tirosh, 1995), our results point out the influence of subject matter knowledge for teaching regarding the way in which these student teachers tried to represent the subject matter to the pupils. However, of the two dimensions considered in subject matter for teaching, *ways of knowing the subject matter* and *images of mathematics teaching and learning*, the influence was different.

Ways of Knowing and the Use of Modes of Representation in Teaching

The student teachers in our study differed in their ways of knowing the concept of function. In particular, they considered different aspects of the concept and the modes of representation in a different way. This influenced their pedagogical reasoning. Juan and Rafael's ways of knowing the concept of function emphasised the operational aspect of functions and the algebraic mode of representation. They considered the graphs as a complement of the algebraic mode of representation. These student teachers gave a priority role to algebraic representation and computational activities over the problems of reading and interpreting graphs. José and Alberto incorporated the use of graphs as an 'instrument' for solving real situations. However, while José seemed to attribute two independent words (*model* and *correspondence*) to the function concept in the organisation of subject matter for teaching, Alberto's emphasis showed the complementary roles of graphs and the algebraic mode. These emphases influenced these student teachers' organisation of content and the types of problems chosen in the teaching sequence. For all four student teachers, their ways of knowing the concept of function as a teaching-learning object influenced what they considered important for the learner and affected their use of the modes of representation in teaching, considered as teacher's tools to obtain his/her teaching goals.

Even (1998) considered that flexibility in moving from one representation to another is intertwined with flexibility in using different approaches to functions. In the same way, our findings indicate the importance of analysing the relationship between different aspects of mathematical content that student teachers emphasised and different use of modes of representation in their pedagogical reasoning. One implication of our results is that the mathematical content should become a context for the contemplation of pedagogical issues, (e.g., discussion about different ways of representing specific mathematics topics, their strengths and limitations linked to aspects of concept emphasised). In this context, student teachers can discuss and evaluate the multiple representations linked to different approaches to a specific concept.

On the other hand, the limitations in student teachers' knowledge of pupils' understanding seems logical in student teachers that have not attended teaching practice yet, and with an education mainly focused on mathematical content. Nevertheless, we think that one implication is that the assessment of different modes of representation should be coupled with discussion on specific knowledge of pupils' understanding of particular concepts during teacher education. Stacey et al. (2001) have shown the need for teacher education to emphasise content knowledge that integrates different aspects of a topic and 'pedagogical content knowledge that includes a thorough understanding of common difficulties' (p. 205). From the results of our study, a focus on the relationships among the different aspects of the concept, modes of representation and knowledge of students' mathematical understanding of a particular topic has been shown as necessary.

Influence of Student Teachers' Images

In relation to the influence of student teachers' images on their pedagogical reasoning the results obtained are not so clear. The four student teachers viewed mathematics as a set of systematic procedures for solving problems. They held similar images about teaching and learning, such as the 'telling and remembering', and 'saw' the 'prior knowledge' necessary to perform school problems as 'prior content' previously introduced. Two of them made general references about motivation in their own learning to justify some aspects on the lesson plans, using real-word problems as tools to motivate pupils.

From these results, we can appreciate that when these student teachers think about the content for teaching their initial decisions are closely related to their ways of knowing the mathematical content. Possibly in the student teaching experiences, where these student teachers reconstruct their knowledge about the mathematical concepts as teaching-learning objects (Jones & Vesilind, 1996), a better identification of the role of images is possible.

Implications for Teacher Education

If we recognise that the subject matter knowledge for teaching derives from a variety of sources; our results point to the fact that these do not all have the same influence when the student teachers think about the subject matter with the purpose of teaching (pedagogical reasoning). Therefore, during the teacher education program the student teachers should approach pedagogical content knowledge in more than one way. In particular, learning situations on the specific cognition of mathematical topics should be introduced in the method courses of mathematics education. Furthermore, mathematics teacher education should consider opportunities for the student teachers to design learning tasks, analyse the mathematical field of these tasks and consider the curricular learning goals their engagement might support. All of this could be used in seeing the mathematical content in practice, generating the possibility of analysis and reflection on the influence the different ways of knowing and images have on decisions in the practice of teaching.

We believe that the use of mathematical knowledge in teaching is a concern of teacher education. Knowing more about the relationship between subject matter knowledge for teaching and pedagogical reasoning provides us with information to design activities in teacher education programmes. Cases, critical incidents and interviews concentrating on the cognition of mathematical topics (Barnett, 1998; Markovits & Even, 1999) can allow us to pose questions of learning and to develop analyses and reflections about mathematics teaching and learning. If we consider that the use of knowledge in teaching (e.g., to examine and sequence different mathematics problems to design a plan for teaching a particular concept) is different from knowing the mathematical content, we should take new decisions in relation to teacher education. These decisions must allow us to prepare teachers who not only know content but make use of it to help students to learn.

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