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The Pohlke-Schwarz Theorem and its Relevancy in the Didactics of Mathematics[1]

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Summary. As soon as the first volume of "Descriptive Geometry" by Karl Pohlke (1810 – 1876) has appeared (Berlin, 1860), inclusive the fundamental theorem of oblique axonometry with a note that "*the proof of the theorem probably could not be accomplished in an elementary way and that was why it was taken off for the second volume of the book*", both synthetic and analytic proofs has been made by many mathematicians within the time period longer than half a century. The paper presents – besides the history of proofs of the Pohlke's theorem – the genial elementary proof of the generalized statement introduced by a young pupil of Pohlke, *H. A. Schwarz* (1843 – 1921) in 1864. The main goal of this paper is to point out the close connection between the method of oblique axonometry and a "free" parallel projection used in school practice within the intuition in *stereometry*. In conclusion there are notes on the problem of the completeness of the oblique image of a geometrical figure (considering the *problems of geometry of position* as well as *problems involving perpendicularity and metrical problems*).

Riassunto. Il teorema fondamentale dell'Assonometria obliqua é stato pubblicato da Karl Pohlke (1830 – 1876) nel primo volume del suo libro "Geometria descrittiva" (Berlin, 1860) senza la dimostrazione con un avviso che "*La dimostrazione di questo teorema sembra non poter essere fatta elementarmente perciò si deve riservarla per il secondo volume*". Subito dopo – nel corso del tempo, per più di mezzo secolo – tanti matematici (persino i più grandi) eseguivano le dimostrazioni del teorema, quelle sintetiche e anche analitiche. L'articolo offre – oltre la storia delle dimostrazioni del teorema di Pohlke – una dimostrazione semplice e luminosa – presentata da *H. A. Schwarz* (1843 – 1921), un giovane studente di Pohlke nel 1864. Lo scopo principale dell'articolo è mettere in evidenza la relazione stretta fra il metodo di rappresentazione di Assonometria obliqua e quello di proiezione parallela "*libera*" utilizzata nella pratica scolastica nel corso dell'insegnamento /apprendimento della *Stereometria* (Geometria elementare dello $\boldsymbol{E}_3$-spazio). In conclusione nel testo ci sono le note riguardante il problema della completezza dell'immagine obliqua di una figura geometrica (riguardo ai *problemi di natura posizionale* e/o ai *problemi di natura metrica* concludendo i *problemi della perpendicolarità*).

Abstrakt. Sotva vyšiel prvý diel učebnice „Deskriptívna geometria" (Berlín, 1860) Karla Pohlkeho (1830 – 1876), v ktorom autor vyslovil základnú vetu šikmej axonometrie s poznámkou, že *pravdepodobne elementárny dôkaz vety neexistuje, a preto sa odkladá do druhého dielu učebnice*, v priebehu vyše polstoročia sa objavilo mnoho dôkazov tejto vety (syntetických i analytických) – aj od najuznávanejších súdobých matematikov. Okrem zaujímavej histórie dôkazov Pohlkeho tvrdenia sa v článku čitateľ môže zoznámiť s geniálne jednoduchým dôkazom *H. A. Schwarza* (1843 – 1921), Pohlkeho žiaka. Hlavným cieľom článku je poukázať na tesný súvis zobrazovacej metódy šikmej axonometrie a tzv. *voľného rovnobežného premietania* používaného v školskej praxi vo vyučovaní *stereometrie*. Záver je venovaný poznámkam súvisiacich s úplnosťou obrazu geometrického útvaru vo voľnom rovnobežnom premietaní (*vzhľadom na riešenie polohových* a/alebo *metrických úloh* vrátane *problémov o kolmosti základných geometrických útvarov*).

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1 From the History of the Demonstration of Pohlke's Theorem

The fundamental theorem of oblique axonometry

"*Any three non collinear segments in a plane with the same endpoint can be considered as the oblique parallel projection of a tripod.*[2]"

Karl Pohlke (1810 Berlin – 1876 Berlin), professor of descriptive geometry at the Institute of technology in Berlin-Charlottenburg, has formulated the fundamental theorem of oblique axonometry in 1853. He was at that time a teacher of the local Academy of civil engineering. Because of some vagueness in the first version of the statement and the fact that the considered projection was assumed normal in most cases, the doubts of its verity have arisen. A letter to Pohlke from Jacob Steiner[3] gives evidence of it. It has stimulated Pohlke to study more in detail the conditions in order for a quadruple of points to have been an oblique projection of vertices of the tetrahedron, the four edges of which constitute a tripod. The results of his research were published in 1858 (*Miscellaneous Theorems and Problems*).[4]

The tuition in descriptive geometry at German secondary schools, Academies and Institutes of technology required textbooks on this subject. Pohlke started the work on one of the first textbooks at the very beginning of his career. His textbook *Descriptive Geometry* (Darstellende Geometrie) in two volumes has gained mark of modernity owing to its contents as well as the consistent application of the scientific procedures established by Steiner in his work *Systematic Treatise* (Systematische Entwickelung). That has brought him a long-range success.[5] Pohlke introduced the fundamental theorem of oblique axonometry in the first volume of his textbook (Berlin 1860, par. 113) with a notice that an elementary proof of the theorem probably did not exist and therefore it will be delayed until the second volume appears. The first edition of the first volume was followed by two another ones in 1866 and 1872, the second one was published only once (1876). This "challenge" of Pohlke has been briefly answered at least by three elementary demonstrations of the *Pohlke's theorem*.[6] They have been – in the chronological order – the demonstration of J. W. Deschwanden (incomplete proof), of H. Kinkelin[7] and of the young H. Schwarz, hardly twenty years old pupil of Pohlke, in that time without the doctor degree yet. The Schwarz's demonstration was so genially simple that Pohlke published it with Schwarz's permission in the second edition of the first volume of *Descriptive Geometry*. The Pohlke's own demonstration has never been published and remained for more than a decade unknown.

H. Schwarz[8] and also T. Reye[9] have proved the following generalized statement of the theorem: "*Any three complanar segments* $O'X', O'Y', O'Z'$ *such that at most three points of* O',

[2] *tripod*: three segments originated from a single point and perpendicular one to another
[3] *Jacob Steiner* (1796 – 1863), professor of geometry at the University of Berlin, one of the greatest German geometers of the 19th century; close friend of Pohlke
[4] Vermischte Sätze und Aufgaben (J. r. ang. Math., LV, 1858, 377)
[5] In the contents is not missing the introduction of principles of any linear method of representation including the relief perspective, applications to elementary surfaces and solids, plane sections of solids, intersections of solids, plane/space curves, surfaces of revolution, ruled surfaces, helicoids, etc.
[6] The fundamental theorem of oblique axonometry was named Pohlke's theorem by mathematicians and descriptive geometers who have attempted to solve this problem.
[7] *J. W. Deschwanden*, the professor of descriptive geometry and the director of Institute of technology in Zürich, *Application of oblique parallel Projection in axonometric Representation* (Anwendung schiefer Paralell projectionen zu axonometrischen Zeichnungen, Natur. Ges. Zürich, VI, 1861, 254 – 284, VII); *H. Kinkelin*, professor of the Academy of civil engineering in Basil, *Oblique axonometric Projection* (Die schiefe axonometrische Projection, Natur. Ges. Zürich, VI, 1861, 358 – 369) – the analytic solution
[8] *Elementary Demonstration of Pohlke's fundamental Theorem of Axonometry* (Elementarer Beweis des Pohlke'schen Fundamentalsatzes der Axonometrie, J. r. ang. Math., LXIII, 1863, 309 – 314); *Hermann Schwarz*

X', Y', Z' are collinear, can be considered as a parallel projection of three non complanar segments OX, OY, OZ with the prescribed ratios and mutual angles." Within five decades were published numerous another demonstrations of the Pohlke-Schwarz's theorem, both synthetic and analytic ones. We present some of them with the names of their authors.

The demonstration of Karel Pelz[10], an outstanding mathematician in the field both synthetic and constructive geometry, was one of the first ones (About new Demonstration of the Pohlke's fundamental Theorem, 1877). As soon as the publication has appeared, Schwarz recognized in it a proof identical to an original proof of Pohlke.[11] G. V. Peschka[12] has succeeded in completion of the Deschwanden's proof; consequently the Peschka's demonstration has been considered as the first elementary demonstration of the Pohlke's Theorem in Austria-Hungary.

An elegant analytic proof of the statement in question has supplied Arthur Cayley[13], an English algebraist, geometer and analyst. Also the top Bohemian mathematician of his time, Jan Sobotka[14], has solved the problem of fundamental theorem of orthogonal axonometry as well as of an oblique one. Felix Klein[15] oneself had no doubt about the significance of the Pohlke's theorem and has accomplished another demonstration (an analytic one) that was published in a collection of his lectures "*Elementary Mathematics from the higher point of view*". The second volume of this book, named *Geometry*, presents all geometry knowledge that is inevitable by Klein's opinion for teacher candidates, especially.

(1843 – 1921) had earned his doctoral degree at Berlin under Weierstrass. After spending a short time at the University of Halle he accepted a professorship at the University of Göttingen (1875 – 1892) and then at the University of Berlin, where he worked until his death. His affection for geometry (though he has not occupied with any research in this domain) and his unusual ability to transform geometrical considerations into the language of analysis has brought him to the most significant results (for example Cauchy-Schwarz's inequality, Schwarz's functions, etc.).

[9] *T. Reye* (1838 – 1919), *Demonstrations of Pohlke's fundamental Theorem of Axonometry* (Beweis von Pohlke's Fundamentalsatz der Axonometrie, Vrtlj. Natur. Ges. Zürich, XI, 1866, 350 – 358)

[10] *Karel Pelz* (1845 Běleč u Křivoklátu – 1908 Prague); *On a new demonstration of the fundamental theorem of Pohlke* (Über einen neuen Beweis des Fundamentalsatzes von Pohlke [6]). K. Pelz took his university degree at the Polytechnic Institute in Prague, he was a student of Fiedler and Küpper. He worked as an assistant lecturer by Küpper for 5 years, a professor at technical secondary school at Těšín and in Graz, where he has entered in a very short time an academic career at the Polytechnic Institute. The years passed in Graz had been the most prosperous and productive years of his life. In 1896 his long-time dream came true when he turned to Prague and became an ordinary professor of descriptive geometry at the Czech Polytechnic Institute and worked at this post until his death. K. Pelz has excelled in the synthetic theory of conics, curves and surfaces (especially the quadric ones) and has been interested in another various contemporary problems. His contribution to the solving several problems has always been excellent.

[11] He made a reference to it in Ges. Mathem. Abhandlungen II, Berlin 1890, 350.

[12] *Gustav Adolf Victor Peschka* (1830 – 1903), at that time the professor of descriptive geometry at the German Polytechnic Institute in Brno, in 1891 – 1901 the professor of descriptive geometry at the Polytechnic Institute in Vienna; *An elementar Demonstration of Pohlke's fundamental Theorem of Axonometry* (Elementarer Beweis des Pohlke'schen Fundamentalsatzes der Axonometrie, Stzgsb. Math. Nat., Akad. Wien LXXVIII, 1878, II Abth., 1043 – 54)

[13] *Arthur Cayley* (1821 – 1895), *On a Problem of Projection* (The quart. J. p. appl. Math., XIII, 1875, 19 – 29)

[14] *Jan Sobotka* (1862 – 1931), the professor of descriptive geometry at the Polytechnic Institute in Vienna, the first professor of desriptive geometry at the Czech Polytechnic Institute in Brno and from 1904 the professor of mathematics at the Czech University in Prague. *On mathematical Study of Axonometry*. (Zur rechnerischen Behandlungen der Axonometrie, Stzgsb. Böhm. Ges. Prag, 1900)

[15] *Felix Klein* (1849 – 1925), the professor at the University of Erlangen and Göttingen, German algebraist, analyst, geometer. In Erlangen, in a famous inaugural program in 1872 (*The Erlangen program*), he introduced the classification of geometries according to invariants under groups of transformations.

2 The Schwarz's Proof of fundamental Theorem of Pohlke

From numerous proofs of the Pohlke-Schwarz's fundamental theorem the following (due to H. Schwarz) is quite elementary. We give it in a broad outline. Schwarz has formulated the statement of Pohlke in a more general fashion:

Theorem 1 *The vertices of any quadrangle*[16] *can be considered as an oblique parallel projection of the vertices of a tetrahedron that is similar to a given tetrahedron.*[17]

The proof of theorem 1 is based upon the very interesting theorem of L'Huilier[18] : *The sections of an arbitrary three-edged closed prismatic surface include all the possible forms of triangles*. (In other words: *Every triangle can be considered as the normal projection of a triangle of a given form*.) The next auxiliary theorem is a simple consequence of this proposition:

Theorem 2
To any n-edged ($n > 3$) *closed prismatic surface* ^nH *and to any n-polygon* P_n *affine*[19] *to the arbitrary n-polygonal plane section of* ^nH *there exists a plane that intersects* ^nH *in an n-polygon that is similar to the given n-polygon* P_n .

The proof of the Pohlke-Schwarz's theorem is now easy. Evidently it is sufficient to prove the following *modified* statement (equivalent to the theorem 1):

Theorem 3
The oblique image of the vertices of any tetrahedron ABCD (*by an oblique parallel projection onto a plane*) *can always be the vertices of a quadrangle* $A_1B_1C_1D_1$ *that is similar to a given quadrangle* $\overline{A}\overline{B}\overline{C}\overline{D}$.

Proof (of the theorem 3)

If an oblique parallel projection φ of the required properties does exist it holds[20]:

- The intersection point $A_1C_1 \cap B_1D_1$ of the diagonals of the quadrangle $A_1B_1C_1D_1$ has to be the parallel projection of two points that are incident separately with just one of two skew edges AC, BD of the given tetrahedron $ABCD$. Let's denote them: M, N ($M \in AC$, $N \in BD$). Consequently $\varphi(M) = \varphi(N)$ ($\varphi(M) = M_1$, $\varphi(N) = N_1$).
- According to the properties of two similar figures as well as the fundamental properties of parallel projection we have: $(A_1C_1M_1) = (\overline{A}\overline{C}\overline{M})$, $(B_1D_1N_1) = (\overline{B}\overline{D}\overline{N})$ (similar quadrangles) and $(A_1C_1M_1) = (ACM)$, $(B_1D_1N_1) = (BDN)$ (parallel segments are projected in the same proportion).

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16 We consider *a simple quadrangle*, i.e. a plane geometric figure *ABCD* consisting of four points *A*, *B*, *C*, *D* no three of which are collinear, and the four segments *AB*, *BC*, *CD*, *DA* (connecting the points in a given order). Four points are *vertices*, and four segments *sides* of the quadrangle.

17 In the original Schwarz's statement there was no mention of the quadrangle; explicitly was excluded only the case of four points that lied on the same line ([1] p. 303). But three collinear points as the parallel projection of the three vertices of tripod is neither the case of axonometry nor is used in representations of solids in school practice in solving of stereometric problems.

18 *Simon L'Huilier* (1750 – 1840), French-Swiss mathematician. He has proved the theorem in 1811.

19 Two *n*-polygons are called affine if and only if there exists an affine transformation which maps one of *n*-polygons onto another.

20 An oblique image of any point X of the space $\boldsymbol{E_3}$ we denote X_1 ($\varphi : \boldsymbol{E_3} \to (\alpha_1)$, $\varphi : X \mapsto X_1$).

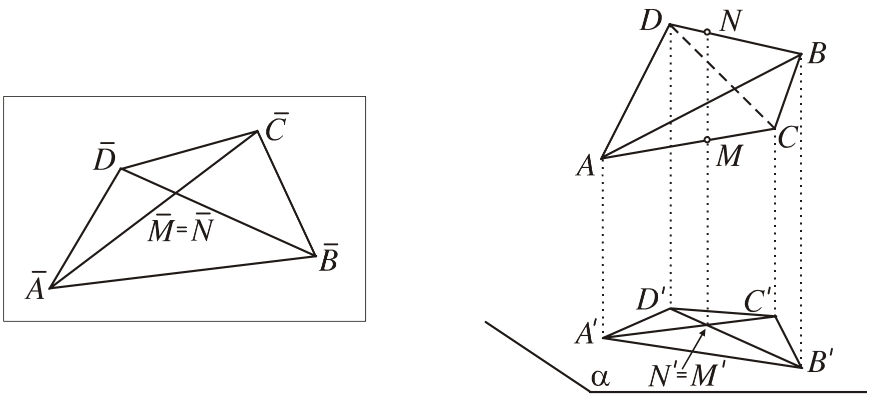

Fig. 1

It is now sufficient to construct the points M, N by following equalities: (ACM) = = $(\overline{A}\overline{C}\overline{M})$, (BDN) = $(\overline{B}\overline{D}\overline{N})$; evidently $M \neq$ N (it results from straight lines AC, BD being the skew ones). (Fig. 1) The line MN represents projection "rays" [21] (the projectors) of the oblique parallel projection that we construct. The projectors of the vertices of the tetrahedron $ABCD$ constitute the edges of the four-edged closed prismatic surface ${}^4H$ (a surface ${}^4H$ is a boundary of the projection figure of the tetrahedron $ABCD$). The plane section of the surface ${}^4H$ by an arbitrary plane α (not parallel with MN) is a quadrangle $A'B'C'D'$, which is affine to the given quadrangle $\overline{A}\overline{B}\overline{C}\overline{D}$; this statement follows from equalities a) $(ACM) = (A'C'M') =$ $= (\overline{A}\overline{C}\overline{M})$, b) $(BDN) = (B'D'N') = (\overline{B}\overline{D}\overline{N})$. The last equalities in expressions a), b) stand for the necessary and sufficient condition in order for quadrangles $A'B'C'D'$ and $\overline{A}\overline{B}\overline{C}\overline{D}$ to be affine. The statement of the theorem 3 is now a direct consequence of the theorem 2.

3 High Relevancy of Pohlke's Theorem in Tuition in Mathematics

The Pohlke's theorem is a fundamental theorem of the axonometric representation of three-dimensional figures through an oblique parallel projection into a plane. The principles of such representation a potential reader can find e.g. in [2], [7] and in all handbooks on descriptive geometry dealing with axonometric mapping for the use by engineers or specialists on computer graphics.

The method is based on oblique parallel projection into an arbitrary plane ε (not at infinity) of the extended Euclidean space $\overline{E}_3$ [22] fixedly connected with a base $\prec O; E^x, E^y, E^z \succ$ of an arbitrary orthonormal coordinate system under a condition that the plane ε is not parallel to any of the axes x, y, z of coordinates and no coordinate axis is a projector of the oblique projection. The plane ε is called the *axonometric plane of projections* and an oblique projection of a point M into this plane is the *axonometric projection* of this point (denotation: M_k).[23] The configuration $\prec O_k; E_k^x, E_k^y, E_k^z \succ$ is the *axonometric coordinate system*. By Pohlke's theorem we can take points O_k, E_k^x, E_k^y, E_k^z arbitrarily, under the condition that they are vertices of a quadrangle (in the axonometric plane of projections). (Fig. 2a, b)

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[21] More exactly: the projectors of a parallel projection are not rays, but a system of parallel non-oriented straight lines, while a ray is an oriented straight line and a system of all "congruently" oriented rays is a *direction*; for example the direction of illumination in the theory of shadows (*the rays of light*).

[22] Euclidean space E_3 is extended with *ideal elements* (points, lines and an ideal plane, so called *elements in infinity*).

[23] The set of axonometric projections of all points of the geometrical figure U is called *axonometric projection of the figure U* and is denoted U_k.

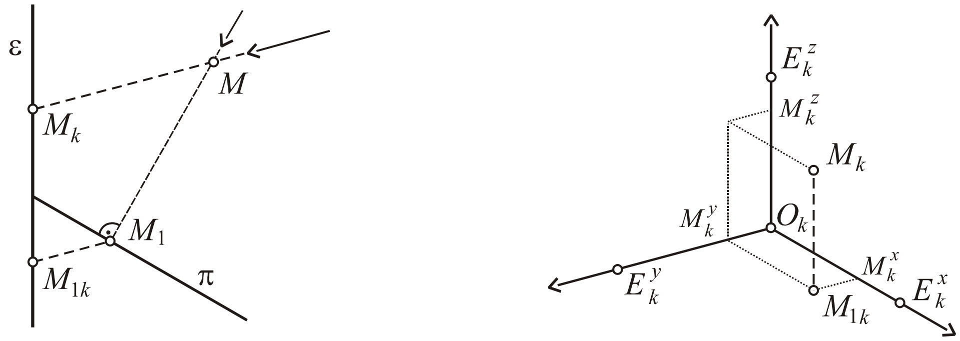

Fig. 2a, b

Let M be an arbitrary non-ideal point of the space $\overline{E}_3$ and x^M, y^M, z^M its coordinates in the chosen orthonormal coordinate system. Then

$$x^M = (M^x E^x O),\ y^M = (M^y E^y O),\ z^M = (M^z E^z O) \qquad (1)$$

where the points M^x, M^y, M^z are the vertices of the parallelepiped (in this case rectangular one), which corresponds to a point M. The representation of a point M in axonometry will be an ordered couple of the points (M_k, M_{1k}), M_1 being a *normal* projection of a point M into *the first auxiliary plane of projections* $\pi = \overleftrightarrow{xy}$[24] (the plane π is the first coordinate plane). The point M_1 is said to be *the first projection* of the point M and the point M_{1k} is *the axonometric projection of the first projection of the point M*, that is $M_{1k} = (M_1)_k$. From the definition of the first projection of a point is obvious:

$$MM_1 \parallel z \vee (M = M_1) \Rightarrow M_k M_{1k} \parallel z_k \vee (M_k = M_{1k}), \text{ respectively.}$$

It is easy to verify that the mapping $\psi : \overline{E}_3 \to \varepsilon \times \varepsilon$, $\psi : M \mapsto (M_k, M_{1k})$ so that $M_k M_{1k} \parallel z_k$ or $M_k = M_{1k}$ (in a way described above) is a *bijection*.

The coordinates x^M, y^M, z^M of a point M establish the corresponding rectangular parallelepiped of the point M as well as its parallel projection (in the given axonometric coordinate system) considering that a ratio in which a point divides a line segment[25] is an invariant under a parallel projection. Consequently the equalities (2)

$$x^M = (M_k^x E_k^x O_k),\ y^M = (M_k^y E_k^y O_k),\ z^M = (M_k^z E_k^z O_k) \qquad (2)$$

follow from (1). The couple (M_k, M_{1k}) is settled by the axonometric projections of two vertices of the mentioned parallelepiped.

Conversely, the fact that an ordered couple (M_k, M_{1k}) determines points M_k^x, M_k^y, M_k^z (Fig. 2b) and consequently the coordinates of the point M (and the point M itself) is evident.

Note. An axonometric method of representation does not require the mastery of any other method of representation used in descriptive geometry. It does require a mastery of *stereometry* (elementary geometry of the Euclidean space $\boldsymbol{E_3}$) with the notion of parallel projection and its invariants/invariant properties, the fundamental notions connected with the parallel projections of simple geometrical objects (prismatic/cylindrical surface, pyramidal/ conical surface, sphere) and of solids derived from that objects (prism/cylinder, pyramid/ cone) in the free parallel projection used in a school practice.

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[24] It can be used also the second or the third *auxiliary plane of projections* in the connection with the second or the third plane of coordinates, respectively.
[25] To tell more exactly it is a ratio of an ordered triple of collinear points.

What is the connection between the axonometric method of representation (by an oblique projection into a plane) and the *free parallel projection*[26] used in a school practice for the illustration of the solving stereometric problems? From the generalized statement of Hermann Schwarz it follows that the correlation is very strong. It can be said, that the method of oblique axonometry forms a background of the representing of 3-space objects in the free parallel projection. The knowledge/understanding at least of its principle makes the work of a teacher in the tuition-instruction/learning process in stereometry much more effective.

Let's notice that the map $\psi: \overline{E}_3 \to \varepsilon \times \varepsilon$, $\psi: M \mapsto (M_k, M_{1k})$ keeps to be bijection also in the case of an arbitrary affine coordinate tetrahedron $OE^xE^yE^z$. A normal (= the first) projection into the plane π is then substituted by an oblique parallel projection, the projectors of which are parallel to z-axis. The first projection could be also the central projection with a centre in an arbitrary non-ideal point. In both cases we have to do with the so-called *inner projection* connected with the representation of prisms/cylinders and pyramids/cones in the free parallel projection.

Stereometric problems are amply of the constructional character. Even if there is no difference between a general *formal structure* of the solving of the construction stereometric problem in comparison with a planimetric one (analysis of a problem, construction, demonstration, discussion), there is an essential new problem in stereometry – namely a presentation of the *construction*. Considering that really it is not possible to accomplish the geometric figure of the required properties (usually by plane means)[27], it is necessary to *postulate explicitly the concept of the construction* in the solving stereometric problems. Under the construction in stereometry we understand the working out an algorithm, which *would enable* us to "accomplish" the space object of the required properties by the use of so-called *elementary constructions*. We postulate elementary constructions as the simple basic problems that proceed towards objects, the existence of which is guaranteed by axioms and their simple consequences. It is necessary to list them at class; but, naturally, their introducing should correspond to the psychic maturity of pupils/students. The elementary constructions could be, for example, the following:

1. To construct a plane that is incident with a given triple of non collinear points.
2. To construct a line of intersection of two non-parallel planes.
3. To solve any planimetric problem in a given plane.
4. a) To choose an arbitrary point that is incident/is not incident with a given line.
   b) To choose an arbitrary point that is incident/is not incident with a given plane.
   c) To choose an arbitrary line that is incident/is not incident with a given point.
   d) To choose an arbitrary line that is incident/is not incident with a given plane.
   e) To choose an arbitrary plane that is incident/is not incident with a given point.

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[26] *Free parallel projection* is called a parallel projection into a plane that is independent of any coordinate system. The solving the stereometric problem is connected as rule with some reference solid/figure by help of which the given geometric figures in the problem (points, lines, planes …) are determined.

[27] Even in the opposite case nothing would be solved, because there is a stupendous difference between the material *model* of the geometrical object and the *abstract concept* of this object (*mathematics notion*). [12] But this note does not want to reduce the importance of the *work with material models* in the tuition in stereometry. The model stands for an inevitable instrument not only at the time of the first contacts of the pupils/students with the space objects; it can be omitted only under condition that pupils/students by this time have managed very effectively the construction of "sketches" of geometric figures, e.g. of the images of the figures in the free parallel projection. This is possible only after having reached the certain level of the *mastery of the system of stereometry* and also it depends on the complexity of the studied problem. For example the study of the regular/semi-regular convex solids and their representation in descriptive geometry – even at the university level – is unthinkable without using models.

f) To choose an arbitrary plane that is incident/is not incident with a given line.

5. To construct a line that passes through a given point and is parallel to a given line.

It does not mean that an algorithm of the solving a stereometric problem has to include only the elementary constructions 1 – 5 from the list above. After having acquired an algorithm of the solving of another simple problem (for example the construction of a plane passing through a given line and parallel to another line or the construction of a plane passing through a given point and parallel to another plane …) also this problem can figure in algorithm as an elementary construction.

To work out an algorithm of the solving the stereometric problem is sometimes rather hard. It can be facilitated by the so-called *sketch*; by this term we understand an oblique image of a geometric object in the method of a free parallel projection. The fact that the free parallel projection is independent of any coordinate system is important too. For example, it enables the teacher to choose images of some figures arbitrarily, when explaining the matter. It also facilitates the taking notes by students. On the other side is important – from the didactic point of view – to formulate problems for students in such a way that the solution should be uniform. It will happen only in such problems, where images of the geometric figures are complete. One can require *the completeness of an image with respect to the solution problems of position* or *the completeness of an image with respect to the solution metric problems*; it depends on the characteristic of a given problem. There is no need to stress the importance of the *training* the solving stereometric problems by *constructions in the image-plane* (on plane images of space objects). This activity is – besides the acquiring the logical fundaments of the geometry of 3-dimensional Euclidean space (on the adequate level corresponding to the psychic maturity of pupils/students) – *a component of an extraordinary importance* in the tuition in Mathematics. The image/sketch of the 3-space object helps us to fill up the distance between the model and the abstract concept of the object. This all put very high requirements on the teacher's work.

The following notes can be understood as an attempt *to define some difficulties* connected with the construction of images of geometrical objects (affected by the free parallel projection) in the solving stereometric problems, as well as to adumbrate the possible ways of their solutions. Some of them will be formulated in the *hypotheses of the research work*.

1. The fact that a teacher of Mathematics masters the system of *elementary geometry of 2-dimensional Euclidean plane* as well as *of 3-dimensional Euclidean space* (system of stereometry) on the reliable level should be taken for self-evident. Without the knowledge of the system she/he would not be capable of the solving any of another problems.
2. The matter of a high importance is a correct formulation of a problem – considering the basic classification of stereometric problems into two groups: a) *problems of geometry of position*; b) *problems involving perpendicularity and metric problems*.[28] For the managing these problems the teacher should be acquainted with a notion of the *completeness of an image* of a given geometric figure with respect to the solving position problems as well as metric ones. Some examples explaining the notions will follow.
3. Every teacher acquainted with the Pohlke's theorem should be aware of the fact, that demands (laid on pupils) which concern the execution of the *correct "drawing" of a cube etc.* (by raising *absurd rules* of the constant shortening of images of some edges, the measuring of angles etc.) are inappropriate. Also demands on pupils/students to

[28] A simple computation problem ("*substitution in formula*") in the solution of which no stereometric construction is required evidently is no metric problem.

identify the geometric object from its image (effected by the free parallel projection) in the case that the image is complete with respect to the solving position problems but *incomplete* with respect to the solving metric problems should be taken as inadmissible.

4. The student of the secondary school would comprehend that the projection in question (in the free parallel projection) is not a normal one. The construction of the normal projection of a tripod is not elementary[29]; it is based on a knowledge that does not belong to the contents of the secondary/high school curriculum on the subject.

The following examples explain some of the notions from the paragraph 2.

We call the oblique parallel image of an *n*-edged prism ${}^{n}H\left({}^{1}A{}^{2}A...{}^{n}A,{}^{1}\overline{A}\right)$ (${}^{1}A{}^{2}A...{}^{n}A$ is the base *n*-polygon P_n and ${}^{1}A{}^{1}\overline{A}$ is a lateral edge of ${}^{n}H$) *complete* with respect to the solving *position problems* if there are given: a) the projection of *any triple of vertices* of any its base polygon; b) the projection of *one vertex* of another base polygon (which lays on the same edge of the solid with a vertex of the given triple); c) the *form* of the *n*-polygon P_n (i.e. an arbitrary polygon that is *similar* to the polygon P_n) (for $n > 3$). The parallel projection of the mentioned quadruple of vertices can be chose – by the Pohlke-Schwarz’s theorem – arbitrarily and also suitably (for example in a way, that any two lateral edges does not lay in the same projecting plane, i.e. the projections of them are not incident, etc.). (Fig. 3a) For the completion of projections of the other vertices of P_n we use the fundamental properties of the parallel projection (*the parallelism* – the invariant property, *the ratio of parallel segments* – the invariant) and the information about its form. Projections of vertices of another base $\overline{P}_n$ can be completed by using the property of the projection of *congruently oriented congruent segments* ${}^{i}A{}^{i}\overline{A}$ (i = 1, 2, …, n). However, every other geometric figure entering into the problem (a straight line, a plane …) has to be determined by points fixedly connected with a given reference solid, in this case the prism ${}^{n}H$. These points may lie on the lines passing through the edges of the prism as well as in the planes containing any couple of its lateral edges (also of non-neighbouring ones), in the planes of the base polygons, etc.

Analogically the image of an *n*-edged pyramid ${}^{n}I\left({}^{1}A{}^{2}A...{}^{n}A,V\right)$ (${}^{1}A{}^{2}A...{}^{n}A$ is the base *n*-polygon P_n and the point *V* the principal vertex of ${}^{n}I$) is called to be *complete* with respect to the solving *position problems* if there are given: a) the projection of *any triple of vertices* of its base polygon P_n ; b) the projection of its *principal vertex V*; c) the form of the *n*-polygon P_n .

[29] *Karl Fridrich Gauss* (1777 – 1855); Gauss’fundamental Theorem of normal Axonometry. In: *Works of Gauss* (Gauss’ Werke II, p. 309, Gauss’ Werke VIII, p. 345 – 347) [1]
Ludwig Julius Weisbach, (1806 – 1871); *The monodimetric and axonometric Method of projection* (*Perspective*) (Die monodimetrische und axonometrische Projectionsmethode (Perspective), in: Volz und Karmarsch, Polytechnische Mitteilungen, Tübingen, 1844) [1]

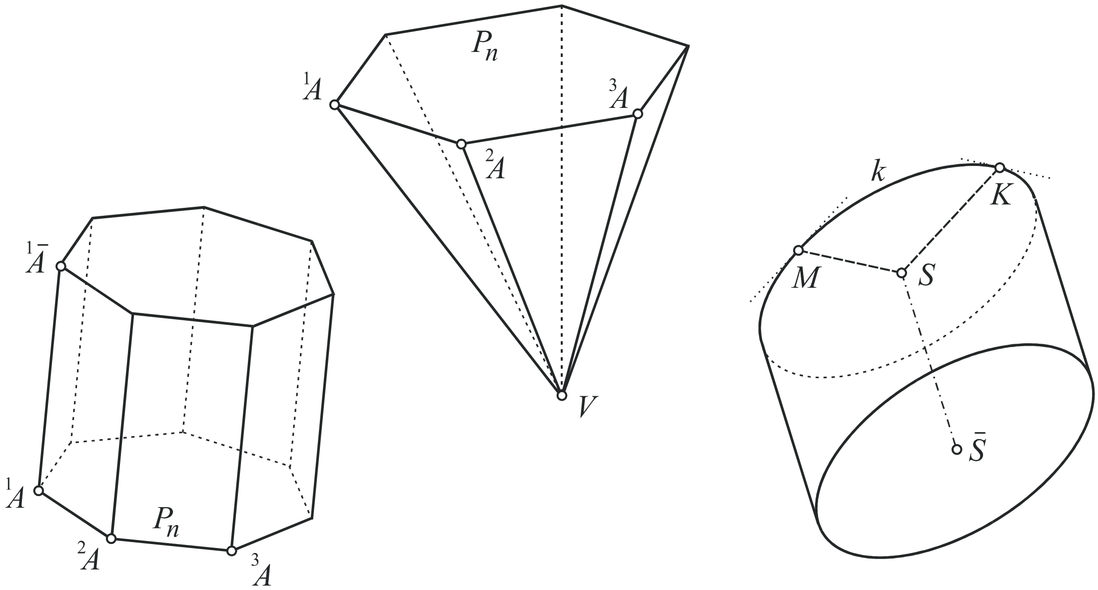


Fig. 3a, b, c

In Fig. 3b can be mapped *an arbitrary* six-edged pyramid, the base polygon of which is for example the regular hexagon. We can complete the projections of the other vertices of this hexagon by the construction of its centre S. In the original six-polygon the quadrangle ${}^1A{}^2A{}^3AS$ is a rhomb; consequently the parallel projection of the point S is the fourth vertex of a parallelogram given by projections of other three vertices. In the final construction we use that the point S is the symmetry centre of the hexagon (an invariant property of a parallel projection). (The point S in the Fig. 3b is not marked; its construction basing on the above description is obvious.)

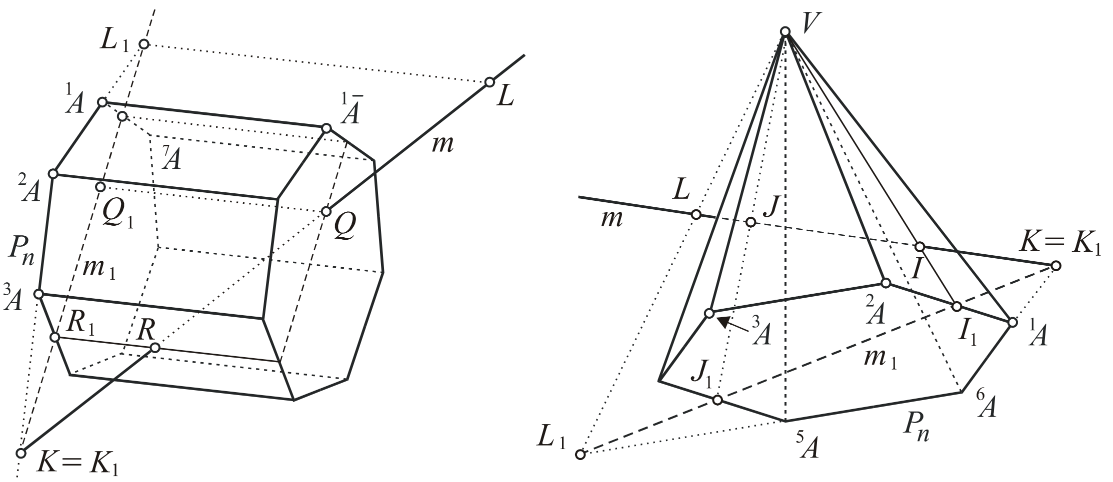
Also in the constructions of images of oval solids (*circular cylinders* and *circular cones*)[30] we can proceed in the same way. Let the directrix k (the *edge* of a solid) be given e.g. by its centre S and by the end-points of its two semi-diameters SK, SM perpendicular each to another. (Fig. 3c) Images of these segments (in a free parallel projection) are two conjugate radii of the ellipse, which is the projection of the circle k; by two conjugate radii is an ellipse fully determined. The image of the cylinder/cone can be completed by the projection of the centre $\bar{S}$ of the other basic circle/of its vertex V. The parallel projection of the quadruple S, K, M, $\bar{S}\,/V$ we can choose arbitrarily (Pohlke-Schwarz's theorem), in a way that every triple of this quadruple is not collinear. The image including the cylinder/cone is complete if every another geometric figure entering into the problem is determined by points fixedly connected with a given reference cylinder/cone. In the case of a circular cylinder these points may lie on the lines that are incident with its *sides*[31], in the planes that are incident with any of its base circle, in its tangent planes etc.

What is substantial, in any case, is an ability to complete each point occurring in the solving problem by its *auxiliary projection* into the plane of its base polygon/circle. This auxiliary projection is a parallel one (with projectors determined by lateral edges or sides of a solid) and a central one (with the centre in the principal vertex/vertex of a solid) in the case of prisms/cylinders and pyramids/cones, respectively. Fig. 4 illustrates the solving of a simple position problem: *to determine the position* of a straight line m and a given solid (prism,

[30] The *directrix* (basic curve) of a circular cylinder/cone is a circle.

[31] The *side* of a cylinder/cone with vertex V is each segment $M\bar{M}\,/\,MV$ ($M \in k$, $\bar{M} \in \bar{k}$, where k, $\bar{k}$ are the basic circles of the cylinder and the segments $M\bar{M}$ and $S\bar{S}$ are parallel / k is the base circle of the cone).

pyramid) and also to construct – if there exists – their intersection. In both cases the line *m* is determined by two its points *K*, *L*; in the case of a prism the point *K* lies on the line containing the base edge ${}^2A{}^3A$ and the point *L* in the plane that is incident with the vertices ${}^1A$, ${}^2A$, ${}^1\overline{A}$,



$m \cap {}^7H = (m \cap \lambda) \cap {}^7H =$
$= m \cap (\lambda \cap {}^7H) = \mathbf{RQ}$

$m \cap {}^6I = (m \cap \lambda) \cap {}^6I =$
$= m \cap (\lambda \cap {}^6I) = \mathbf{IJ}$

Fig. 4a, b

in the case of a pyramid the point *K* lies on the line ${}^1A{}^6A$ and the point *L* in the plane of the face ${}^5A{}^6AV$.

In the case of a pyramid the auxiliary projection is a central one from the centre *V* (the principal vertex of a pyramid) into the plane of its base polygon. For the central projection of the points *K*, *L* we get analogically: $K_1 = K, L_1 \in \overleftrightarrow{{}^5A{}^6A}$ [32]; the projecting plane $\lambda = \overleftrightarrow{mL_1}$ of the line *m* intersects the surface of the pyramid in a triangle I_1J_1V and consequently the line *m* intersects the given six-edged pyramid in the segment *IJ*. The points *I*, *J* are determined by ordered couples of points: $I = (I,\ I_1), J = (J,\ J_1)$. [33]

The axonometric method of representation (by an oblique projection) also answers the question of the *completeness of the image* of a geometric object *U with respect to the solution metric problems* (*problems involving perpendicularity* inclusive): it is necessary to have given the *parallel projection of a tripod* [34], which is fixedly connected with an object *U*. In the case of a polyhedron it proves to be convenient to choose one of its faces (2 or 3 faces – if possible) in a plane containing a face (2, 3 faces) of a tetrahedron which corresponds to the given tripod. This knowledge makes immediately clear a reason why a *cube*, a *rectangular* of

[32] The lower index "1" indicates the central projection of an object.

[33] Evidently, the interpretation of these solving problems can be another. In the case of the prism, all common points of the line *m* with the prism are exactly common points of *m* with an intersection of a *direction plane of the given prism* (that is incident with *m*) with the prism. In the case of the pyramid the points in question are analogically all common points of *m* with an intersection of a *vertex plane of the given pyramid* (incident with *m*) with the pyramid. However, the solution can be interpreted in this way only after a profound acquainting students with the problem of the classification of the position of the direction/vertex plane of a prismatic/pyramidal surface and the surface itself and its transformation to the planimetric problem of the classification of a position of *the intersection line* of this plane with a plane of the base polygon P_n and the polygon P_n itself.

[34] In axonometry a tripod is represented by the *axonometric coordinate system* (p. 1, 5). End points of edges of this tripod can be taken for the vertices of a tetrahedron which is *orthonormal*; we say that this *orthonormal tetrahedron* corresponds to the given tripod.

given *dimensions* or a *regular pyramid* (given by the length of a *base edge* and its *height*) – consequently *also a cube* (implicitly) – are the most favourite reference solids in the solving metric problems (but *unfortunately* also problems of position).

In what way could we avoid an unintentional stereotype? For to gain it is necessary:

a) In the solving position problems to use in a greater extent *parallelepipeds*, *oblique prisms*, *any pyramids* (with a given regular base polygon or a base polygon of a given form) including *tetrahedrons*; a convenient reference figure could be any *trihedral*.[35]
b) In metric problems the training of working out algorithms of the problems solving (on basic geometric elements) on plane images of space objects should start from the image of an orthonormal tetrahedron $(O; E^x, E^y, E^z)$ as a reference object. Once again we have to do with three edges of a cube (ending in the same point); but a fact of the great advantage in this case is that the other three faces of the cube "*does not shade*" the solving/solution. By the suitable location of a solid also does not take place the problem of the visibility.[36] Only after *having mastered* the solution stereometric problems with this reference model it is possible pass from it to the solving problems on plane images of more diverse and complex reference solids.

All these proposals will be components of the research in a certain project. If we take in consideration all we have referred to till now we can see that several very serious problems regarding an instruction in stereometry are going to arise. Many of them are directly connected with the problems we have dealt with in this paper; is *impossible* to separate them from the tuition/learning process in stereometry. We have to do with the representation of the $\boldsymbol{E}_3$-space objects by the free parallel projection. Let us recall at least one problem that is connected with *indication* of parallel projection of objects. The free parallel projection evidently is no one-to--one correspondence between space objects and their plane images and in the case of the incompleteness of the image of an object with respect to the solving metric problems is not possible to make any conclusions as to its form as well as to its metric characteristics. On the other hand, in initial lessons on stereometry – i.e. at the secondary and high school level – is established to use the identical indications of a parallel projection of a geometric figure and that of the original. It may simplify the wording and is fully justified in primary schools[37] but can also be the cause of many misunderstandings. What it brings to practice in class, every teacher of this subject should know very well; maybe because of it the tuition in stereometry became a nightmare for many teachers of mathematics and often has been reduced to the training of *plane sections of the cube* and the *calculation* of the *volumes* of elementary solids. The main role of the teacher in stereometry is to form at students *abstract* (mathematics) *concepts of the geometric figures*, the *logical sequence of notions*, and their *mutual continuity*[38] as well as to train abilities and skills of the students to solve the base problems connected with this concepts. The teacher *should not leave* the last aim out of account in tuition in stereometry on *any level* even when in the didactic transposition of knowledge (at

[35] Trihedral is a figure formed by three noncomplanar lines which intersect in a point. A simple trihedral is the union of three noncomplanar rays with the same initial point.
[36] At the beginning of the tuition in stereometry the visibility of reference objects – as a rule – is not considered (the wire models) with the exception of the *result* of the solving problem in simple cases (e.g. when constructing the intersection point of a line with a plane the determination of the visibility of two rays of this line with respect to the given plane may be demanded).
[37] Also in primary school the children have to be aware of the fact that they have to do with two objects: geometric figures and their images. The use of models of elementary solids is inevitable.
[38] This is intended to mean students of high (grammar) schools and the university students.

that very moment) it is not actual. The great stimulus and source of motivation can be the *computer techniques* background.[39]

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[39] We will mention this problem yet; it will be one component of the research in progress.