The role of History of Mathematics in research in Mathematics Education

Filippo Spagnolo[1]

The point of view of Mathematics: Mathematics are languages.

The paradigm of reference is the "Theory of situations"[2].

The point of view for communication of Mathematics: The mathematics are languages with semiotic interpretation[3].

What is a semiotic interpretation of mathematics languages?

- Syntactic point of view: syntax of mathematics languages, Bourbaki and structuralism, Formalism;
- Semantic point of view:
  - In the mathematics languages are the "set theory" as base of structure. For example: The group is defined in the which-ever set: the group of Integer, of symmetry, etc.
  - In the algebraic language, for example: $4x+2$ and $2(2x+1)$ different norm (sense) but they denote the same function (same set of ordinate couples). $(x+5)^2=x$ and $x^2+x+1=0$ they denote the same objet (empty set) but have a different sense[4].
  - In the physic language: $F=ma$ and $F=ma^2/a$ are sintacticly correct but the second relation not have sense in the physic language.
  - The relation of mathematics language as an interpreter of way of mind: the Gauss problem: $1+2+3+4+5+6+7+8+9+10$ $(n+1)n/2$ and $n+n/2+(n/2-1)n$ are equivalent but the sense is different.

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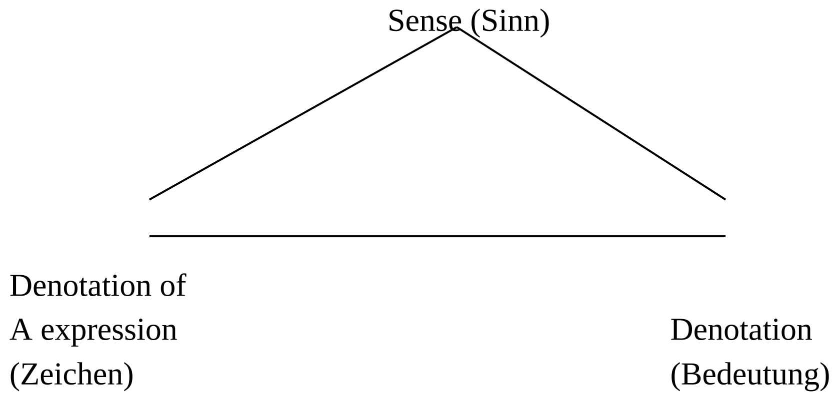
[1] G.R.I.M. (Gruppo di Ricerca sull'Insegnamento delle Matematiche, Department of Mathematics, University of Palermo). INTERNET: http://dipmat.math.unipa.it/~grim. E-Mail: spagnolo@dipmat.math.unipa.it.

[2] - G. Brousseau G., "Theory of Didactical situations in mathematics". 1970-1990" (304 pages) traduction M. Cooper, N. Balacheff, Rosamund Sutherland et Virginia Warfield. (KLUWER Academic Publishers), 1997.
 - G. Brousseau, Théorie des Situations didactiques, La pensée Sauvage, Grenoble, 1998.

[3] F. Spagnolo, Insegnare le matematiche nella scuola secondaria, La Nuova Italia, Firenze, 1998, Italia.

[4] Ferdinando Arzarello - Luciana Bazzini - Giampaolo Chiappini, *L'algebra come strumento di pensiero (Analisi teorica e considerazioni didattiche)*, Quaderno n.6 Progetto strategico C.N.R. Tecnologie e Innovazioni didattiche, Pavia, 1993

Use of Frege triangle



- Pragmatic point of view: communication point of view, didactics point of view

In the history of mathematics[5]*:*

- History of Syntax of mathematics languages: Bourbaki (History of mathematics[6]): Evolution of Algebra:
  - Law of composition: Egyptian and Babylonian have a complex system of norms calculation on Natural numbers >0 and Rational numbers >0, Commutativity of product of rational numbers (Euclid, Theory of magnitude), Diofanto, - 2 pages - , XVII century law of composition in algebra (Gauss), theory of substitutions (Lagrange), Galois (groups of substitutions, XIX century (2 pages). (9 pages)
- History of semantic of mathematics languages: Are the books with titles "History Mathematical Thought"[7]. The history of thought scours mathematics languages analysing the "senses" attributes to mathematics concepts, before organically they could to play the role in mathematical language organized.
  - In this way is the book "History of mathematics, history of problems[8]" (The inter-irem commission, Ellipse, Paris)
  - The history of function concept (also in Piaget, Epistémologie et psycologie de la fonction, Etudes d'épistémologie génétique, 1968, Presses Universitaries de Frances). The point of view of psychology is privileged. Every study of mathematics concept are completed with history study.

[5] F. Spagnolo, Storia e Didattica, Ricerca in Didattica, n.2, IRRSAE-Sicilia, to appare.
[6] N. Bourbaki, Elémentes d'histoire des mathématiques, Hermann, Paris. (Italian : Elementi di storia della matematica, Feltrinelli, Milano, 1963).
[7] Examples: M. Kline, Mathematical thought from Ancient to Modern Times, 1972 (Italian: Storia del pensiero matematico, Einaudi, 1991, Torino).
[8] The inter-irem commission, History of mathematics History of problems, Ellipses (32, rue Bargue, Paris 15°), 1997. (Version française: Histoire des mathématiques, histoire des problèmes)

- Morris Kline, Mathematical Thought from Ancient to Modern times, 1972. The history of semantic and syntax are not completly separated in this occasion.
- The history of Eudoxe-Archimede Postulate (see Spagnolo, Les obstacles epistemologiques: Le postulate d'Eudoxe-Archimede, 1995);

- History of pragmatic of mathematics languages: There is the history of communication of mathematics.
  - What was the Know (Savoir, Sapere) in a determinate historical period?
  - What was the real know of students?

In this perspective they are many important the historical sources: books, official curruculum, register of teachers, reviews of mathematics and reviews of mathematics education. (In Italy they are reviews of mathematics education since 1870. In Palermo the review "Il Pitagora" (1874-1919)[9])

In history of Algebraic language in western culture they are 3 periods:

- Rhetorical algebra: no symbol, natural language (Eastern Arabs);
- Syncopate algebra: natural language with abbreviations for the operations and for relations more frequent (Diofanto, West Arabs);
- Symbolic algebra: symbol in all relations (Indian, after XVII century in Europe, Viète).

A classification more subtle of algebraic languages[10]

	Natural languages	Geometry	Arithmetic	Examples
Rhetorical Algebra 1	Yes	· They argue with pre-Euclid instruments · They solve a problem at a time	Language of support: procedural	Chinese, Babylonian, Egyptian
Rhetorical Algebra 2	Yes	· They argue completely with Euclid instruments · They solve a problem at a time	Language of support: procedural	Classic Greeks, Euclid
Syncopate	Yes	· They solve a problem at a time	Introduction of abbreviations for	Diofanto

[9] F. Spagnolo-T. Marino, Alcune considerazioni storiche su "IL PITAGORA" (Giornale di Matematica per gli alunni delle scuole secondarie), Comunicazione convegno storia della didattica, Milano aprile 1991. Quaderni del Gruppo di ricerca didattica di Catania e Palermo, settembre 1991.

F. Spagnolo, T. Marino et Alii, Considerazioni su alcuni articoli di Didattica della matematica della rivista "Il Pitagora", La Matematica e la sua didattica, ed. Pitagora, Bologna, n.4, 1994.

[10] E. Malisani, Storia dell'algebra, Quaderni di Ricerca in Didattica, n.5, Palermo, 199

Algebra 1			unknown and its powers	
Syncopate Algebra 2	Yes	· They argue completely with Euclid instruments · They solve classes of problems	Introduction of names for the unknown and its powers	Fibonacci [11], Trattato d'algibra[12] (anonymous of XIV century[13])
Syncopate Algebra 3	Yes	· They argue completely with Euclid instruments · They solve classes of problems	Introduction of abbreviations for unknown, its powers and some relations as the 4 operations, equality and root	Algebristi of 500
Syncopate Algebra 4	yes	· They argue completely with Euclid and algebraic instruments · They solve classes of problems expressing formulas	Introduction of a particular notation for the unknowns, its powers, some relations as the 4 operations, equality and root	Bombelli
Symbolic Algebra[14]	no	· They argue completely with Euclid and algebraic instruments · They solve classes of problems expressing formulas	Introduction of symbols for the unknowns, its powers and some relations	Viète

The development of algebraic thought in the Frederic's period: A dispute between Abacists and Algorithmists.

The historic contest.

The target of this paragraph is to try to understand the passage to syncopated algebra 2 about the period of Fibonacci. To do this we will try to introduce the develop of Algebraic Thought in Sicily in the Frederic's period (Frederic II was born 26.12.1194, dead in the 1291[15]) and such thought had developed around the algebra's history.

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[11] They uses some names to call the unknown in the ending of Liber Quadratorum, but they argue completely with geometry.

[12] They associate in all work the unknown and its powers with particular names, but they argue with geometry.

[13] Anonymus, *Il Trattato d'Algibra (manuscript of XIV century)*. With introduction of R. Franci and M. Pancanti, Siena, Quaderno del Centro di studi della Matematica Medioevale, 18, 1988.

[14] They comes introduced the symbolic algebra without other levels since the present work stops to introduction of symbol.

[15] D. Abulafia, *Frederic II (Un imperatore medievale)*, Einaudi, Torino, 1995.

To argue this hypothesis is a enough difficult enterprise as they no exist historical documents to permit we have direct assertions. The are only indirect documents:

1) Documents relative to cultural situation before Frederic II [16];

2) Testimony of Leonardo Pisano (called Fibonacci), mathematical lived between second half of 12° century and second half of 13° century, through our works about questions asked by "Giovanni from Palermo" and Teodoro in occasion they visit at Pisa with the emperor Frederic II[17]. (Giovanni from Palermo and Teodoro was part of court of Frederic II).

The problems suggested by Giovanni from Palermo and by Teodoro, expressed in symbolic language of contemporary algebra, are:

1- To find a number x such that the quantity x^2+5 e x^2-5 they be squares.
2- To solve equation $x^3+2x^2+10x = 20$.

The 2° problem (suggested by Teodoro) was in Euclid's geometric tradition and wasn't particularly innovative about procedure resolutive. Fibonacci proves only root can't to belong to irrational numbers studied in Euclid's Book X and then expresses the solution in sexagesimal fractions according Egiptian tradition.

The analysis of 1° problem is more interesting why it individualizes two different conceptions: Rhetorical algebra of abacists and syncopate algebra of alghoritmists.

The Arabs of eastern (abcists) solved the problems one by one without to suppose the possibility of generalizations (they no posses instruments by symbolism.

Giovanni from Palermo, in line with this tradition, lays the problem to Fibonacci too knew the solution of other similar problems (example: $x^2-6=a^2$, $x^2+6=a^2$).

In the resolution of problem, Fibonacci excludes immediatly the integer solution (see the preceding propositions of XIV of *Liber quadratorum.* The reasoning expressed with contemporary language is:

All numbers of form $a^2 - b^2$ they say congruous but more accuracy[18]:
"A number C, integer or rational, they say congruous if exists a square number, integer or rational, such that added and subtracted C they obtain still a square. A number C is so congruous if and only if the system: $\begin{cases} y^2 - C = x^2 \\ y^2 + C = z^2 \end{cases}$

[16] 1) Ernest Kantorowicz, *Frederic II imperatore*, Ed. Garzanti, Firenze, 1988. 2) David Abulafia, *Frederic II (Un imperatore medievale),* Ed. Einaudi, Torino, 1995. 3) Gino Loria, *Storia delle Matematiche,* Vol. I (Antichità, Medio Evo, Rinascimento), Ed. Sten, Torino, 1929. 4) Michele Cipolla, *Il contributo Italiano alla rinascita della Matematica nel Duecento,* Discorso della seduta del 14.1.1934 presso la R. Accademia di Scienze, Lettere e Belle Arti di Palermo.

[17] Léonard de Pise, *Le Livre des nombres carrés (Liber Quadratorum),* Libraire Scientifique et Tecnicque Albert Blanchard, Paris, 1952.

[18] La definizione di numero congruo la riportiamo dal lavoro di R. Franci (*Numeri congruo-congruenti in codici dei secoli XIV e XV*, Bollettino di Storia delle Scienze Matematiche, Anno IV, n.1, La Nuova Italia Editrice, 1984);

admits solutions integers or rationals. If C is a congruousus number the pertinent power square number tey say congruousus square or congruousus."

To solve the proposal problem is to solve the system of equations:

1) equation $x^2 -5 = a^2$, equivalent to $x^2 - a^2 = 5$;
2) equation $x^2 +5 = b^2$, equivalent to $x^2 - b^2 = -5$.

In the first equation, we consider the possible cases:

x	A	$x^2 - a^2$
2	1	3
3	1	8
4	1	15
5	1	24
3	2	5
4	2	12
4	3	5
...	...	...

The numbers 3 and 2, 4 and 3 are possible solutions for first equation since the solution is 5, but they are solutions of second equation?

We must to count on the other expression $x^2 +5=-a^2$. This expression they haven't integer roots is sufficient to transform $x^2 -b^2 =-5$. In the table already written they see immediately that they can't negative solutions. They will must to make tables of congruous numbers to can to locate the solution. Fibonacci follows a analogous reasoning. The solution of Fibonacci is a rational number express in the fraction form 41/12. Fibonacci not provide a demonstration , of course he have make use of tables of congruous numbers were known at the time and they not came related in the Liber Quadratorum (cfr. Franci, 1984). Fibonacci say that the numbers of his numerical check 31, 41, 49 are in the arithmetic sequence of ratio 720:

(1) $(41/12)^2 +5 = 1681/144 +5\cdot 144/144 = (49/12)^2$ while x=41/12 and a=49/12;

(2) $(41/12)^2 -5 = (1681-5\cdot 144)/144 = (31/12)^2$ while x=41/12 and b=31/12;

The expressions (1) and (2) they can also to write respectively:
$41^2 + 5\cdot 12^2 = 49^2$ 1681+720=2401
$41^2 - 5\cdot 12^2 = 31^2$ 1681-720=961
they give to can locate, in table for research of congruous numbers, a possible solution is:

x	a	$x-a^2$		x	a	$x- a^2$

41^2	31^2	$720=5\cdot 12^2$		41^2	49^2	$-720=5\cdot 12^2$

The research of congruous number in our case is $720=5\cdot 12^2$ consequently they are determining to next research of solutions and also they are useful the consideration that the square of numbers 31, 41, 49 are in sequence of ratio 720. In the proposition XIII Fibonacci proofs the number finded must be in the form $5\cdot k^2$. He proofs in XII proposition the congruous number by power square is congruous number.

Single problem Class of problems.

History - Pupils - Teacher - Research in Mathematics Education

Point of view of Researcher in Mathematics Education (Communication of mathematics). By to argue the researches and for a possible reproducibility.

- History of syntax: conceptions that they use to syntactical adjustment of mathematics language.
- History of semantic: conceptions that to use to a-priori analysis of behaviour of pupils.
- History of pragmatic: study of phenomena of teaching concerning "didactic transposition".

Point of views of Teacher:

- For to restatement (focus) significative "didactics situations" for teaching/learning.
- For the epistemology of teacher.

- History of syntax: personal use of teacher but also for to check the analysis a-priori and/or "didactic transposition".
- Chronological history concerning mathematics languages.
- Thematic history: history of demonstration (to argue, to deduce, to infer), history of limit concept, of infinity, of rigour, of function, of natural number, etc.
- The recovery of know of discipline in the point of view more general: historical-philosophical

Point of view of pupil

- He inserts the study of mathematics languages in cultural dimension
- He inserts a temporal dimension in the construction of mathematics languages