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WHY IS MODELLING NOT INCLUDED IN THE TEACHING OF ALGEBRA AT SECONDARY SCHOOL?

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Abstract

The kind of algebraic activity which is most carried out in current Spanish compulsory secondary education corresponds to the predominant view that algebra is a "generalised arithmetic". This implies that algebra is confined to the field of arithmetic and the work with numbers (as opposed to the working with magnitudes) which does not benefit the emergence of algebra as a modelling tool. The analysis of the constraints that the didactic transposition process imposes on school mathematics practices highlights that the institutional limitations on the teaching of algebraic modelling are not only due to the students' cognitive difficulties.

Résumé

Le genre d'activité algébrique qui se développe actuellement dans l'enseignement secondaire obligatoire espagnol correspond généralement à une interprétation dominante de l'algèbre comme « arithmétique généralisée ». Cela comporte un enfermement de l'algèbre dans le domaine du calcul arithmétique et du travail avec des nombres (par opposition au travail avec des grandeurs) qui ne favorise pas l'émergence de l'algèbre comme instrument de modélisation. L'analyse des contraintes qu'impose la transposition didactique sur les pratiques mathématiques scolaires montre que certaines des limitations qui pèsent sur l'enseignement de la modélisation algébrique vont bien au-delà des difficultés cognitives des élèves.

Riassunto

Il genere di attività algebrica che si sviluppa attualmente nell'insegnamento secondario obbligatorio spagnolo corrisponde generalmente ad una interpretazione dominante dell'algebra come « aritmetica generalizzata ». Questo comporta un sconfinamento dell'algebra nel dominio del calcolo aritmetico e del lavoro con i numeri (in opposizione al lavoro con le grandezze) che non favoriscono l'emergere dell'algebra come strumento di modellizzazione. L'analisi dei vincoli che impone la trasposizione didattica sulle pratiche matematiche scolari mostra che alcune limitazioni che pesano sull'insegnamento della modellizzazione algebrica vanno al di là delle difficoltà cognitive degli allievi.

Resumen

El tipo de actividad algebraica que se desarrolla mayoritariamente en la enseñanza secundaria española actual corresponde a una interpretación dominante del álgebra como "aritmética generalizada". Esto conlleva un encierro del álgebra en el ámbito del cálculo aritmético y del trabajo con números (en oposición al trabajo con cantidades de magnitud) que no favorece la emergencia del álgebra como instrumento de modelización. El análisis de las restricciones que el proceso de transposición didáctica impone sobre las prácticas matemáticas escolares muestra que las limitaciones que pesan sobre la enseñanza de la modelización algebraica van mucho más allá de las dificultades cognitivas de los alumnos.

1. Teaching elementary algebra as *generalised arithmetic*

Mathematics at secondary school has its own particular features that differentiate it in many aspects from the mathematical works where they were firstly developed before being introduced at school through a complex process of *didactic transposition* (Chevallard 1985). When we deal with the problem of teaching elementary algebra in secondary school –which kind of algebra has to be taught and how, so that the pupils learn–, we are driven to study not only what we, as researchers, understand *algebra* to be in theoretical terms (Bolea, Bosch, Gascón 1989), but also what the educational system itself understands algebra to be, that is, which kind of mathematical activities related to algebra are developed in its teaching and learning processes.

It is a common fact, established by numerous research projects (Sutherland *et al.* 2001) that the first learning of algebra takes place within the core of arithmetic, an area of mathematics that is closer to the student and which acts as a reference point for the algebraic work thereafter. As such, algebra is developed closely with arithmetic and quite apart from the rest of mathematical contents. Algebra arises at school as a kind of "algebraic language": a way of expressing the general properties of arithmetical operations and its rules are reduced to a limited extension of the working of numerical calculations. This prevailing and more or less explicit interpretation of algebra as a *generalised arithmetic* is expressed in a fairly specific group of types of school mathematical tasks which are not necessarily connected to each other and that can be summarised in the following four points:

1. Writing numerical-verbal expressions using symbols that describe and/or generalise arithmetical calculation techniques.
2. Manipulating algebraic expressions in a formal way to simplify or transform them into a pre-established form (developing, simplifying, rationalising, etc.).
3. Establishing and manipulating algebraic expressions where the letters represent unknown numbers. In particular, solving equations interpreted as equalities between algebraic expressions that are true for certain concrete values of the unknowns.
4. Solving word problems with equations through a translation of the verbal formulation of the problem, assigning a name to the unknown quantities and numerical values to the data.

We have shown in a previous work (Bolea 2002) that, at least concerning current Spanish secondary education (12-16 years old), this prevailing understanding is mainly followed by official curriculum guidelines and institutional instructions for teachers, as well as by the key text books and other kind of teaching materials. We have also studied how teachers describe their algebraic teaching practices as well as their personal understanding about what school algebra is and what it should be. Our study corroborates the above-mentioned theory that introductory algebraic practices are based on arithmetical contents, the understanding of algebraic symbols which are given in class are almost always presented in numerical terms, and that the abstraction level assigned to algebra is always higher than that of arithmetic.

As a result of the previous study, we wish to emphasise that in school mathematical practices, algebraic symbolism could be present in other blocks of contents, such as geometry or functions, but it is rarely present in arithmetic. This proves that, although the algebraic instrument is introduced through the formalisation of arithmetical calculations, it hardly ever serves to enrich the numerical field. In school mathematical practices, the relation between arithmetic and algebra is unidirectional: first comes arithmetic and, then, algebra.

2. The absence of the teaching of algebra as a *modelling tool*

Beside the point of view of algebra as a generalised arithmetic, we can also see algebraic activity as essentially a mathematical modelling tool (in the sense of Chevallard 1985, 1989, 1990). In this case, algebra is not considered as a content of its own, but as a tool for modelling mathematical systems, what we called (Bolea *et al*. 1998) the *algebraisation process of mathematical organisations*. According to this, teaching algebra at school should incorporate mathematical tasks which include the following attributes:

1. Algebra should serve to model mathematical systems. In particular, it should allow us to pose and solve problems in different mathematical fields (arithmetic, geometry, etc.) which are otherwise hard to pose and solve without algebra.
2. Algebraic modelling should provide answers to questions related to the scope, reliability and justification of the mathematical activity which is carried out in the initial system. The algebraic model that is constructed should especially allow for the description, the generalisation and the justification of problem-solving processes, and should bring together techniques and problems that, at first, appear to be unrelated.
3. Algebraic modelling should lead to an expansion and a progressive transformation of the initially studied system, with the incorporation of new kinds of problems, new techniques to solve problems, new interpretations, new links to other systems and other fields, etc.
4. In the algebraic modelling process, expressions should include letters that designate magnitudes (and no only numbers) and the manipulation of these expressions does not require any preliminary distinction between known and unknown quantities.
5. This process facilitates the study of relationships between magnitudes of any kind (geometrical, physical, commercial) and evolves towards functional modelling.

From the findings of the aforementioned empirical study, we can confirm that the interpretation of algebra as a modelling tool has a very weakly presence in current Spanish secondary education. For instance, when we look at the kind of tasks proposed by one of the most common textbooks as algebraic problem-solving tasks, we can see that (Bolea 2002):

– Less than 20% of the tasks (including those that are to be completed by the teachers) correspond to global modelling activities. Furthermore, in these cases, the part of the activity that has to be carried out by the students is often reduced to the solving of an equation.

– Even half of students’ tasks correspond to formal manipulation of algebraic expressions.

– The final stage of the modelling process –the interpretation of the obtained results in order to formulate new problems– is totally non-existent. This highlights that the initial system studied is not “taken seriously enough” in problem solving at a school level.

– ‘Second order’ questions never appear, i.e. questions related to the possibility of the modelling work itself, the scope of the model which is constructed, the information (data) needed to continue the modelling work, etc.

– The most complete modelling activity occurs when dealing with arithmetical systems, in other words, when algebra is introduced as a tool for describing and justifying numerical properties. It never serves as a tool for creating new numbers (for example, to expand numerical fields in response to the need to have solutions to certain types of equations, etc.).

These findings allow us to corroborate Chevallard’s thesis about the “disalgebrisation” of the school curriculum, a phenomenon that can be summarised into a number of general facts, as following:

(a) Algebra is not used to relate problems that appear in different content blocks: first grade equations and proportionality, etc. We have observed what Chevallard (1989) described as ‘the maintenance of a strong autonomy between the content blocks, and the resulting *disintegration* of algebraic corpus’.

(b) Formulas, which only appear in volumes and surfaces calculations or in commercial arithmetic, are never obtained as the fruit of a previous algebraical work. Their only role is to facilitate “rules” to automate certain calculations, but rarely to create new mathematical objects or new properties of the old ones.

(c) The different *number systems* don’t appear as the consequence of an algebraic construction.

(d) Apart from the special case of ‘word problems’ in which letters are used to designate unknown quantities, the nomination or ‘re-nomination’ activity, that is the introduction of new letters in the course of the mathematical activity, is completely absent. For example, variable changes are not used to simplify expressions, or to solve equations or inequations, or to interpret functions in different reference systems from those determined by original variables.

(e) Working on algebraic objects, taking them as objects to study, is practically non-existent in secondary education. Thus, for instance, certain algebraic objects (equations, expressions, formulas, and functions) can be manipulated (solved, simplified, represented or transformed) but they are never properly studied.

According to Chevallard (1989), the phenomenon of 'disalgebraisation of the mathematical curriculum' which, on the whole, responds to the cultural 'belittling' of algebra, is in turn a consequence of the 'logocentrism' of western culture.[*] In this case, our purpose is to highlight the existence of transpositional constraints which are most closely linked to the mathematical education system. To this end, we will show that the tasks that correspond the understanding of algebra as 'generalised arithmetic' adhere to the restrictions that didactic transposition imposes on school mathematical activities much more closely than those involving *algebraic modelling*. In this way, we are able to clarify the nature of the 'transpositional difficulties' that hinder the teaching of algebraic modelling at school.

3. Transpositional restrictions on school algebra

As shown in a previous work (Bolea *et al.* 2001[a]), didactic transposition theory establishes the existence of different kinds of generic restrictions which are imposed on the taught knowledge at the heart of any educational system (Arsac 1988). We consider four kinds of generic restrictions that are strongly interrelated:

A. Restrictions which originated from the need to adapt school mathematical activities to the institutional representation of the mathematical knowledge. In other words, adapting that which the educational system considers as mathematics with that which is understood as the teaching and learning of mathematics.

B. Restrictions due to the need to evaluate the mathematical activity which students have to carry out, and the related mathematical knowledge. This necessity tends to lead to an internal differentiation and independence of the mathematical block of contents, and a bigger algorithmisation of its techniques, with the resulting loss of functional sense of the mathematical activity.

C. Restrictions that arise from the need that all taught knowledge must appear as definitive and unquestionable. This 'didactical' necessity conflicts with the need for the dynamic of any research process to reconsider the previously studied mathematical organisations to show their limitations and contradictions, and to restructure and integrate them into larger and more complex organisations. This restructuring may be so in-depth that it becomes necessary to 'correct' the effects of a previous transposition through a kind of 'de-transposition' (Antibi and Brousseau 2000).

D. Restrictions imposed by the *didactic time* in various senses, such as structuring of taught knowledge into an *ordered series of subjects*, the *ageing* of the teaching system which involves the need for constant reforms, the *internal obsolescence* of the didactic process (Artigue 1986), the need for *fast learning* or learning over a very short period of time (which could even lead to the illusion of *instant learning*) and, finally, restrictions concerning the availability of the *didactic memory of the system* (Brousseau and Centeno 1991).

If we consider these major kinds of constraints, it is easy to verify that they are much more compatible with the types of tasks which correspond to the view of algebra as 'generalised arithmetical' rather than with this that give rise to algebra being seen as a 'modelling tool'.

A. Our findings on how teachers interpret school algebra allow us to corroborate the hypothesis of the dominance of the understanding of algebra as 'generalised arithmetic' in the educational institution. This hypothesis arose from the analysis of classroom materials, textbooks and other kind of documents for teaching guidance. If the educational system 'understands' elementary algebra as a generalised arithmetic, then it is clear that the tasks which conform with this view will be more present in school practices related to algebra. However, algebraic modelling also corresponds closely to the 'modern' understanding of mathematics as a problem solving activity. This 'higher level' understanding (because it concerns mathematics as a whole instead of looking only one of its fields) seems to favour 'modelling-tool' tasks. We have found this bias to be true in the results of our questionnaire to teachers. Even if a majority of teachers describe taught elementary algebra as having most of the characteristics of generalised arithmetic, many of them consider that the study of elementary algebra *should be* more closely related to problem-solving or, even, modelling activities.

B. As regards restrictions due to evaluation exigency, it can be noted that algebraic modelling techniques are among the less visible, less 'algorithmisable', more difficult to 'split up' and are, in conclusion, more difficult to evaluate. On the contrary, due to their limitation to the arithmetical field, tasks which are characteristic of generalised arithmetic can be easily divided and organised in quite a flexible manner, allowing them to be split, thereby facilitating their evaluation by the delimitation of concrete learning objectives.

C. The third kind of restrictions also directly affects algebra as a modelling tool because they involve a process of reorganising the mathematical contents. In fact, the need for all taught knowledge to appear as *definitive* and *unquestionable* hinders the study of its limitations and contradictions and, therefore, the need to restructure, modify, correct and integrated the mathematical contents studied, in order to make them larger an more complex. On the other hand, arithmetic is a pre-defined construction that is no questioned nor 'deconstructed' by the introduction of new problems or new algebraic treatments. It is only relatively enlarged, through the introduction o f a more extensive symbolism which enables new problems to be solved without the need to modify the solution or the earlier established structures. The initial numerical field is also not transformed through the introduction of the algebraic instrument, since this latter is not enlarged by the inclusion of new numbers.

D. We have seen that algebraic modelling work begins with setting questions regarding a mathematical system that cannot be answered using only the techniques which pertain to that system. This questioning highlights the limitations of the system and the need to 'complete it' in some way. Taking into account that the work in this system can be done in a previous course or with another teacher, then the ability to carry out the algebraic modelling is restricted due to the availability of the didactical memory. Closely related with these constraints, there is the need to take into consideration *long-term objectives* that enable one to study the initial system, question it and carry out all the stages of the algebraisation process (studying the initial system, questioning of the work that has been done, model building, working the model, formulating answers and new questions, etc). Nonetheless, this condition, that can be identified with a kind of 'epistemological patience' and which is essential to carrying out the algebraisation process successfully, conflicts with the demands of 'instant learning'. As a consequence, it is quite impossible to carry out in-depth algebraisation processes in the school environment, in an adequate, explicit and detailed enough way.

4. Is it possible to integrate algebraic modelling in secondary education?

Up to this point, we have tried to explain the absence of algebraic modelling activities in secondary education. Inevitably the question that arises now is if it will be possible, and didactically viable, in the current mathematics teaching system, to design a mathematical curriculum that could incorporate elementary algebra as a modelling tool. We have not found as yet the solution. However, we could try to suggest a possible direction to deal with this important problem.

In a recent work, Yves Chevallard (2000) shows a particular way to reconstruct algebra in school that, on the one hand, integrates those elements which are classically considered as *algebra* (equations, inequations, polynomials, equations systems, etc.) and, on the other hand, taking into account the main restrictions shown above, allows for an enlargement of arithmetical calculations using algebra as a modelling tool. In a sense, the proposal can be considered as the construction of the minimal algebraic model that fully models elementary arithmetic, including general problem-solving techniques, instead of being limited to just formalising specific numerical calculations.

Chevallard's proposal begins by characterising elementary arithmetic as the construction and carrying out of what he call 'calculations programs' (such as: ‘15×2 + 123 – 348×13’ o ‘12300 – 3%×12300’, etc.). He then considers the set of 'calculations programs' as the initial mathematical system to be studied. Questions that can be formulated are such as: the possibilities of developing a calculation program; ways to determine the equivalence between two formally distinct programs; determining some elements of the program (unknowns) given is final result; etc. To answer these questions, we cannot only remain in the specific area of the initial system. The need arises to introduce algebraic expressions (with parameters and variables) as models for these programs. These models turn out to create a new mathematical organisation that includes and completes the initial system of 'calculation programs'.

At this stage, we cannot take for granted the ‘viability’ of this kind of teaching proposals based upon the introduction of elementary algebra as a modelling tool (research which, in the above case, is still pending experimental cross-checking). In other words, the possibility that modelling activities can exist in a generalised and stable manner in current mathematical educational systems cannot be guaranteed just by the quality of the teaching materials or by the cognitive characteristics of the students. Any curriculum proposal which aims to directly influence the didactic transposition of algebra should take into account the conditions which are imposed by the educational system itself. Moreover, because the algebraisation process ends up affecting all mathematical content blocks, it is foreseeable that any curricular proposal that pretends to integrate algebra as a modelling tool will lead us further than the specific area of the teaching of algebra, in order to encompass all of the mathematical secondary school curriculum.

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[*] The French philosopher Jacques Derrida highlighted this western metaphysical standpoint which supports implicitly that the view that the 'thought' resides in the 'head', it is expressed by the *voice* and the *word* and is preserved through *writing*. Nonetheless, the written word is only a degradation of thought or, in short, a by-product. Common culture is not aware of the essential fact that *scientific formalisms* are languages that *do not come from any oral language*, but are born as writings and are difficult to 'oralise'. This causes specific didactical problems when teaching algebra, for example. In this way, all that *is said* or *can be said* (the 'reasoning') is overvalued and all that can only *be done* is negatively considered, in particular, that which is *only written* without being enounced orally. *Logocentrism* involves a deep lack of understanding of the nature of the scientific activity as it underestimates, and even the existence, of *the written formalisms as instruments of scientific thought.* (Bosch and Chevallard 1999)