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AN EXTENSION FOR BALLIEU INEQUALITY

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Abstract. We shall generalize in a *Lemma*, a very old result of R. Balllieu (see [1]), relating to the *Triangle Geometry*, and using notions from *Real Analysis* and *Jensen inequality* for a *concave function.* So we shall prove that in a triangle ABC, for every $t \in \mathbf{R}$ we have: $2^{t-1}\sin^t\frac{A}{2} \leq \frac{a^t}{b^t + c^t}$.

Résumé. Nous généralisons dans une *Lemme* un résultat ancien, introduit par R. Ballieu (v.[1]). Celui-ci este rélatif à la *Géometrie du Triangle* et il utilise des notions de *l'Analyse Réelle* et également, *l'inégalité de Jensen* pour les *fonctions concaves*. Par conséquent, nous démontrerons que dans un triangle ABC, pour tout $t_\theta\mathbf{R}$, nous avons la suivanté inégalité: $2^{t-1} sin^t\frac{A}{2} \leq \frac{a^t}{b^t + c^t}$.

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0. INTRODUCTION

Sometimes, we say that the Trigonometry is a closed domain... Because apparently, we know all about the relations which reign over the sides and the angles of a triangle. And this is a consequence of the studies made during the time, by Ptolomeu (the 2-nd century), by the representants of the Arabian Middle Ages mathematic school (between the 8-th and the 13-th centuries), or by Johannes Müler, who wrote in 1464, the first fundamental trigonometry book, entitled *Regiomontanus*. The aspects in the fascinate world of a triangle, were improved by Fr. Viète, in the wonderful time of Renaissance, and later, by strong mathematicians as L.Euler, J. Lagrange, A.Cagnoli (the 18-th and the 19-th centuries).

The Trigonometry is at the base of many Mathematic disciplines, as Geometry, Mechanics, Analysis etc.

The students of the high school level generally know very well, to use the trigonometric formulas proving identities and a good part of the inequalities which sometimes ask special technics. Surely, the university students, learning superior knowledge from Analysis and Function Theory, have the possibility to refine and to obtain sharp inequalities in Trigonometry! One of these, which I shall generalize in the next rows, is Ballieu inequality (see [1]) whose applications give us, a lot of elementary, but strong results (see [2]).

1. PRELIMINARIES

We shall use the classical notations in a triangle ABC, namely, *a, b, c* for sides and *A, B, C* for its angles. From the *folklore* of Mathematics, you know the following:

Proposition 1. Let ABC be a triangle. Then, the next sentences are true:

i) $\sin\frac{A}{2} \le \frac{a}{b+c}$;

ii) $2\sin^2\frac{A}{2} < \frac{a^2}{b^2+c^2}$, iff ABC is acute.

Proof. i) Using the side formula of $\sin(A/2)$, where $p = (a+b+c)/2$, we have:

$$\frac{(p-b)(p-c)}{bc} \le \frac{a^2}{(b+c)^2} <=>$$

$$\frac{a^2-(b-c)^2}{4bc} \le \frac{a^2}{(b+c)^2} <=>$$

$$a^2(b-c)^2 - (b-c)^2(b+c)^2 \le 0 <=>$$

$$(b-c)^2\left[a^2-(b+c)^2\right] \le 0 <=>$$

$a \le b+c$, obviously in a triangle.

ii) Because $2\sin^2(A/2) = 1-\cos A$, with the *cosinus theorem*, our inequality becomes:

$$1-\frac{b^2+c^2-a^2}{2bc} < \frac{a^2}{b^2+c^2} <=>$$

$$\left(a^2-b^2-c^2\right)\left[\frac{1}{2bc}-\frac{1}{b^2+c^2}\right] < 0 <=>$$

$$\left(a^2-b^2-c^2\right)\frac{(b-c)^2}{2bc\left(b^2+c^2\right)} < 0 <=>$$

$$a^2 < b^2+c^2,$$

from where, with the *sinus theorem* we find:

$$\sin^2 A < \sin^2 B + \sin^2 C,$$

or equivalently:

$$1-\cos 2A < 2-\cos 2B - \cos 2C.$$

Transforming the sums in products, it occurs:

$$2\cos^2 A > -2\cos A\cos(B-C) <=>$$

$$2\cos A\cos\frac{A+B-C}{2}\cos\frac{A-B+C}{2} > 0 <=>$$

$$\cos A\sin C\sin B > 0 <=>$$

$$\cos A > 0 <=> m(A) \in (0, \pi/2),$$

because in a triangle, $\sin B > 0$, $\sin C > 0$. ■

Remark 1. i) The relation from *ii)* is true with equality, in a rectangular triangle.

ii) Permuting the sides and the angles, for *i)*, we also obtain:

$$\sin\frac{B}{2}\le\frac{b}{a+c},\ \sin\frac{C}{2}\le\frac{c}{a+b},$$

and for *ii)*, we also have:

$$2\sin^2\frac{B}{2}<\frac{b^2}{a^2+c^2},\ 2\sin^2\frac{C}{2}<\frac{c^2}{a^2+c^2}.\blacksquare$$

From [2], I discovered the result of Ballieu, , in the:

Proposition 2. In a triangle ABC, for every $t\in(0,1]$, the following inequality:

$$(1)\ \ 2^{t-1}\sin^t\frac{A}{2}\le\frac{a^t}{b^t+c^t}$$

is true.■

Its proof can be studied in [1].

Ballieu inequality permit us to prove quickly the next result:

$$2^t(\cos A+\cos B+\cos C-1)^t\le\frac{8a^tb^tc^t}{(a^t+b^t)(a^t+c^t)(b^t+c^t)},\quad(\forall)t\in[0,1],$$

from where, for $t=1$, it follows the very known inequality:

$$\cos A+\cos B+\cos C\le 3/2\ .$$

With the last this, because:

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1+\frac{r}{R},$$

where r (respectively R) is the radius of the inscribed (respectively circumscribed) circle, in a triangle ABC, we obtain the famous Euler inequality:

$$2r\le R.$$

Its generalization, through the anterior upper start point, is:

$$\left(\frac{2r}{R}\right)^t\le\frac{8a^tb^tc^t}{(a^t+b^t)(a^t+c^t)(b^t+c^t)},\quad(\forall)t\in[0,1].$$

Surely, we immediately notice that (1) is also true for $t=0$. Adding now the values given by **Proposition 1**, we conclude that (1) is true for $t\in[0,1]\cup\{2\}$. The question, which appears from here, is: *are these the unique values which satisfy* (1) ?

2. THE MAIN RESULT

We shall generalize Ballieu relation in the next:

Lemma. In a triangle ABC, for every $t\in\mathbb{R}$, the inequality (1) is true.

Proof. α) Firstly, we prove that, (1) occurs for $t<0$. Computing $2^{t-1}\sin^t(A/2)$, we have:

$$2^{t-1}\sin^t\frac{A}{2}=2^{t-1}\sqrt{\left[\frac{(p-b)(p-c)}{bc}\right]^t}=\frac{1}{2}\sqrt{\left[\frac{(a+c-b)(a+b-c)}{bc}\right]^t}\le$$

$$\le\frac{1}{2}\sqrt{\left(\frac{2a\cdot 2a}{bc}\right)^t}=2^{t-1}\frac{a^t}{\sqrt{(bc)^t}},$$

keeping account that $|b-c|\le a$. So, we found:

(2) $2^{t-1}\sin^t\frac{A}{2}\le 2^{t-1}\frac{a^t}{\sqrt{(bc)^t}}$,

and if we want having (1), with (2), we must impose that:

$$2^{t-1}\frac{a^t}{\sqrt{(bc)^t}}\le\frac{a^t}{b^t+c^t} <=>$$

(3) $2^{t-1}\le\frac{\sqrt{(bc)^t}}{b^t+c^t}$.

From the geometric average for $b^t>0, c^t>0$, it holds:

(4) $\frac{\sqrt{(bc)^t}}{b^t+c^t}\le\frac{1}{2}$.

Therefor, if (3) is true, with (4) it occurs:

(5) $2^{t-1}\le\frac{1}{2} <=> t\le 0$.

β) Secondly, we shall study (1), for $t\ge 0$, using on $[0,\pi]$, the *concavity* of *sin*^{t}. So, we must study the geometrical form of the function:

(6) $f(x)=\sin^t x,\ x\in[0,\pi]$.

Immediately, we have:

$$f'(x)=t\sin^{t-1}x\cos x,$$

$$f''(x)=t\sin^{t-2}x\left[t\cos^2 x-1\right].$$

But (6) is concave, only if:

(7) $t\sin^{t-2}x\left[t\cos^2 x-1\right]\le 0.$

Surely $\sin^{t-2}x\ge 0,$ because $x\in[0,\pi]$. Therefore, (7) is true, only if:

$$\beta_1)\ \begin{cases} t\le 0 \\ t\cos^2 x-1\ge 0 \end{cases} \quad \text{or} \quad \beta_2)\ \begin{cases} t\ge 0 \\ t\cos^2 x-1\le 0 \end{cases}.$$

β₁) is excluded, because we supposed $t\ge 0$. For *β₂)*, denoting $g(u)=tu^2-1$, when $u=\cos^2 x\in[0,1]$, we must impose the conditions:

$$\begin{cases} t\ge 0 \\ \Delta_u>0 \\ g(-1)g(1)\ge 0, \end{cases}$$

from where, $t>0$.

Now, knowing that *sin*^{t} is a *concave function* for $t>0$, we can use *Jensen inequality* (see [3]), which gives us:

$$\sin^t\left(\frac{B+C}{2}\right)\ge\frac{\sin^t B+\sin^t C}{2} <=>$$

(8) $\cos^t\frac{A}{2}\ge\frac{\sin^t B+\sin^t C}{2}$.

With the *sinus theorem*, we replace $\sin B$ by $\frac{b}{a}\sin A$, and $\sin C$, by $\frac{c}{a}\sin A$. So, (8) becomes:

$$2\cos^t \frac{A}{2} \geq \frac{b^t + c^t}{a^t}\sin^t A <=>$$
$$1 \geq \frac{b^t + c^t}{a^t} \cdot 2^{t-1} \sin^t \frac{A}{2},$$

keeping account that $\cos^t (A/2) \geq 0$. Therefor, it holds (1). Cumulating now the cases *α)* and *β)*, we conclude that the inequality (1) is true, for every $t \in \mathrm{R}$. ■

Remark 2. i) Supposing that *β₁)* could realize it, when $t \leq 0$, we must impose:

$$\begin{cases} t \leq 0 \\ \Delta_u > 0 \\ g(-1)g(1) \geq 0 \end{cases} <=> t \in \{\Phi\}.$$

So, no $t \leq 0$ gives a concavity case for sin^tx, in the situation $x \in [0,\pi]$.

ii) Although $t \leq 0$ has excluded the concavity case of $sin^t x, (\forall) x \in [0,\pi]$, however, the inequality (1) holds ! This fact must be understood as one in which the function $sin^t x$ is a combination between convex and concave.

iii) Permuting the sides and the angles of the triangle, for (1), we also obtain two similar inequalities:

$$2^{t-1}\sin^t \frac{B}{2} \leq \frac{b^t}{a^t + c^t}, (\forall) t \in \mathrm{R},$$
$$2^{t-1}\sin^t \frac{C}{2} \leq \frac{c^t}{a^t + b^t}, (\forall) t \in \mathrm{R}. \blacksquare$$

We must note here at the end, that before to discover and to generalize Ballieu inequality, I published in [4], as problem, the following results, which I present to the readers, for solving:

$$2^t \sin^t \frac{A}{2} \leq \frac{a^t}{\sqrt{b^t c^t}}, (\forall) t \in \mathrm{R},$$
$$(9)\quad 2^t \sin^t \frac{B}{2} \leq \frac{b^t}{\sqrt{a^t c^t}}, (\forall) t \in \mathrm{R},$$
$$2^t \sin^t \frac{C}{2} \leq \frac{c^t}{\sqrt{a^t b^t}}, (\forall) t \in \mathrm{R}.$$

REFERENCES

[1] **Ballieu R.** *„Simon Stevin*", nr.26, 1949, pg. 129-34;

[2] **Bencze M.** *„Inequalities in Triangle"*, Octogon, Math. Magazine, v.7, nr.1, 1999, pg. 99-107;

[3] **Modan L**. *„Proposed Problem"* , Octogon, Math. Magazine, v.11, nr.2, 2003;

[4] **Rogai E**. *„Tables and Mathematical Formulas"* (in Romanian), Editura Tehnică, 1984.