"Quaderni di Ricerca in Didattica", n14, 2004.
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THE NATURE OF MULTIPLE REPRESENTATIONS IN DEVELOPING MATHEMATICAL RELATIONSHIPS[1]

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Abstract

This study focuses on the representations and translations of mathematical relationships that are emphasized and taught at school, and discusses two theoretical models that may explain the pattern and difficulties in translating from one form of representations to another. Data were obtained from 79 students of grade 6. Analyses using structural equation modeling were performed to evaluate the two theoretical models. Results provided support for the hypothesis that multiple representations of mathematical relationships constitute different entities, and thus multiple representations do not by themselves help sixth grade students develop mathematical understanding.

INTRODUCTION

There is strong support in the mathematics education community that students can grasp the meaning of mathematical concepts by experiencing multiple mathematical representations (e.g., Janvier, 1987; Sierpinska, 1992). In this context, NCTM (2000) refers to a new "process standard" that addresses representations. The term "representations" is interpreted as the tools used for representing mathematical ideas such as tables, graphs, and equations (Confrey & Smith, 1991). By a translation process, we mean the psychological processes involving the moving from one mode of representations to another (Janvier, 1987).

During the last two decades, the critical problem of translation between and within representations, and the importance of moving among multiple representations and connecting them are addressed in several studies (Sfard, 1992; Yerushalmy, 1997; Goldin, 1998; Reading, 1999). Most of them are based on the assumption that students´ ability to understand mathematical concepts depends on their ability to make translations among several modes of representations. Others examine the construction and transformation of representations through the process of mathematical problem solving (Yamada, 2000). The present study purports to throw some light about the nature and the contribution of multiple representations to mathematical learning and problem-solving. It is investigated how the translations among and within the several modes of representations contribute in the development of students' understanding of various

[1] A poster on the same subject has been presented by Athanasios Gagatsis and Iliada Elia. Another larger form of this text has been published by Athanasios Gagatsis, Iliada Elia and Antigoni Mougi in a special issue on representations of Scientia Pedagogica Experimentalis (2002).

mathematical relationships. For this reason it discusses two models that may explain the pattern and difficulties in translating from one form of representation to another. More specifically, the nature of translations among multiple representations is examined by attempting to provide answers to two questions:

a) Do multiple representations and translations from one mode of representation to another relate to each other in such a way as to help students abstract the underlined concepts, and apply these representations to problem solving?
b) Do different representations of mathematical relationships constitute different entities that may not convey the expected mathematical concepts, and need to be taught explicitly to students in order to deepen their understanding in mathematical concepts?

Both of these questions are interrelated and are motivated by practical and theoretical concerns. These questions are focused on the way of using multiple representations in teaching mathematical functions and on the organization of didactical approaches to promote students' understanding. As for the theoretical needs, these emerge from the lack of a theoretical framework of representations capable of supporting the kinds of understandings necessary to translate form one representation mode to another. Both practical and theoretical concerns are interwoven in understanding the relations and connections between the multiple representations of mathematical relationships.

METHOD

To examine students' understandings in mathematical relationships, we generated questions presenting an idea in one representational mode and students were asked to represent the same idea in another mode. Each question focused on students' abilities to perform translations from one representational system to another. A test was administered to 79 Cypriot students in grade 6 ranging in age from 120 to 132 months (Average age=123 months, SD=2.1 months). Each of the four factors of the study (i.e., graphical, tabular, verbal, and symbolic) involved three problems that represented relations of the following type: $y=ax$, $y=ax+b$, and $y=x/a$.

The assessment of the proposed models was based on investigating the fit of the relevant confirmatory factor models. Model fitting was tested by computing three fit indices; the chi-square to its degree of freedom ratio (?^{2}/df), the comparative fit index (CFI), and the root mean square residual (RMR). These three indices recognized that observed values for ?^{2}/df <2, values for CFI>.9, and RMR values close to 0 are needed to support model fit (Marcoulides & Schumacker, 2001).

A content analysis of pupils' responses to the test items was conducted, in order to examine how students actually comprehend and interpret the symbolic representations of mathematical concepts and relationships. The constant comparative method (Denzin & Lincoln, 1998) was used to analyze the qualitative data.

Content analysis of students' responses to the test tasks was focused on students' misconceptions in relation to the meaning of mathematical symbols. These misconceptions seem to reveal students' incapability to relate the symbolic to the verbal form of representation. For example, students translated the verbal expression "four times fewer" as "– x 4". The term fewer was seen as an indication of the fact that they have to subtract and therefore the symbol "-" was used. Another interesting response given by some students, included the use of the expression "N=KxO" as a symbolic representation of the fact that the number of cubes didn't change among players. The particular finding indicates that students considered the symbol "xO" as the identity element for multiplication. Examples of students´ answers reflecting the above misconceptions are given below.

Verbal	**Symbolic**
Five children share cubes among them. Each child gets four times fewer from the next one.	N=-4xK*

Example 1

Tabular		**Verbal**	**Symbolic**
Children	*Cubes*	Five children have some cubes to share. Each one will get 8.	N=KxO*
1st	8		
2nd	8		
3rd	8		
4th	8		
5th	8		

Example 2

**Letter "N" was agreed to represent the number of cubes that any child would get, and letter "K" to stand for the number of cubes the previous child would get.*

These examples of misinterpretation and misuse of the particular mathematical symbols are perhaps due to the fact that students use them frequently and therefore theorize them as the most appropriate symbols to express their mental representations.

RESULTS

In order to answer the questions of the present study, the assessment and comparison of model 1 (figure 1) and model 2 (figure 2) was the focus of the analysis of the data. Although both models fit the data, the ratio of $?^2$/df is smaller in model 2 (1.35) than in model 1 (1.87), and the CFI of model 2 (.98) is much greater than that of model 1 (.91).

Chi-squared difference test was also used to compare the fitting of the two models. The difference of $?^{2}$ between the two models is 23.59, which is statistically significant (df=1, $p<.001$). It can be therefore claimed that model 2 fits the data in a better way. The critical difference between the two models is in the relations between the factors. This means that model 2, which has an excellent fitting, explains better than model 1 the structure of the relationships between the different modes of representations, and the relations between the particular indicators and the corresponding latent constructs (factors).

The factor loadings of each task in model 2 are shown in figure 2, and the variances explained by the corresponding latent factors are illustrated in Table 1. All loadings were found to be significant ($p<0.01$). This implies that the hypothesized factors can be substantiated by the observed variables as measured by the test. Figure 2 also presents the effects of the factors in model 2 that highlight several important relations. The size of the effects (parameter estimates) in model 2 were tested through t-tests and all the paths were significant ($p<.01$). The graphical factor strongly exerts a direct effect on the verbal (0.67) and tabular (0.76) modes of representations. However, the effect of the graphical factor to the symbolic factor is indirect through the tabular factor of representation (0.53). The latter factor exerts a strong direct effect on symbolic factor (0.70).

Table 1: Explained variance in each Latent Factor by their corresponding observed variables

Latent Factors	**Observed Variables**	**R^2**
Graphical (GRA)	GRA&TA* GRA&VE GRA&SYMB	0.75 0.84 0.76
Verbal (VE)	VE&TA VE&GRA VE&SYMB	0.91 0.87 0.36

Latent Factors	**Observed Variables**	**R^2**
Tabular (TA)	TA&GRA TA&VE TA&SYMB	0.75 0.90
Symbolic (SY)	TA&SYMB SYMB&TA* SYMB&GRA	0.89 0.94 0.86

The graphical factor accounts for 45% of the variance on verbal factor and for 58% of the variance on tabular representations, while the tabular factor accounts for 50% of the variance on symbolic factor. The graphical factor also accounts indirectly for 28% of the variance on the symbolic representations. These results are in line with those reached by Aspinwall et al., (1997), indicating that in some cases the visual representations create cognitive difficulties that limit students' ability to translate between graphical and algebraic representations.

THEORETICAL MODELS FOR MULTIPLE REPRESENTATIONS

Two models were hypothesized as appropriate to answer the questions of the study. Both models included four factors representing four types of representations in mathematical relationships, namely, the graphical, the verbal, the tabular, and the symbolic. The first factor involves tasks in which a relationship is given in its graphical form and students are asked to translate it to its verbal, tabular, and symbolic form. The verbal factor includes tasks that provide students with the verbal description of a problem. Students are asked to translate the verbal description of the problem to its graphical, tabular, and symbolic representations. In the same way, the tabular and the symbolic factors include problems that are given in their tabular and symbolic forms, respectively, and students are asked to translate them to the other three forms of representations. These four factors were considered important in studies on representations (e.g., Janvier, 1996).

However, the two models differ in their structure. Model 1, which is presented in figure 1, views translations from one mode of representation to another as interrelated. Thus, model 1 provides support to the argument made in the first question of the study since it is based on the hypothesis that students are able of connecting different representations of a relationship. This model indicates not only that multiple representations are valuable for learning mathematical relationships but also that the differences among these representations can contribute to this value. Since all the factors in model 1 are interrelated, we hypothesize that each representation and translation makes clear the meaning of the mathematical relationship, and that students have the ability to move from one mode of representation to another one with relative facility.

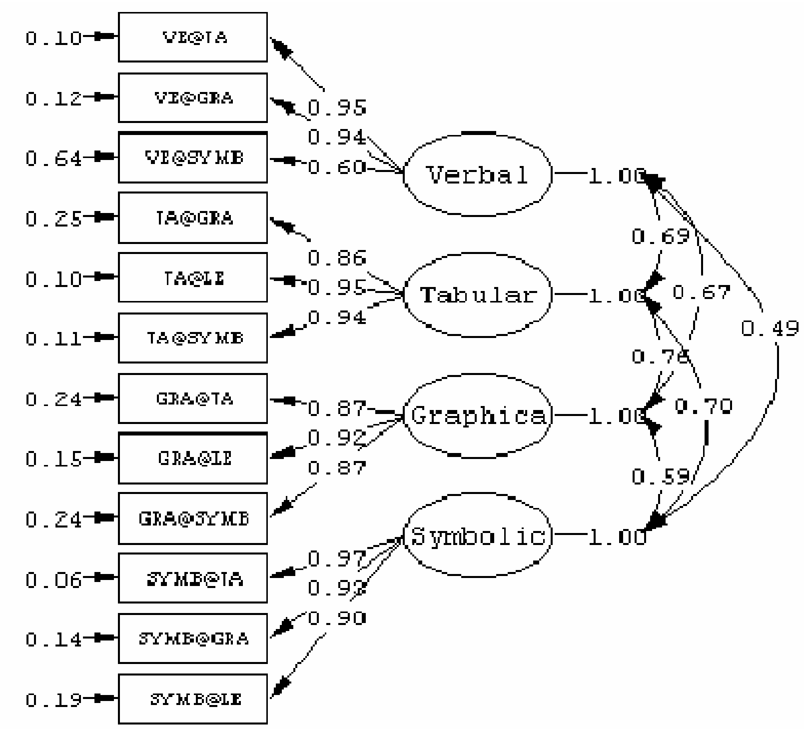

Figure 1: Model 1 Representing the Interrelations among the Translations.

(Goodness of fit indices: X^2*=89.63, df=48,* X^2*/df=1.87, CFI=.91, RMR=.007)*

On the other hand, model 2 (see figure 2) is based on the theoretical assumption that there are modes of mathematical representations that are prerequisites for other representations that are more complicated or sophisticated. It refers to the second question of the study, which hypothesizes that multiple representations function as distinct entities and that each representation yields its own insights into mathematical relationships. It also hypothesizes that multiple representations and translations constitute a hierarchical system, and that not all of them contribute to the development of mathematical relationships in the same way. It assumes that there are cases in which students do not recognize that the translations among multiple representations refer to the same concept, and in some cases different translations may create confusion.

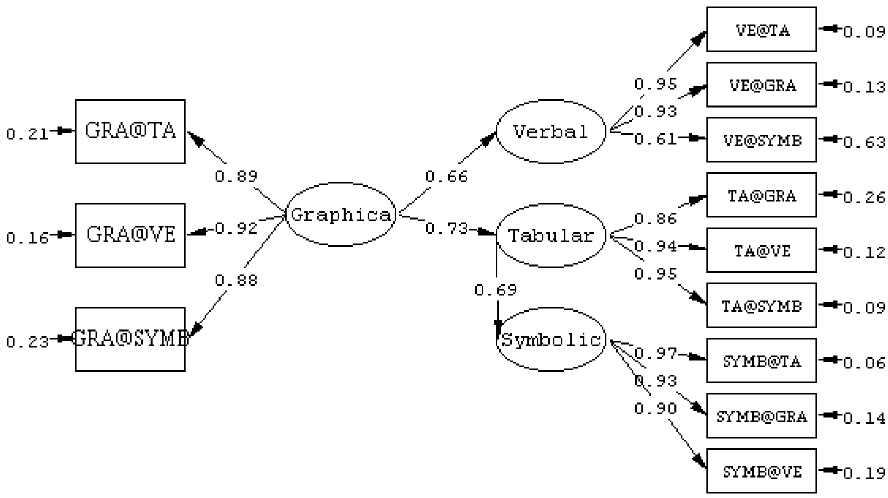

Figure 2: Model 2 Representing the Hierarchical Structure of Translations

(Goodness of indices: X^2=66.04, df=49, X^2/df=1.35, CFI=.98, RMR=.007)

DISCUSSION

The present study is in the framework of the ongoing discussion about the nature of mathematical representations. Basic research in this area provides support to two tentative hypotheses that: (a) mathematical representations are interrelated, and (b) mathematical representations constitute different entities (Duval, 2001). Our results provide support to the second research question, implying that multiple representations do not by themselves help students develop mathematical understanding. This is exemplified by the hierarchical structure of multiple representations supported by the data of our study. The hierarchy of multiple representations has theoretical and practical implications. From a theoretical perspective, this hierarchy means that some representations function as prototypes (Schwarz & Hershkowitz, 1999). Prototypes are the representations that have a set of characteristics most highly correlated with the characteristics of other representations, and serve as the basis for understanding and connecting a number of representations of the same content. In this study, the graphical factor seems to act as a prototype for understanding the representations starting from the verbal and tabular descriptions (verbal and tabular factors). At the same time, the tabular factor acts as a prototype for enabling students to handle the translations starting from symbolic forms (symbolic factor). Prototypicality further indicates that the cognitive demands of the translations among representations are not quite the same, and thus each one needs special attention during instruction (Duval, 2001). Moreover, findings based on the qualitative analysis, indicate students' difficulties concerning the relation between the symbolic and the verbal form of representation. Therefore, teachers should be aware of the fact that a symbolic system in mathematics education does not always perform its functions in the expected way and should give special importance to the passage from natural to symbolic language (Bazzini, 1999). These results also support Miller's (2000) conclusions that each representational system has its own regularities and transferring between representational systems and can sometimes be a stumbling block to learning new concepts.

Finally, the above findings have some implications for instruction. The structure of the translations between different representations may facilitate instruction by following the paths of the proposed model and by sequencing correctly the domain of function representations. It can be therefore claimed that the proposed paths of model 2 can be used more explicitly in the process of teaching and learning mathematical relationships.

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