

A nonlinear model of the unsteady heat transfer through an electron tube*

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Abstract

This paper provides a didactic experience concerning a second order nonlinear ordinary differential equation (ode) met in a heat transmission problem. The zero-dimensional temperature transient is studied in an electron bulb wall, radiated by a filament and twice convectively coupled. The lumped parameter approach, neglecting any inertia of the filling gas with respect to the wall, leads to:

$$\ddot{y} + \beta \dot{y}(1 + y^3) = 0, \quad \beta > 0,$$

with initial, assigned values to y and \dot{y} , where y is a unknown positive time-function, proportional to the wall temperature. The relevant Cauchy problem has been solved in closed form, after a previous discussion which reduces only to one the five possible analytical cases. Our solution has been compared with a numerical output showing a perfect superimposition. By our example the student should appreciate-among other things- the immense capabilities of the computer algebra packages of Mathematica® in solving the physical models.

KEYWORDS: Nonlinear initial value problem, heat conduction, heat radiation.

Riassunto

Questo articolo trasforma in esperienza didattica la simulazione di un fenomeno di trasmissione del calore che si esprime nella integrazione di una equazione differenziale non lineare. Si studia il transitorio termico zerodimensionale nello spessore di parete dell'ampolla di un tubo elettronico a gas, parete irradiata dal filamento, e due volte convettivamente accoppiata. Si è fatto uso dello schema modellistico a parametri concentrati, mentre da un punto di vista dinamico è stata trascurata l'inertza del gas di riempimento rispetto alla parete. L' equazione differenziale ordinaria non lineare a cui si perviene è:

$$\ddot{y} + \beta \dot{y}(1 + y^3) = 0, \quad \beta > 0,$$

con valori iniziali assegnati a y e \dot{y} , essendo y una incognita, e sempre positiva, funzione del tempo proporzionale alla temperatura di parete. Il corrispondente problema di Cauchy è stato risolto in forma chiusa, previa discussione che riduce a uno solo, avente significato fisico, i cinque possibili casi del calcolo. La soluzione che si presenta è stata confrontata con metodi numerici di riferimento, ottenendo una perfetta sovrapposizione. Da questo nostro esempio lo studente dovrebbe, fra l'altro, cogliere e apprezzare l'immenso aiuto dei programmi di computer algebra di Mathematica® nella integrazione e nel calcolo simbolico di modelli fisici.

PAROLE CHIAVE: Problema di Cauchy non lineare, conduzione del calore, irraggiamento termico.

1 Introduction

This paper introduces a didactic experience concerning the closed form integration of a second order nonlinear ordinary differential equation (ode) which we have been led to by a heat transmission problem.

Mathematica® will play an essential role for computing the effective solution: it allows to make the previous factorization of a fourth degree polynomial in order to split a certain rational integrand into a sum of fractions. By our example the student should appreciate the computer algebra immense capabilities. But the packages available in Mathematica® - as it will be clear throughout the lesson - are not sufficient for solving the problem, because no full automation will be possible, and the human mind use will be recommended anyway.

Our problem will concern the heat transfer in a quite general sense: conduction, convection, and radiation. Conduction is the process that conveys thermal energy (heat) through a solid, or in a fluid at rest. As long as a temperature difference is kept (by any possible way), heat will continue to flow from the high-temperature to the low-temperature regions.

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The mathematical analysis of the heat conduction is a remarkable milestone of the first years of the XIX century, when the French mathematician Jean B. Fourier (1768-1830) experienced and quantified the general principles of it. His first relevant article was submitted in 1807; successively he won the Academy *Grand Prize* of 1812 with a further paper on the same subject again, but which raised some criticism by Lagrange, Laplace and Legendre for its lack of rigor. Fourier’s notions can be seen in the context of the general power balance:

$$\frac{dE_{st}}{dt} = E_{in} - E_{out} + q_v,$$

and the capability to “receive or transmit heat across their surfaces” is represented by the E_{in} and E_{out} terms. The rate-of-energy-storage term $\frac{dE_{st}}{dt}$ for sensible energy storage can be represented through the system volume V , specific heat c_p and volumic density ρ :

$$\frac{dE_{st}}{dt} = \rho V c_p \frac{dT}{dt}.$$

The analytical theory of the conduction disregards the molecular structure of a body and considers it a continuum, even if much time after Fourier it has been understood that it occurs just due to particles collisions. In such a way, Fourier combined the energy balance on a differential control volume with the constitutive relationship, after him known as “Fourier’s Law” (linking the rate of a flow of heat to the thermal gradient) and established a partial differential equation (pde) for the temperature within a conductor, i.e.:

$$\rho c_p \frac{\partial T}{\partial t}(t, x) = \chi \frac{\partial^2 T}{\partial x^2}(t, x),$$

is the one-dimensional case, being χ the body’s thermal conductivity.

The transient problems ($\partial T/\partial t \neq 0$) do need *initial thermal* conditions for every position in the system. Mathematically, the heat equation is a parabolic pde insofar as time is considered. In such a case the heat propagation has an *infinite speed*. But some investigators ([6], p. 865) pointed out that the transient heat conduction must proceed with heat propagation *finite speed*: then the pde is of hyperbolic nature (*telegraph* type), see [1], and the excellent treatment [4].

The above unsteady purely conductive equation, can be enlarged to include an internal source $q_v = \text{const}$, or a given forcing function¹ of t :

$$\rho c_p \frac{\partial T}{\partial t}(t, x) = \chi \frac{\partial^2 T}{\partial x^2}(t, x) + q_v(t),$$

where each term has the dimension of a energy for unit time and volume.

Nevertheless, the higher are the temperatures the more one is required to take into account the radiation by, or to, the body. Its mechanism allows heat propagates (by electromagnetic waves) even in lack of medium. The power radiated by a T -body to a T_0 -environment, follows the Stefan-Boltzmann law of the fourth powers of both, the body and the medium temperatures. In such a way, we don’t have a *linear* conduction equation any longer, but a mixed conduction/radiation *nonlinear* pde:

$$\rho c_p \frac{\partial T}{\partial t}(t, x) = \chi \frac{\partial^2 T}{\partial x^2}(t, x) - k_r (T^4(t, x) - T_0^4) + q_v, \quad (1.1)$$

a much more difficult problem², which we are going to treat in a (unsteady) meaningful case, where the spatial dependence of T on x can be ignored.

2 A two-body convecting/radiating system

Let us now consider a case of transient heat flow in an electron tube (Figure 1). A filament is heated at temperature $T_f = \text{const}$. by an electric current. By convection, the heat transfer mechanism when a moving fluid carries the heat away, some heat will be freely convected to the surrounding gas, and from the gas to the bulb thin wall, which receives further heat directly from the filament by thermal radiation.

All these thermal inputs to the bulb wall, from which a certain amount of heat flows out to the atmosphere, will cause an unsteady thermal behavior to the wall itself, until some equilibrium is reached.

¹Such is the case of a radioactive source, whose thermal emission is obtained by the thermal yield of each transition multiplied for the “activity”, namely the decaying nuclei number for unit time and volume.

²Mind that $k_r > 0$ is a radiation coefficient. Anyhow the problem (1.1) has not to be confused with a conduction problem with radiative boundary condition. E. g. for an infinitely long R -cylinder the conduction pde collapses into a *linear* ode in $T(r)$: in such a case the radiation towards the surroundings will not involve $T(r)$, but only $T(R)$ with a *nonlinear* condition on it.

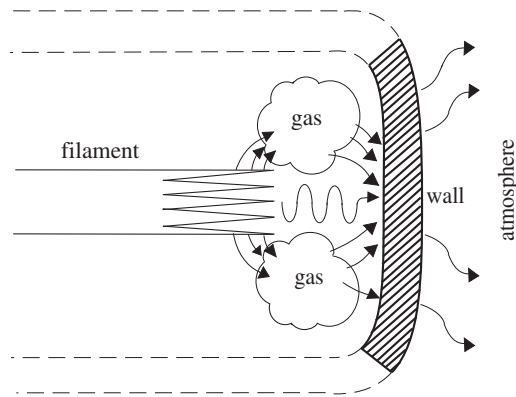


Figure 1: A sketch of the system

The atmosphere around the tube is modeled as a sink of infinite thermal capacity, namely capable of absorbing whichever amount of heat without changing its temperature, say T_a . Consequently, if T_f and T_a are fixed, we will model the 4-components system as a two-body one (gas and wall).

The tube is thought long enough with respect to its diameter, so that no effects will be taken into account as due to the distance. Thus we will single out both the gas and the wall temperature ($T_g(t)$ and $T_w(t)$) by means of one thermal point for each (*lumped parameters* approach). Such punctual temperatures will be assumed as state variables depending solely on time.

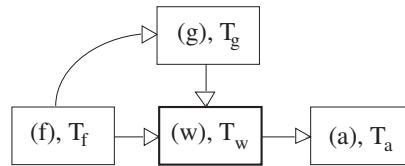


Figure 2: Block diagram of our 4-components system: filament (f), gas (g), wall (w), atmosphere (a), and their mutual thermal exchanges: (f, g): convection; (g, w): convection; (f, w): radiation; (w, a): convection.

The system equations are coming down from the thermal balance for the gas and the wall, with some adjustment of (1.1):

$$\begin{cases} C_g \dot{T}_g = k_{fg} (T_f - T_g) - k_{gw} (T_g - T_w), & (\text{gas}) \\ T_g(0) = g_0, \\ C_w \dot{T}_w = k_r (T_f^4 - T_w^4) + k_{gw} (T_g - T_w) - k_{wa} (T_w - T_a), & (\text{wall}) \\ T_w(0) = w_0. \end{cases} \quad (2.1)$$

The dots mean a derivative with respect to time, each term being a specific power and where all the quantities:

- C_g , thermal capacity of the gas,
- C_w , thermal capacity of the wall,
- k_{fg} , convection coefficient (filament \rightarrow gas),
- k_{gw} , convection coefficient (gas \rightarrow wall),
- k_r , radiation coefficient (filament \rightarrow wall),
- k_{wa} , convection coefficient (wall \rightarrow atmosphere),

are fixed and positive.

Focusing on the wall, the system (2.1) can be read as a nonlinear *first* order ode in T_w with only one coupling term wall/gas, i.e. $k_{gw}(T_g - T_w)$, with T_g ruled by the first ode of (2.1), which is linear, but holding T_w again. The system's two unknowns shall be decoupled: in order to do so, let us take the derivatives of both equations (2.1) with respect to the time:

$$\begin{cases} \frac{C_g}{k_{fg} + k_{gw}} \ddot{T}_g + \dot{T}_g = \frac{k_{gw}}{k_{fg} + k_{gw}} \dot{T}_w, \\ \ddot{T}_w + \frac{4k_r}{C_w} T_w^3 \dot{T}_w + \frac{k_{gw} + k_{wa}}{C_w} \dot{T}_w = \frac{k_{gw}}{C_w} \dot{T}_g. \end{cases} \quad (2.2)$$

Looking at the first equation of (2.2), we assume that the coefficient of \ddot{T}_g is negligible with respect to that of \dot{T}_g , namely:

$$\frac{C_g}{k_{fg} + k_{gw}} \ll 1. \quad (2.3)$$

Consequently the first equation of (2.2) can be approximated as:

$$\dot{T}_g \simeq \frac{k_{gw}}{k_{fg} + k_{gw}} \dot{T}_w,$$

which is immediately integrated:

$$T_g(t) \simeq \frac{k_{gw}}{k_{fg} + k_{gw}} T_w(t) + g_0.$$

On the smallness of the ratio (2.3), beyond the effectively low gas thermal capacity C_g , some influence is exerted by the not low values of k_{fg} and k_{gw} . The inequality (2.3) declares that the filling gas has a thermal capacity so small that its temperature follows *promptly* the same dynamics of the wall which it laps on, without any phase displacement or delay: $\dot{T}_g \propto \dot{T}_w$. Then our ode to be solved is:

$$\ddot{T}_w + \frac{4k_r}{C_w} T_w^3 \dot{T}_w + \left(\frac{k_{gw} + k_{wa}}{C_w} \right) \dot{T}_w = \frac{k_{gw}}{C_w} \left(\frac{k_{gw}}{k_{fg} + k_{gw}} \right) \dot{T}_w. \quad (2.4)$$

The equation (2.4) is of the *second* order: nevertheless we do not need any supplementary initial condition for it: see the third equation of (2.1). Notice that equation (2.4) does not hold any longer a direct and explicit reference to the thermal sinks at levels T_f and T_a . Putting:

$$\frac{4k_r}{C_w} = \alpha > 0, \quad \frac{1}{C_w} \left(k_{gw} + k_{wa} - \frac{k_{gw}}{k_{fg} + k_{gw}} \right) = \beta, \quad (2.5)$$

and passing from T_w to $y > 0$:

$$T_w = y \left(\frac{\beta}{\alpha} \right)^{1/3},$$

we are faced with the 2nd order, nonlinear ode:

$$\begin{cases} \ddot{y} + \beta \dot{y}(1 + y^3) = 0, \\ y(0) = w_0 (\alpha/\beta)^{1/3} := y_0, \\ \dot{y}(0) = v_0. \end{cases} \quad (2.6)$$

Equation (2.6) has a physical meaning if the initial conditions will be capable to induce a dynamics with $y(t)$ always greater than zero. The next explicit integration will single out such conditions. Notice that the β -expression in (2.5) can be written as:

$$\beta = \frac{1}{C_w} \left[k_{gw} (1 + a) - \frac{1}{b + 1} \right],$$

where:

$$a = \frac{k_{wa}}{k_{gw}} > 0, \quad b = \frac{k_{fg}}{k_{gw}} > 0,$$

and then, for $k_{gw} > 1$, surely there will be $\beta > 0$ for whichever triple of values k_{gw}, k_{wa}, k_{fg} .

3 The wall temperature ode: discussion and integration

3.1 Stability analysis

Equation (2.6), not included in the handbooks, is a second order one, and due to its initial conditions, leads to a Cauchy initial value problem like those of particle Mechanics. We first deal with the stability analysis of (2.6), by writing the second order equation as a two dimensional nonlinear system:

$$\begin{cases} \dot{y} = x, \\ \dot{x} = -\beta x (1 + y^3). \end{cases}$$

In such a way we will see that (2.6) is a unstable equation. In fact all the points of the form $(x, y) = (0, y)$ are equilibria, where the eigenvalues of the variational matrix:

$$V = \begin{bmatrix} 1 & 0 \\ -\beta(1 + y^3) & 0 \end{bmatrix},$$

are $\lambda = 0$ and $\lambda = 1$. A zero eigenvalue implies that the Lyapunov theory cannot be applied and has to be replaced by the centre manifold theory, [2], but of course this theory is not known at any undergraduate level. Nevertheless, the explicit integration we are going to perform, will entitle us, though in presence of the zero eigenvalue, to predict when the solution of (2.6) will exhibit a stable behavior, without applying the centre manifold theory.

3.2 The time equation

Putting $\dot{y} = p(y)$, and $\beta = 4\beta_1$, the differential equation (2.6) becomes:

$$\begin{cases} p(y) [p'(y) + 4\beta_1 (1 + y^3)] = 0, \\ p(y_0) = v_0. \end{cases}$$

Notice that $p(y)$ cannot be the null function, if we seek a non-steady solution of (2.6). Then, a first integration gives:

$$\dot{y} = K_0 - 4\beta_1 y - \beta_1 y^4, \quad (3.1)$$

where $K_0 = v_0 + 4\beta_1 y_0 + \beta_1 y_0^4$ and we have used $\dot{y} = p(y)$. Really K_0 could be nonpositive only if $v_0 < 0$, namely if the wall started its transient with a cooling, what would be a nonsense. Then we will assume $K_0 > 0$. The 4th degree y -polynomial:

$$f(y) = K_0 - 4\beta_1 y - \beta_1 y^4.$$

cannot any way have four real roots, because $f'(y)$ has one real root only. Now integrating (3.1) we find the time equation:

$$t = t(y) = \int_{y_0}^y \frac{du}{f(u)} = \int_{y_0}^y \frac{du}{K_0 - 4\beta_1 u - \beta_1 u^4}. \quad (3.2)$$

The same result could have been achieved writing the equivalent system (3.1) in the phase plane:

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = -\frac{1}{4\beta_1(1 + y^3)},$$

and integrating by separation of variables.

3.3 Stability analysis again

Minding that β_1 is positive in our framework, we infer that the quartic polynomial $f(y)$, which cannot have four real roots, may have:

1. two real distinct roots and two complex coniugate roots, if $K_0 + 3\beta_1 > 0$. Moreover:
 - (a) if $K_0 > 0$ one real root is positive (say r_+), and the other real root is negative (say r_-);

- (b) if $K_0 = 0$ one real root vanishes and the other is negative;
 - (c) if $K_0 < 0$ both the real roots are negative;
2. one real double root and two complex conjugate roots, if $K_0 + 3\beta_1 = 0$;
 3. no real root if $K_0 + 3\beta_1 < 0$.

Of five situations above listed for analytic completeness, the first only has a physical meaning.

As a matter of fact, the cases 1.-(b) and 1.-(c). have to be left out because $f(y)$ would be negative, for $y > 0$. Such a negativity would imply, through the (3.2), a negative (absurd) evolution of the time during the wall heating. Alternatively, we remind the K_0 positivity, for a prompt exclusion of cases 1.-(b) e 1.-(c).

The cases 2. and 3. shall be once again rejected due to the integrand negativness for $y > 0$.

Alternatively, we could remember again the K_0 positivity; adding the β_1 positivity too, one is led to leave out cases 2. and 3.

Having got rid of the analytic situations lacking of physical content, we go finally to analyze the only case where, despite of the +1 eigenvalue, the system shows a stable behavior if we set up suitable (physically consistent) initial conditions. As a matter of fact, with an initial condition $y_0 > 0$, the assumption 1.-(a) rules that it shall be $y_0 < r_+$. For increasing y , the system can reach r_+ , but-see (3.2)- after an infinity of time. Then the wall will get an asymptotic temperature value (see Fig. 3), given by $T_\infty = r_+ (\beta/\alpha)^{1/3}$.

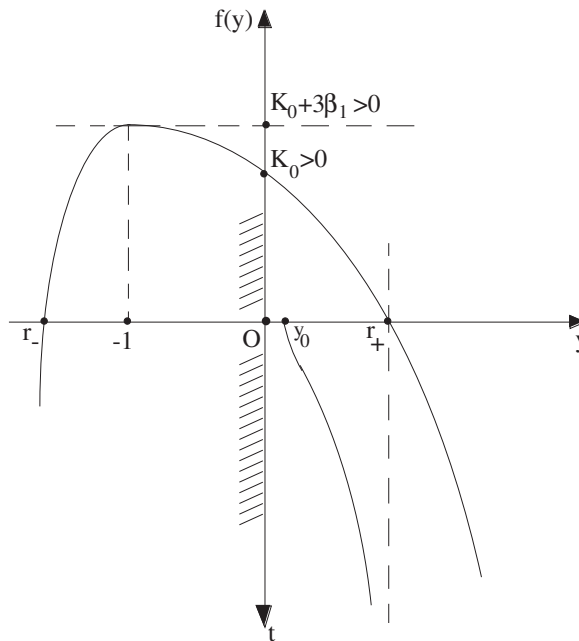


Figure 3: A qualitative sketch of the lines $(f(y), y)$ and $(t(y), y)$ and their relationship with the y -asymptotic behavior at r_+ .

In such a way the unsteady solution, with $K_0 + 3\beta_1 > 0$ and $K_0 > 0$, exhibits a stable and physically acceptable behavior.

3.4 Integration

Accordingly, we will develop the case 1.-(a) only. Then we can assume that $f(y)$ splits in:

$$f(y) = -\beta_1 (y - r_1) (y - r_2) (y^2 + py + q),$$

where p and q ($p^2 < 4q$) come from the $f(y)$ factorization. We will see later how to compute such constants: now it is enough to know they can be computed, i. e. by Mathematica®, solving the fourth degree equation $f(y) = K_0 - 4\beta_1 y -$

$\beta_1 y^4 = 0$. Next, one shall proceed to find A, B, C, D in the partial fractions expansion:

$$\frac{1}{f(y)} = \frac{A}{y - r_1} + \frac{B}{y - r_2} + \frac{Cy + D}{y^2 + py + q}. \quad (3.3)$$

Mathematica® has a specific command: **Apart**, but it is uneffective in our case, giving the output, shown, as well as all the following computer statements, in Table 1.

$$\begin{aligned} \text{In}[1] := & \text{Apart}\left[\frac{1}{(y - r_1)(y - r_2)(y^2 + py + q)}\right] \\ \text{Out}[1] = & -\frac{1}{(q + py + y^2)(y - r_1)(-y + r_2)} \end{aligned}$$

Table 1

Then we shall provide with suitable instructions Mathematica® for obtaining the partial fractions expansion, shown in Table 2.

$$\begin{aligned} \text{In}[2] := & \text{Together}\left[\frac{A}{(y - r_1)} + \frac{B}{(y - r_2)} + \frac{Cy + D}{y^2 + py + q}\right] \\ \text{Out}[2] = & \frac{(Aqy + Bqy + Dy^2 + ApY^2 + BpY^2 + Ay^3 + By^3 + Cy^3 - Bqr_1 - Dy r_1 - BpY r_1 - By^2 r_1 - Cy^2 r_1 - Aqr_2 - Dy r_2 - ApY r_2 - Ay^2 r_2 - Cy^2 r_2 + D r_1 r_2 + Cy r_1 r_2) / ((q + pY + Y^2)(y - r_1)(y - r_2))}{(q + pY + Y^2)(y - r_1)(y - r_2)} \end{aligned}$$

Table 2

By the command **Together** one computes the sum of the algebraic fractions whilst the **Numerator** one allows to take apart the numerator, see Table 3.

$$\begin{aligned} \text{In}[3] := & \text{num} = \text{Numerator}[\%] \\ \text{Out}[3] = & Aqy + Bqy + Dy^2 + ApY^2 + BpY^2 + Ay^3 + By^3 + Cy^3 - Bqr_1 - Dy r_1 - BpY r_1 - By^2 r_1 - Cy^2 r_1 - Aqr_2 - Dy r_2 - ApY r_2 - Ay^2 r_2 - Cy^2 r_2 + D r_1 r_2 + Cy r_1 r_2 \end{aligned}$$

Table 3

By defining as **num** the output, such a polynomial is stored. The four constants A, B, C, D will allow the partial fractions expansion if the polynomial at numerator is identically equal to 1. By the identity principle, we have to take apart each coefficient of the fourth degree polyomial **num** setting that its constant term equals 1, and each of the remaining four equals zero. The key **Coefficient** takes apart each of the polynomial coefficients while its constant term is given following the command shown in Table .

$$\begin{aligned} \text{In}[4] := & \text{eq3} = \text{Coefficient}[\text{num}, y^3] \\ & \text{eq2} = \text{Coefficient}[\text{num}, y^2] \\ & \text{eq1} = \text{Coefficient}[\text{num}, y] \\ \text{Out}[4] = & A + B + C \\ \text{Out}[5] = & D + Ap + Bp - Br_1 - Cr_1 - Ar_2 - Cr_2 \\ \text{Out}[6] = & Aq + Bq - Dr_1 - Bp r_1 - Dr_2 - Ap r_2 + Cr_1 r_2 \\ \text{In}[7] := & \text{eq0} = \text{num} /. y \rightarrow 0 \\ \text{Out}[7] = & -Bqr_1 - Aqr_2 + D r_1 r_2 \end{aligned}$$

Table 4

The partial fractions expansion then will follow solving the linear system in the unknowns A, B, C, D :

$$\begin{cases} \text{eq3} = 0 \\ \text{eq2} = 0 \\ \text{eq1} = 0 \\ \text{eq0} = 1 \end{cases} \Leftrightarrow \begin{cases} A + B + C = 0 \\ (p - r_2)A + (p - r_1)B - (r_1 + r_2)C + D = 0 \\ (q - pr_2)A + (q - pr_1)B + r_1r_2C - (r_1 + r_2)D \\ -qr_2A - qr_1B + r_1r_2D = 1 \end{cases}$$

It can be accomplished by the key **Solve** plus **Simplify** which has an obvious effect. It is necessary to define to Mathematica® those variables which we wish to solve to, see Table 5.

`In[8]:= Simplify[Solve[{eq3 == 0, eq2 == 0, eq1 == 0, eq0 == 1}, {A, B, C, D}]]`

$$\text{Out[8]} = \left\{ \left\{ C \rightarrow \frac{p + r_1 + r_2}{(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)}, A \rightarrow \frac{1}{(q + p r_1 + r_1^2)(r_1 - r_2)}, \right. \right. \\ \left. \left. B \rightarrow -\frac{1}{(r_1 - r_2)(q + p r_2 + r_2^2)}, D \rightarrow \frac{p^2 - q + p r_2 + r_1(p + r_2)}{(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)} \right\} \right\}$$

Table 5

The partial fractions expansion (3.3) then will be found through the key of Table 6.

$$\text{In[9]} := \frac{A}{(y - r_1)} + \frac{B}{(y - r_2)} + \frac{Cy + D}{y^2 + py + q} /. \left\{ C \rightarrow \frac{p + r_1 + r_2}{(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)}, \right. \\ \left. A \rightarrow \frac{1}{(q + p r_1 + r_1^2)(r_1 - r_2)}, B \rightarrow -\frac{1}{(r_1 - r_2)(q + p r_2 + r_2^2)}, D \rightarrow \frac{p^2 - q + p r_2 + r_1(p + r_2)}{(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)} \right\}$$

$$\text{Out[9]} = \frac{1}{(y - r_1)(q + p r_1 + r_1^2)(r_1 - r_2)} - \\ \frac{1}{(y - r_2)(r_1 - r_2)(q + p r_2 + r_2^2)} + \frac{\frac{y(p + r_1 + r_2)}{(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)} + \frac{p^2 - q + p r_2 + r_1(p + r_2)}{(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)}}{q + p y + y^2}$$

Table 6

Now the integral (3.2) is soon computed in Table 7 (by the symbol **%** Mathematica® is told to execute the instruction on the last output).

$$\text{In[10]} := \int \% \, dy$$

$$\text{Out[10]} = \frac{\text{Log}[y - r_1]}{(q + p r_1 + r_1^2)(r_1 - r_2)} - \frac{\text{Log}[y - r_2]}{(r_1 - r_2)(q + p r_2 + r_2^2)} + \\ \frac{\text{Log}[q + y(p + y)](p + r_1 + r_2)}{2(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)} + \frac{\text{ArcTan}\left[\frac{p + 2y}{\sqrt{-p^2 + 4q}}\right](p^2 - 2q + p r_2 + r_1(p + 2r_2))}{\sqrt{-p^2 + 4q}(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)}$$

Table 7

Asking simply for the integral (3.2) the Mathematica® compiler would provide the right solution at once (Table 8). But in

$$\text{In[11]} := \int \frac{1}{(y - r_1)(y - r_2)(y^2 + py + q)} \, dy \\ \text{Out[11]} = \frac{\text{Log}[y - r_1]}{(q + p r_1 + r_1^2)(r_1 - r_2)} - \frac{\text{Log}[y - r_2]}{(r_1 - r_2)(q + p r_2 + r_2^2)} + \\ \frac{\text{Log}[q + y(p + y)](p + r_1 + r_2)}{2(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)} + \frac{\text{ArcTan}\left[\frac{p + 2y}{\sqrt{-p^2 + 4q}}\right](p^2 - 2q + p r_2 + r_1(p + 2r_2))}{\sqrt{-p^2 + 4q}(q + p r_1 + r_1^2)(q + p r_2 + r_2^2)}$$

Table 8

such a way Mathematica® would operate as a handbook, and all the computation would be lost out of the view. Therefore, minding the didactic completeness of all our treatment, we preferred to face the Mathematica® single steps for arriving at the computational details.

The inverse of the solution of (2.6) has been found to be:

$$t = -\frac{1}{\beta_1} (\Phi(y) - \Phi(y_0)), \quad (3.4)$$

with:

$$\begin{aligned} \Phi(y) = & \frac{1}{r_1 - r_2} \left[\frac{\ln(y - r_1)}{(r_1^2 + p r_1 + q)} - \frac{\ln(r_2 - y)}{(r_2^2 + p r_2 + q)} \right] + \\ & + \frac{(p + r_1 + r_2) \ln(y^2 + p y + q)}{2(r_1^2 + p r_1 + q)(r_2^2 + p r_2 + q)} + \\ & + \frac{[p^2 + (r_1 + r_2)p + 2(r_1 r_2 - q)] \arctan \frac{p + 2y}{\sqrt{4q - p^2}}}{(r_1^2 + p r_1 + q)(r_2^2 + p r_2 + q)\sqrt{4q - p^2}}. \end{aligned} \quad (3.5)$$

Notice that $\Phi(y)$ depends on the quantities r_1, r_2, p, q , which depend only on the initial conditions y_0, v_0 and on the system characteristic β_1 , but not on α , which holds the radiation effects. For getting the thermal problem full solution, the roots' search of the equation $f(y) = 0$, till now postponed, shall be faced.

We could try, by means of Mathematica[®], to get a symbolic solution, being available the algebraic L. Ferrari (1522-1565) formula. But the relevant Mathematica[®] output is of poor effectiveness even if the quartic is in reduced form, lacking of the 3rd and 2nd degree terms. As a matter of fact, positing $X = -K_0\beta_1^{-1}$ in Table we have a look of only one of the solutions of:

$$X - 4y - y^4 = 0.$$

`In[12]:= Solve[X - 4 y - y^4 == 0, y] [[1]] // Simplify`

$$\begin{aligned} \text{Out [12]= } \{ & y \rightarrow \frac{\sqrt{\left(27 - 3\sqrt{3}\sqrt{27 + X^3}\right)^{1/3} - \frac{3^{2/3} X}{(9 - \sqrt{3}\sqrt{27 + X^3})^{1/3}}}}{\sqrt{6}} - \\ & \frac{1}{2} \sqrt{\left(\frac{2 X}{\left(27 - 3\sqrt{3}\sqrt{27 + X^3}\right)^{1/3}} - \frac{2(9 - \sqrt{3}\sqrt{27 + X^3})^{1/3}}{3^{2/3}} - \right.} \\ & \left. \frac{4\sqrt{6}}{\sqrt{\left(27 - 3\sqrt{3}\sqrt{27 + X^3}\right)^{1/3} - \frac{3^{2/3} X}{(9 - \sqrt{3}\sqrt{27 + X^3})^{1/3}}}} \right)} \} \end{aligned}$$

Table 9

For using the Ferrari's symbolic solution, one should tackle the impact of variations of X (which is carrying all the problem data) on the reality and values of each of the roots, and anyhow to plug the complicated formulas for r_1, r_2, p, q in (3.5).

We deemed much more meaningful to fix a sample case and solve $f(y) = 0$ numerically in order to compute the constants r_1, r_2, p, q to be plugged in (3.5)

4 A sample problem

Then let us go to make a discussion deliberately not founded upon technical meaningful and credible values as the filament temperature, or the k 's convective coefficients again and again introduced. By means of the user-friendly structure of Mathematica[®], each aware user will be promptly allowed to compute solution to his own problem.

Let our values be: $\beta_1 = 1, y_0 = 1, v_0 = 5$, then $K_0 = 10$. We factorize the polynomial of 4th degree $f(y) = 10 - 4y - y^4$ using Mathematica[®]. First we identify its roots using the **NSolve** command, see Table 10, obtaining: $r_1 = -2.06744, r_2 = 1.43624, r_3 = 0.315602 - 1.8078 i, r_4 = 0.315602 - 1.8078 i$.

```
In[13]:= NSolve[10 - 4 y - y^4 == 0]
```

```
Out[13]= {{y -> -2.06744}, {y -> 0.315602 - 1.8078 i}, {y -> 0.315602 + 1.8078 i}, {y -> 1.43624}}
```

Table 10

In such a way, the coefficients p and q arising in the factorization formula (3.3) are computed by the command shown in Table 11, so that $p = -0.631204$ and $q = 3.36775$.

```
In[14]:= Expand[(y - (0.315602 - 1.8078 I)) (y - (0.315602 + 1.8078 I))]
```

```
Out[14]= (3.36775 + 9.74274 x 10^-18 i) - (0.631204 + 0. i) y + y^2
```

```
In[15]:= Chop[%]
```

```
Out[15]= 3.36775 - 0.631204 y + y^2
```

Table 11

Notice that **Chop** eliminates all the negligible terms coming from numerical approximation. The $t = f(y)$ computation will be done by means of (3.4). The relevant plot of y as a function of time, is obtained first defining the function Φ (Table 12), and then by inversion giving the **ParametricPlot** instruction, Table 13, with reversed coordinates.

```
In[16]:=  $\Phi[y_] := \frac{\text{Log}[\text{Abs}[y - r_1]]}{(q + p r_1 + r_1^2) (r_1 - r_2)} - \frac{\text{Log}[\text{Abs}[y - r_2]]}{(r_1 - r_2) (q + p r_2 + r_2^2)} +$   

 $\frac{\text{Log}[q + y (p + y)] (p + r_1 + r_2)}{2 (q + p r_1 + r_1^2) (q + p r_2 + r_2^2)} + \frac{\text{ArcTan}\left[\frac{p + 2y}{\sqrt{-p^2 + 4q}}\right] (p^2 - 2q + p r_2 + r_1 (p + 2r_2))}{\sqrt{-p^2 + 4q} (q + p r_1 + r_1^2) (q + p r_2 + r_2^2)};$   

sol =  $\Phi[y] - \Phi[1] /. \{r_1 -> -2.06744, r_2 -> 1.43624, p -> -0.631204, q -> 3.36775\};$ 
```

Table 12

```
In[18]:= solplo =
```

```
ParametricPlot[{-sol, y}, {y, 1, 1.436}, PlotRange -> {-0.2, 1.7}, AxesOrigin -> {0, 0}];
```

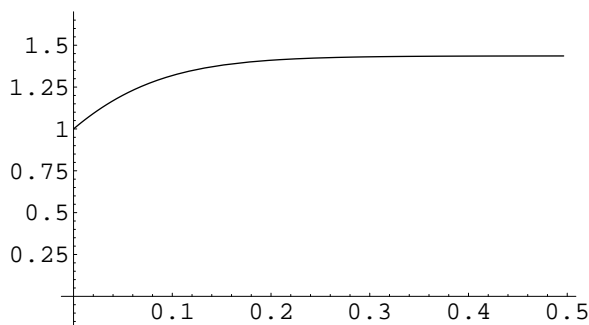


Table 13

Our exact solution can be successfully tested by matching with efficient finite difference techniques: we use for instance the **VisualDSolve** package given in [7]. This package is loaded with the command `<< VisualDSolve.m`, notice that after the package is loaded the number of input changes. So we can plot, in Table 14 the numerical output and check how it is perfectly superimposed with our exact reverse solution using the command **Show** (Table 15).

In[696]:=

```
solwag = SecondOrderPlot[y''[t] == -4 y'[t] (1 + y[t]^3),
    y[t], {t, 0, .9}, InitialValues -> {1, 5}, MaxSteps -> 30000,
    PlotVariables -> {{y}}, AxesOrigin -> {0, 0}, PlotRange -> {-1, 2.2}];
```

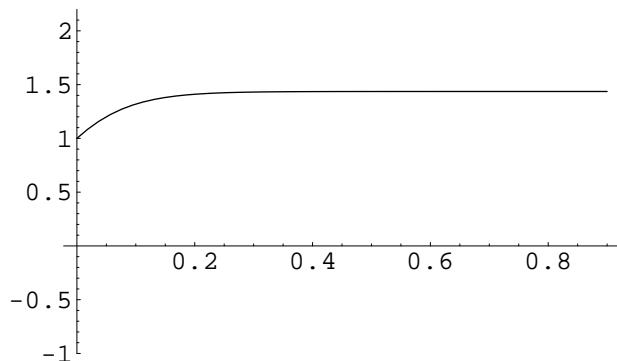
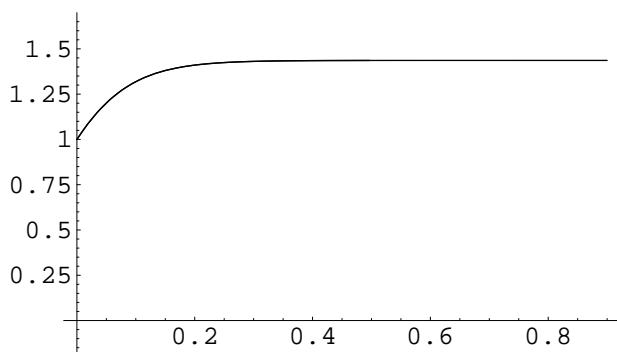


Table 14

In[697]:=

```
Show[solplo, solwag]
```



Out[697]=

```
- Graphics -
```

Table 15

Even if we show there the relevant plot of y versus the time, it should be clear that nobody can invert *formally* the function $t = t(y)$: but by virtue of our explicit formulae for $t = t(y)$, we did an easy tabulation of $y = y(t)$.

5 Conclusions

The authors hope to have succeeded in introducing a learning public to carry out a model till to produce somewhat mathematically treatable but meaningful. In fact we didn't deem useful to add details to the model causing the relevant mathematical problem cannot be exactly solved, needing a numerical treatment. In such a way, we met an analytical (new) problem consisting of a nonlinear ordinary differential equation whose integration we performed in closed form in all its steps, helped by the computer algebra. A full discussion with the y -quartic would have been unacceptably long, and for this only we approached numerically an algebraic sub-problem inside the major problem of solving exactly an unsteady heat transmission. A physical discussion completed our phenomenological picture. Through the final sample problem our closed form solution has been compared with a numerical output showing a perfect indistinguishableness.

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References

- [1] Baumeister K. J., Hamill T. D.: *Hyperbolic heat-conduction equation-A solution for the seminfinite body problem*, Journal of heat transfer, 1969, p. 543-548.
- [2] Carr J. *Application of Centre manifold Theory*, New York, 1981.
- [3] Carslaw H. S., Jaeger J. C.: *Conduction of heat in solids*, Oxford, 1959.
- [4] Cattaneo, C.: Sulla conduzione del calore, *Atti del seminario matematico e fisico dell'Università di Modena*, **III**, (1948-49), 83-101.
- [5] Jain-Krishnan, Proc. Roy. Soc.: **A 225** (1954), 1-32.
- [6] Morse, P. M., Feshbach H.: *Methods of theoretical Physics*, New York, 1953.
- [7] Schwalbe D., Wagon S.: *VisualDSolve. Visualizing Differential Equations with Mathematica*, New York, 1997.