COMENIUS UNIVERSITY BRATISLAVA
FACULTY OF MATHEMATICS AND PHYSICS
DEPARTMENT OF DIDACTIC MATHEMATICS

EMBODIMENT AND A-DIDACTICAL SITUATION

IN THE TEACHING-LEARNING

OF THE PERPENDICULAR STRAIGTH LINES CONCEPT.

Doctoral Thesis by

Giannamaria Manno

ADVISOR: PROF. FILIPPO SPAGNOLO

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ABSTRACT

My research was born from the idea that difficulties and problems students of different grades have mainly come from language and, in general, from the formal aspect of mathematical concepts.

It appeared extremely important to me to consider two sides apparently divergent

1) The specific quality of mathematics and its own language;

2) The role of the context (space, time, people) in communicating mathematics.

Do not forget that language is not only a source of trouble; it is also a necessary player in every learning process, this point is really discussed in recent works about mathematics education (Maier, Radford, Duval).

My personal idea is that, in order to make easier the mathematical communication, it is necessary to create a proper context.

The theoretical idea I go with is the one from Guy Brousseau[1] who defines the milieu: the environment where the student and his knowledge building process happen.

The choice of this theoretical framework is due to this statement:

Learning-teaching process can follow two different ways, the first one is a well known one, based on frontal lessons that are the traditional way to make people learn contents, the second one refers to a way of learning based on the emotional side (that is going to be discussed in chapter 4 about neuro-sciences) that teachers difficultly control but, on the other side, allows the building of cognitive-conflict based situation (chapter I).

I built an a-didactical situation where the relation learning-teaching into the knowledge-pupil-teacher triangle is controlled and analysed in connection with the outside environment and the emotional sphere of the student.

But I did not use only the Theory of Didactical situations; this research aims to link two different theoretical frameworks:

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[1] Brousseau G., *Elementi per un Ingegneria Didattica,* La matematica e la sua didattica, n. 1, 2000, Pitagora Editrice, pp. 6-9.

1) The Theory of Didactical situations that structures the a-didactical situation and has a methodological control role.

2) The Embodiment theory regarding the body experience learning, that leads to the process of creating metaphors and learning into an emotional context.

3) A look at Neuroscience which can give suggestion about new hypothesis of work.

Once again I have to say that the theory of situation comes at a methodological level, the play context is built according to the theory, this play choice has been made with a look at this theory. But in this scenario another element is crucial, the play situations has also some corporal times, so that also the embodiment theory and its approach is very important as well as a theoretical reference and a way to analyse data.

It is possible to link these two theories?

We can probably say that the link between the two theoretical frameworks is that the student is the only player of his knowledge process; by the "devolution" act according to the theory of Didactical situations, by his own senses, brain and mind according to the embodiment theory.

What I tried to build recalls the book "*Cognitive space of action production and communication*" by F.Arzarello[2], whose basic elements are:

✓ The body and the brain;

✓ Physical world;

✓ Cultural environment[3];

Here we can find culture, sense and motion related experiences (embodiment), languages, representations, signs and objects such as pens and computers. But Arzarello does not include in his work the theory of Didactical situations.

What does embodiment mean?

The concept of *Embodiment* is relatively new within the field of mathematical education, I would like to clarify, according to Lakoff and Nuñez, the use of the term in this thesis, and distinguish it from other notions concerning the role of the "physical" and "concrete" in mathematics learning. Embodiment is not simply about an individual's conscious experience of some bodily aspects of being or acting in the world. Embodiment does not

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[2] Arzarello F., *Mathematical, land scapes and their inhabitants: Perpeptions, languages, theories,* ICME 10, 2004.
[3] In learning processes we find that many things interact and link to and within our body

necessarily involve conscious awareness of its influence. Nor does embodiment refers to the physical manipulation of tangible objects, or to the virtual manipulation of graphical images and objects instantiated through technology. Although there is a relation between such experiences and the technical concepts of embodiment, and an embodiment perspective does not constitute a prescription for teaching in a "concrete" way. Similarly, although embodiment may provide a theoretical grounding for understanding the teaching and learning of "realistic" or "contextualized" mathematics, it is not directly concerned with "contextualization" or "situatedness" in subject matter teaching.

Rather, embodiment provides a deep understanding of what human ideas are, and how they are organized in vast (most unconscious) conceptual systems grounded in physical, lived reality.

Some other important elements I consider are the conceptual metaphors. They are "mapping" that preserve the inferential structure of a source domain as it is projected onto a target domain. Thus the target domain is understood, often unconsciously, in terms of the relations that hold in the source domain. For instance, within mathematics, Boolean logic is an extension of the container scheme, realized through a conceptual metaphorical projection of the logic of containers. So a mathematical concept is build via physical experience, and later unconsciously mapped to a set of abstract mathematical concepts.

The "projections" or "mappings" are not arbitrary, and can be empirically studied and precisely stated. They are not arbitrary, because they are motivated by our everyday experience, especially bodily experience, which is biologically constrained. Unlike traditional studies of metaphors, contemporary embodied views don't see conceptual metaphors as residing in words, but in thoughts.

When facing the problem of the comprehension of the concept of "being perpendicular", through body experience, it is a duty to take in account the biological laws on human beings' perception of the concepts of "being vertical" and "being perprendicular".

To better understand this point the thesis provides an analysis of the Vestibular System (A.Berthoz)[4] inside whom there are some receptors sensible to gravity direction.

Gravity can indeed be measured by specialized receptors: the otoliths; gravity is an external reference system, a "plumb line" the body refers to in a geocentric reference system.

The vestibular system is a very important "egocentric" reference tool that allows the perception of the plumb line model.

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[4] Berthoz A., *Il senso del movimento*, Mc Graw Hill, Milano, 1998

Mittelstaedt[5] (1995/96) would say that we needed a new sense to add to the ones involved in gravity vertical perception, he discovered some neural receptors placed in the stomach that react to gravity (he recently ended up saying that these structures were rather placed on kidneys or blood system).

The experimental work has been led within the S.P.O.R.A. project, where I was teacher for the lessons of "Curriculum with structure", organized by the file-leader school "D.D.F. Ferrara" of Palermo, in collaboration with G.R.I.M. (Research Group for Teaching of Mathematics) of the University of Palermo.

The schools involved were at a quite high degree of risk (based on social and economic indicators). The students involved were from 3 to 11 years old (first grade schools).

The field research has required some answers on pupils' concepts about perpendicularity:

1. Do pupils have the inner model of the plumb line?

2. Does the misunderstanding of the notion of perpendicular come from a linguistic misunderstanding?

The quantity analysis has been done with the Chic software, the quality one according to the theory of the situations.

Final considerations have been done by the process of metaphors building through bodily activities that pupils have joined.

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[5] See Berthoz, already quoted, pp 87-103

Introduction

A brief history about my research path

1.0 The switch from the old hypothesis to the new one

The first research hypothesis has concerned the understanding of the function concept by students from the last year of their secondary school. According to this point of view, learning a concept is not a dynamical depending relation between two variables but a mathematical static object that has to be defined into a theory.

Let's think about the definition of a function:

Consider f as a relation of A in B (A and B are two different sets) with dom f = A. If for every x ∈ A exists only one y ∈ B so that (x, y) ∈ f, we will say that f is a function of A in B.

It is possible to say that a function is a kind of relation that has a defined property (the uniqueness of the image).

The research hypothesis was this one:

H1 If students have had a clear idea of what is the difference between a relation and a function then they would have been able to operate on their own when put against a problem concerning this subject; and this attitude could have been reached without paying any particular attention to the language set used to express the problem.

After the submission of the test and the analysis of the statistical data it has clearly appeared that the understanding of the subject is strictly linked to the language set that students know the best, and it is moreover related to the coordination between the used language sets.

Language sets used in these tests are:

- Natural language
- Algebraic language
- Graphic representation

From this test emerged the following hierarchy:

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Algebraic
Language  ────►  Natural Language

          ────►  Graphic
                 Representation
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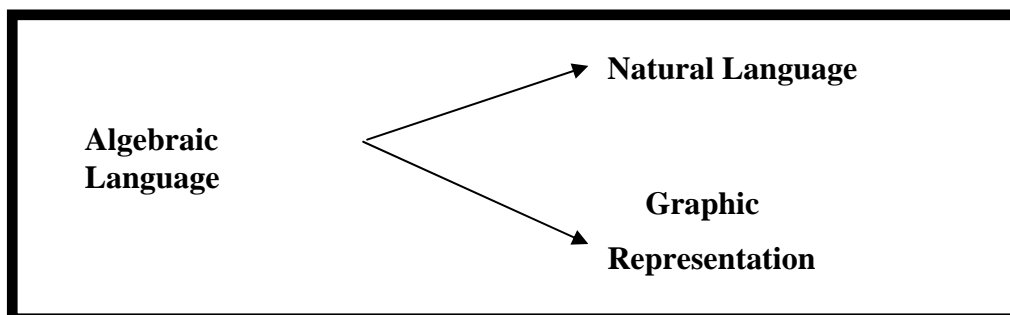
These results have been compared to the ones from Gagatsis[6] and Duval[7] experience; they both agree saying that:

"to the whole comprehension of a mathematical concept we need at least two different sets of representations".

In particular Duval says:

*It is not sufficient that there is a development of every linguistic set; the coordination of the different language sets which a subject has or the teacher wants the pupil learns (for example the algebraic language) needs their coordination. Moreover, one can asks to oneself if the new sets have to be introduced in the framework of a coordination work. In every case, the coordination of the sets is the condition for understanding, because it is a condition for a real differentation between the mathematical objects and their representation. The attitude in front of a domain or a type of activity is an obstacle whose exceeding changes totally the behaviour of pupils. But this coordination is not natural. At different levels of learning one can register the permanence of a subdivision of representation sets among them. An important factor of such a phenomenon of subdivision is the **not-congruence** between a converting representation and the one of the same set.*

Finally it has been possible to say that:

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[6] Lectures presented during a private seminar of the GRIM, Palermo, 9-5-2000.
[7] Duval R., *Quale cognitivo per la didattica della matematica,* La matematica e la sua didattica, n. 3, Pitagora Editrice, Bologna, 1996, pp. 257-261.

- The understanding of the concept is not mainly caused by the difference between relation and function but by the ability students have to recognise the Y image as an operative entity

- It is necessary that students recognise the language environment they are in, and that they can coordinate and handle at least two different language sets. This coordination is non simple and easy. The algebraic set is the hardest to use, students that can handle it, can easily handle other sets, the vice-versa is not very common to see.

At this point the problem arises:

How to go on the research, in relation to the mathematical language?

The possible directions to follow were two:

➤ A study about the coordination of different language sets;

➤ A study about the difficulty about the relation between mathematical and natural language.

2.0 And now let's go working .... it is time to communicate!

I went toward the second direction because I find it as preliminary to the first one, in fact the analysis of these topics must consider some problems linked to the communication in a classroom environment. It is necessary to say that mathematics is here intended as a whole language with its own syntax, semantic and pragmatic. According to this position the teaching act falls into the more complex and wider problem of communication.

In the triangle knowledge-pupil-teacher, mathematics communication uses natural language but some words have a particular meaning.

I believe that the analysis of the relation between natural language and mathematical language should start from the early classes, where the degree of structured mathematical concepts is due to students' degree of evolution development. This is why I decided to study the relation and problems connected to the use of mathematical language and natural language at primary school (children aged 5 to 10).

Considering the wideness of the question and the impossibility of a 360°degrees study I chose a micro-problem, a word that has one meaning in natural language and another one in mathematical language. My choice is the relation between perpendicularity and

verticality; these two words are at the same time a property and a relation (there are studies that show how students of second Italian grade have still problems in understanding relations while they easily understand properties).

New studies[8] about mathematics teaching have found that the most common comprehension problems are caused by the similarity between mathematical language and every day language.

Let's think about the word "vertical"; in mathematical language it refers to a specific relation between two lines, a perpendicular relation, which has to satisfy two statements

1) System of reference

2) Four 90° degrees angles

In every day language it refers to the direction of the centre of the earth.

Students often use this second meaning in a mathematical context. If you change the position of the line considering what you ask students to find the perpendicular line, they give wrong answers.

Hypothesis for a new research were born from these questions:

1) Do students have a plumb line model?

2) Is the misunderstanding of the concept of perpendicularity due to language-linked troubles?

So the hypothesis I start from are:

1) Are the words perpendicular and vertical synonyms to the student mind?

2) When the teacher explains the concept of perpendicularity, students get into the concept, only if they conform in a proper way the concept of the plumb line to the new situation designed by the teacher. If they do not break the point and overcome this epistemological obstacle (plumb line model experimented by students) they will not be able to solve problems where the system of reference is not more a line parallel to the one that passes from the centre of the earth.

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[8] I am refering to Maier's studies quoted in the bibliography.

3.0 Some notes about the national and international research context[9]:

But before getting into the work I have done, I will try to give a panorama of the most important researches about the specific quality of mathematics and its own language.

For example, Usiskin (1996) has radical ideas about it:

"Because mathematics is a living second language, for most students, we should at least teach mathematics as we are teaching living foreign languages, in contexts and, starting the soonest we can and immersing students in the language". It is quite common to link together mathematics and language. think also about Devlin 2000 *The Language of Mathematics: Making the Invisible Visible"*.

In some way opposite is the idea of Lakoff according to him the linguistic activity is a reflection of the cognitive one, and it depends on it. The difference from the other theories, that contain similar statements (Sfard, Radford), consists in that for Lakoff every language is a sign of deep cognitive activity. It is possible to split a concept from its linguistic sign; a language reflects its culture, but it does not contribute to determine it. According to language experts (Eco) metaphor does not own to the linguistic sphere but to the cognitive one

"Unlike traditional studies on metaphors, contemporary embodied views don't see conceptual metaphors as residing in words, but in thought" (Nuñez).

Anna Sfard is very far from Lakoff and Nuñez, she takes metaphors in a high consideration:

"The basic tenet of the communicational approach to the study of human cognition is that thinking may be conceptualised as a case of communication, that is, as one's communication with oneself. Indeed, our thinking is clearly a dialogical endeavour, where we inform ourselves, we argue, we ask questions, and we wait for our own response. The conceptualisation of thinking as communication is an almost inescapable implication of the thesis on the inherently social origins of all human activities. Anyone who believes, as Vygotsky did, in the developmental priority of communicational public speech over inner private speech (e.g. Vygotsky, 1987) must also admit that whether phylogenies or ontogenesis is considered, thinking arises as a modified private version of interpersonal communication." (Sfard,2001,p.26).

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[9] For the references see Ferrari P., www.fnunipmn.it

Radford refuses the opposition between expression and content (deep structure/surface structure). According to him every theoretical scheme that uses this opposition is inadequate. In fact it explains the relation subject/object as not culture-linked or explained, and the meaning construction ends like a result of a relation between a lone subject and an a-storical object. But Radford tells us two important ideas:

"The first one is the Vygotskian idea according to which our cognitive functioning is intimately linked, and affected by, the use of signs. ... As a result, there is a theoretical shift form what signs represent to what they enable us to do. The second basic idea on which our framework is based deals with the meaning of signs and stresses the fact that the signs with which individuals act and in which individuals think belong to a cultural symbolic system which transcends the individual qua individual. Signs hence have a double life. On one hand, they work as tools, allowing individuals to engage in cognitive praxis. On the other hand, they are part of those systems, transcending the individual and through which a social reality is objectified. The sign-tools with which the individual thinks appear then as framed by social meanings and rules of use and provide the individual with social means of semiotic objectification."

Also Radford opinion, that underlines the function of signs and their double life (on an individual level and on a social level), is far from the one by Lakoff and Nunez; maybe because of their different consideration of "culture".

Bruner for example gives culture a wider meaning:

"...meaning itself is a culturally mediated phenomenon that depends on the prior existence of a shared symbol system."

Last one is the Dubinsky school that is quite critic on Lakoff Nunez and Sfard saying that:

"Perhaps the fundamental difference I have with these authors rests on Sfard's contention that "... the way we speak...shapes our way of looking at the world. "My experience and research suggests that it is the other way around."

"Although the idea of using metaphors to create mathematical concepts in the mind of individuals is very attractive, in that it can give mathematics an aesthetic literary flavour, I remain convinced that language is a tool whose real value is the expression of ideas which an individual already has constructed. I do not feel there is any reason to believe that it is powerful enough to be useful in creating more than the simplest mathematical concepts.

Now let us go on with the question regarding the creation of a context in mathematics.

The theoretical idea I go with is the one from Guy Brousseau who in "Theory of Situation" defines the milieu: the environment where the student and his knowledge building process happen.

"In the general idea of teaching, knowledge is a relation between good questions and good answers...Socratic philosophy limits these relation to the ones that a student can make by his own, so that the student can understand the knowledge because he is the maker of the knowledge, but the Socratic scheme can be improved if we think that students can learn from their own experiences and from the interaction with the milieu even if this milieu is not designed to learning; the student learns by looking at the world (empiric-senses hypothesis) or by making some hypothesis to choose among; or thanks to experience (a-priori hypothesis), or also thanks to a complex interaction made of learning and adapting, just like Piaget reports. The student learns by adapting himself to a milieu rich of contrasts, troubles and un-balance, just like it is for human society. A knowledge that comes from the ability to adapt shows itself with new answers and attitudes that are the real proof of a learning process"

It's possible resume the theoretical framework in this research as follow:
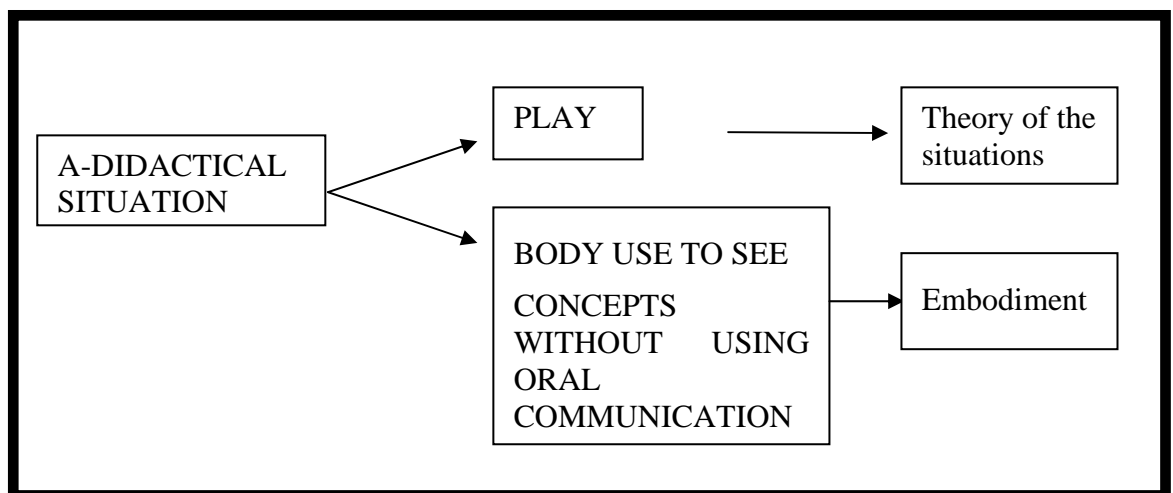
As I previous said, is it possible to link these two theories? It is not hard to answer this question, nor my ideas want to be a full positive answer.

The consideration I have done is that both theories come from a constructive approach; in fact both of them bear:

1) Positive role of the person in building the knowledge

2) Previous cognitive structure that informs the experience of any single person

3) Human being as an auto-organized system that protects and preserve his integrity

And the main idea of the constructive approach is that human knowledge, experience and ability to adapt and react are due to the person itself. The system is not only reactive (able to classify and organize incoming information) but also active, with the capability of anticipating events and situation.

A basic element of the corporal experience (apart from sight and senses) is motion: in neurological term "psyco-moving personality". It represents the result of all perceptive and pragmatic attitudes that together define a specific and personal print of every human being. As regards, Arzarello, at the tenth Icme, quoted Renè Thom, who wrote:

"The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of "meaning", of the "existence" of mathematical object".

The same scholar writes:

"More specifically, the real problem consists in focusing the genesis of mathematical objects in the classroom. To do that, many tools are necessary, some of which appear far from the formal mathematical frame: form epistemology to neurology, through history, psychology, ergonomy.[...] Moreover they (some examples that Arzarello analyses in the article) *show how to fill in the gap between the worldly truth in which our students make their concrete experiences and the logical truth, which represents the rigorous official side of mathematics. In the end it results in a more complex landscape, where the two aspects will not appear so dramatically contrasted and where mathematical object do live. In fact our mind activity depends on an integrated and dynamic set, which acts as a whole, whose components are: the brain, the body, the cultural and physical world. When students learn mathematics all those components (and possibly others, e.g emotional ones) are active and must be taken into consideration by teachers".*

What are the embodiment elements considered?

I try to answer to this question with the following example:

Image schema: are the perceptual-conceptual primitives that allow the organization of experiences involving spatial relation.	Some exampl es are	The *container* schema
		The *souce-path-goal* schema
		The *contact* schema
		The *verticality* schema
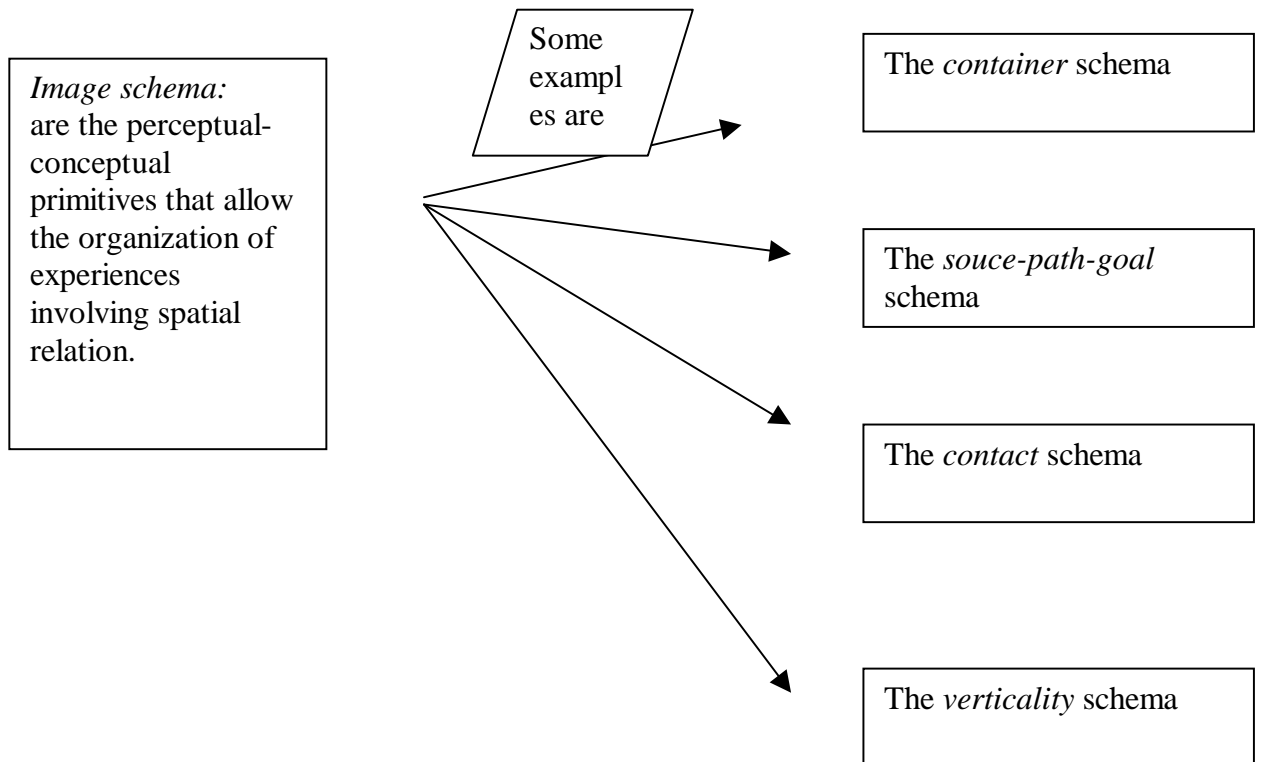
The images in the scheme are not static propositions characterizing abstract relations between symbols and objective reality. They are rather dynamic recurrent patterns that order our actions, perceptions and conceptions. These patterns emerge as meaningful structures for us mainly through the bodily experience of motion in the space, manipulation of objects, and perceptual interaction.

"They are a primary means by which we construct or constitute order and not mere passive receptacles into which experience is poured"(Johnson, 1987,p.29). "Much of what is "abstract" in mathematics...concerns coordination of meanings and sense making based

on common image-scheme and forms of metaphorical thought. Abstract reasoning and cognition are thus genuine embodied process"[10].

Some other important elements I consider are the conceptual metaphors. They are "mapping" that preserve the inferential structure of a source domain as it is projected onto a target domain. Thus the target domain is understood, often unconsciously, in terms of the relations that hold in the source domain. For instance, within mathematics, Boolean logic is an extension of the container scheme, realized through a conceptual metaphorical projection of the logic of containers. So a mathematical concept is build via physical experience, and later unconsciously mapped to a set of abstract mathematical concepts.

The "projections" or "mappings" are not arbitrary, and can be empirically studied and precisely stated. They are not arbitrary, because they are motivated by our everyday experience, especially bodily experience, which is biologically constrained. Unlike traditional studies of metaphors, contemporary embodied views don't see conceptual metaphors as residing in words, but in thoughts.

For example in Italy we can consider the work of Longo about the *"continuous that suggests an idea of the developing of mathematics, where some natural intuition find place"* (linkable to the building metaphors), and Boero and Garuti[11] job about inequality where the embodiment is the theoretical framework that lets identifying, through the theoretical construction of the metaphor, a common matrix implied in the ability to handle the concept (not only in mathematics places) and to find out experience field more suitable to the developing of such actions.

The embodiment comes into when Piaget and Vigotsky-like analysis stops, just because it tries to explain the reason why some operators come out and the way concepts are linked together and inside, what is the basic knowing and understanding role that language has in mathematics.

Another work[12] is the relation between the perceptual world of embodiment and the conceptual world of mathematical symbolism with particular reference to the concepts of vectors. In the article the authors introduce a general development that highlights three distinct models of operation in mathematics, it follows the natural development of the individual, building from physical interaction with the world, developing increasingly

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[10] Lakoff G., Nùñez E.R., 2000, *Where Mathematics comes from*. Basic Books, p 30

[11] P,Boero, R.Garuti,Le disequazioni come ambiente di sviluppo dei concetti di variabile e funzione, in vista dell'analisi matematica, sviluppo del testo di SFIDA-16, 2001

[12] Anna Watson, Panayiotis Spyrou, David Tall, The relation between phisical embodiment and mathematical sybolism: the concept of vector, Mediterranean Journal of Mathematics Education, Vol.I, 2002.

sophisticated meanings for real world phenomena: at the same time, mathematical symbolism is developed to allow greater accuracy of computation and greater precision in solving problems. So there are three mathematical worlds, each one with its own way of constructing concepts and its own way of proof:

✓ Embodied world;

✓ Symbolic world;

✓ Formal world.

4.0 Thesis structure

In the first chapter I recall the basic aspects used to build the a-didactical situation, from the theory of the situation.

The second one is about the main aspects of the mathematical language.

The third one is the experimental endeavour where I describe the a-didactical situation and the results.

The fourth is about the embodiment theory and contains conclusions.

The fifth one is about the suggestions coming from Neuroscience as a work hypothesis.

CHAPTER I:

Theoretical framework, The Theory of Didactical Situations

1.1 The triangle teacher-knowledge-pupil

The field research had been led from March 2004 to April 2004 (SPORA project, school centre Ferrara, Italy) following the idea that Brousseau (1996) proposed in his *"Theory of the Didactical situations"*.

This paragraph will try to analyse and discuss the research hypothesis trying to offer a full analysis of the theoretical work behind it, under the explanation of Prof. Spagnolo.

Research activity over didactic of mathematics has so far studied the teaching-learning phenomenon within the triangle teacher-knowledge-pupil (scheme of existing relations between subjects involved in the Theory of the Didactical situations)
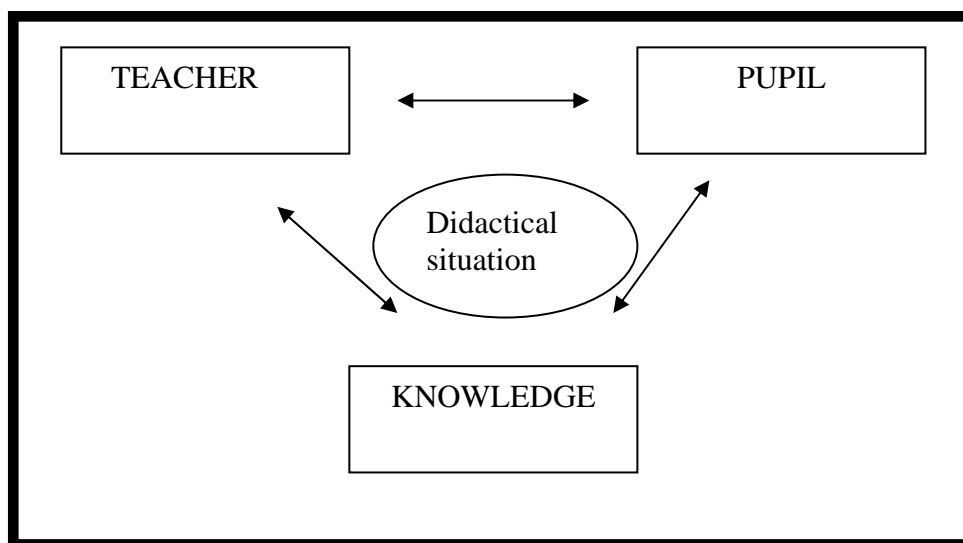
The first step in this theoretical approach is the analysis of this triangle, known since 1982 when it firstly appeared in Yves Chevallard[13] works, where the word "knowledge" would mean the academic and standard knowledge, object of the mathematical research. The teacher role is to make possible a didactical transposition; in other words, teachers have to change the "knowledge" that comes from the research world into a "knowledge taught"

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[13] Chevllard Y., Johsua M.A., *La transposition didactique*, R.D.M., 1991 (Quaderni monografici)

(the one of the daily classroom practice) thanks to another intermediate step that is the "knowledge that has to be taught".

| Math knowledge | $\Longrightarrow$ | Knowledge that has to be | $\Longrightarrow$ | Taught knowledge |
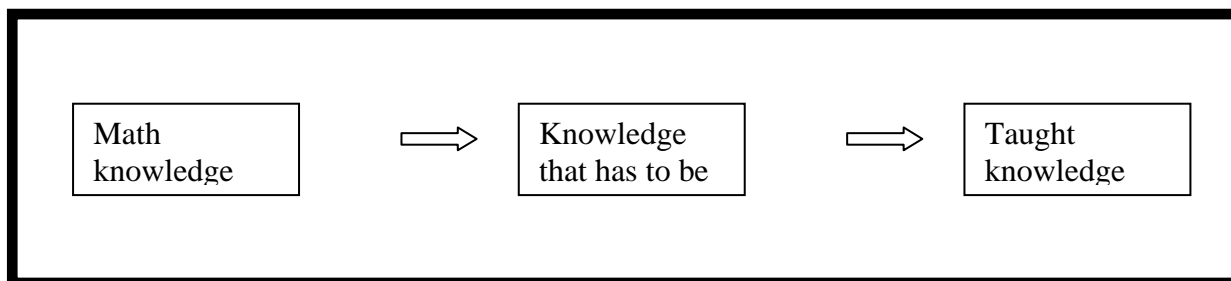
The double implication of this triangle suggests that we are facing a complex interaction that works back and forward. When analysing the triangle it is important that no one of the members takes a main role, every study of the topic teaching-learning has to consider the three members at the same level.

Cornu and Vergnioux[14](1992), say the same thing and also specify that the three subjects analysis is not a matter of didactical science but something that didactical experts have to do, students are studied from four different approaches: as biological entities, epistemic entities (learning psychology), affective entities and social entities, moreover teachers can be regarded as a social, institutional, pedagogical or affective entities. We also have to say that the triangle-scheme is not isolated from the rest of the world; it acts and reacts to the didactical world, social and cultural environment (noosphere)[15]. The teacher has to consider the noosphere and its didactical activity too, because this is what we find between school system and foreign environment. Noosphpere is a privileged place of observation because not only teachers (the first line involved in teaching act) but also researchers and society members are inside it and they keep asking answers and improvements from the school system.

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[14] Spagnolo F., Ferreri M., *l'apprendimento tra emozione ed ostacolo,* lavoro eseguito nell'ambito del contratto C.N.R n. 9001293CT01, pp 27-28

[15] Godino defines the noosphere as a place that keeps together all people studying the contents and ways of theaching.

1.2 The didactical contract

The first time we learn about the idea of a didactical contract is in the early seventies thanks to Brousseau who was studying mathematics failure from students otherwise well performing in subjects other than mathematics, this is what Brousseau writes:

" *students tend to make any information or limitation clear using what the teacher, whether consciously or unconsciously, produces in his teaching activity. We think about the most common habits in teaching, and we define a didactical contract as the specific behaviour that students expect from teachers and teachers expect from students too*" (Brosseau, 1980, pp.127-128).

The expectations Brousseau talks about are not due to fully and clear expressed statements coming from teachers or school, he thinks of the idea itself of school and mathematics and their continuous repeating.

So we can say (Spagnolo 1998):

"*the didactical contract comes out as a result of a negotiating process of relations, whether explicit or not, between a student or a group of them, an environment and an educational system with the aim of letting students get a given, or under construction, knowledge*".

The didactical contract gives rules during the learning process; it is in fact a whole made of expectations and behaviour of students and teachers towards knowledge.

It states, in a not explicit way most of the times, what students and teacher have to do, their roles and their responsibilities one to another.

The didactical contract is different from the pedagogical contract (1973/74 J.Filloux)[16] because of the main role of knowledge, in fact:

1) The didactical contract is about knowledge
2) There is a didactical contract for every kind of knowledge
3) To acquire knowledge you always have to break the contract
4) It is implicit, and never fully explained
5) A contract fully based on acting rules of teachers and students, totally explicit will lead the didactical relation to a failure.

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[16] The pedagogical contract is a more general and social matter than cognitive and is about explicit negotiation of rights and duties of teachers and students apart from knowledge. It is unique and involves both students and teachers.

Knowledge learning process needs several breaks of the contract itself, a circumstance that can be overcome by the act of "devolution".

Devolution[17] is the process that a teacher uses to make students accept (in an implicit way) the responsibility of the learning situation or a new problem, always conscious of the consequences of this transfer.

Brousseau speaks about the consequences of breaking this contract:

1) The teacher might create right circumstances to make students learn and he should be able to recognise when learning happens.

2) Students might be able to learn.

3) The didactical relation has to keep going and be alive no matter what.

4) Teachers have to be sure of previous learning and set a place where a new one is possible.

A teacher role is to be responsible of the results accomplishment and to offer students all the necessary means to learn and gain knowledge; on the other hand it is also necessary that students accept the burden to deal with new problems no one told them how to solve and what strategies to use.

Like every contract they do exist clauses, but in this case they are not predictable, knowledge will be the recipe that will solve any crisis.

1.3 Theory of didactical situations

This theoretical path starts in 1986 thanks to the work of Brousseau. The theoretical idea that stays behind this approach[18] is the main role given to the relation between students learning process and the environment where the learning happens.

"Students start their learning process in an environment that is unbalanced and full of difficulties and obstacles just like human society. The new knowledge comes from the skill to adapt to the new circumstances and stimuli and a new reaction to the environment is the proof that a learning process has taken place."

The student knows that:

" the problem he has to face has been chosen in order to make him learning and gaining a new knowledge, this knowledge is justified by the inner logic of the situation (..)".

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[17]The devolution act (from French) consists in making conscious the student of a problem and his role and responsibility in coping with the new situation; this is the core of the theory of situation

[18] From *"l'apprendimento tra emozione ed ostacolo" pp 27,* Spagnolo/Ferreri

(He will be able to handle it in a proper way when he finds himself in a situation outside a teaching context and without any guide).

This kind of context is called a-didactical situation[19].

A didactical situation involves two features:

1) the a-didactical situation
2) the didactical contract

In Brousseau works we can find a new role for teachers inside the teacher-pupil system. The teacher has to build an environment that allows students a specific learning at the end of every activity. Students and the object of the new knowledge are important in every a-didactical activities.

The situation is built in such a way that the student receives many inputs and tries to solve the problem by attempts and errors, in order to find the most suitable winning strategy. This process happens in a duty-free context where students learn interacting with the environment adapting their knowledge system to many different possible strategies, without the teacher support.

According to me, the most important step of the a-didactical situation is the "validation" moment where students' strategies are discussed in order to find an agreement on one strategy that leads to a winning strategy. Students build new theorems and they all together decide whether to approve or deny them. This process leads to a new awareness (the mathematical concepts learning) not asked by the teacher and out from the standard school knowledge (we will talk about it later).

Once again it is better to say that an a-didactical situation is not against a didactical context (what I mean for situation is a set of circumstances where a person or a group of people find in, it is also the set of relations that brings together people and environment) but it is a part of it.

The model Brousseau uses for his a-didactical situation is the play model[20] where:

"students are put in a context of free and rich interaction, where they decide whether to share or not to share information, questions, learning methods etc.. Teachers are so involved into an interaction-play with students and their incoming problems".

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[19] The a-didactical situation has not to be confused with the non-didactical situation, that is a pedagogical situation with no particular link to a specific knowledge. Students can play with a mathematical tool but the strategies they use are not specific to a cognitive aim. In a non-didactical situation the teacher does not create an environment suitable to learn a specific knowledge.
[20] We will come back later on this point

The didactical contract is the only rule and strategy of the didactical situation and it is strictly related to knowledge. Often students do not answer teachers' questions on the basis of the content that teachers mean to give them, but on the basis of what they think teachers expect from them.

The effectiveness of the a-didactical situation is in the fact that students have the responsibility to get into the problem, whatever it is, and teachers give them this responsibility. Students are the builders of their own knowledge. When something like this happens we face the act of "devolution" (par 1.2).

This is time when the metaphor of the strategy game[21] comes in; there are many different strategies, but only some of them lead to a solution of a problem and so make possible to a student to gain a new knowledge. Learning happens only through problem solving and for that reason we can say that the theory of the situation is a constructive theory[22].

The a-didactical situation needs some important steps in order to create a new mathematical learning:

1) Action situation: an action situation is into the environment and makes easy to build implicit theories that work as proto-mathematical models.

2) Formulation situations: it makes easy to gain new explicit languages and models, if its social shape is explicit then we talk about communication situation.

3) Validation situation: students are required to solve problems and they make clear and fully explanations about theories and any means they have used to solve the problem.

4) Institutionalisation situation: this situation gives value of truth to knowledge learnt in a classroom; it is usually related to concepts, symbols and knowledge likely to be used at different times and to different purposes. These situations go together with the act of "devolution"; the institutionalisation of knowledge is basically a process that allows students changing their previous knowledge into a new official knowledge thanks to the approval of the teacher that gives them a value of truth and makes it

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[21]D'Amore B, "Elementi di didattica della matematica", Pitagora Editrice,1999, pp. 234-281
[22]Cognitive studies have described the learning process as a student that builds his knowledge interacting with the context and organizing his mental schemes. His role is active because he works on learnt knowledge. This is a constructive approach and is the most widely followed by learning experts, it requires two statements:(Kilpatrick,1987):
✓   Learn and know are both active processes
✓   knowing is an adaptive process and by it someone can organize a rich set of experiences.

possible to use the acquired new knowledge to solve future problems (knowledge and transfer capability).

During the "devolution" phase a paradox makes its appearance into the teaching relation.

"If the teacher demands what he/she wants and the student is not able to get it" the teacher sees it as a pragmatic paradox, while the student finds it a paradox in teaching[23]: "if he accepts that, according to the contract, teachers have only to teach results the pupil has not been able to learn (if he has not reached results on his own he will never get and learn them)". On the opposite if he denies any information coming from the teacher then the teaching relation is broken.

A student learns when he accepts the teaching relation and tries to confute it at the same time, not considering it as a fixed statement.

Only if the didactical contract is broken students can finally get into the learning moment.

But how can we produce an a-didactical situation?

1) Students have to find themselves into a situation where they are not sure of what strategy is the best to use, they can start using a known procedure, but this is not what the teacher wants them to learn.

2) This procedure has to show its ineffectiveness, so that the student can come to a new winning strategy, modifying his knowledge. It actually exists an environment where the student can deny and accept a strategy also thanks to a kind of feedback. A feedback is a correction, an acceptance or a denial of different solutions with the goal of finding the winning strategy in a set of possible strategies. This is an a-didactical environment.

3) The situation has to be repeatable; it is teacher's job to prepare an a-priori analysis considering all the possible strategies that students might use, which allows a quantity and statistical analysis.

The teacher has to be in control of the situation and of every possible change in it. In order to count and control every possible strategy Brousseau suggests using the play model as a synonymous of situation. To do that, he studies all the meanings of play:

1) Free body or mental activity, fiction-based, with no other aim than itself and the pleasure that it gives.

2) Play is the organization of such activity under a system of rules that states a success or a lost.

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[23] Spagnolo F. *Insegnare le matematiche nella scuola secondaria*, La Nuova Italia, 2000

26

3) A play is also the set of tools needed to play; a step of game caused by a specific way to put together the tools.

4) Play is the way of playing; we talk about procedures, such as tactics and strategies.

5) It is also a set of stages in which a player can choose to find himself - see number 2; giving rules and shape to the model is important not only to understand the problem but also to prepare an "emergency" strategy that can modify a-didactical situation.

Describing different a-didactical situation Brousseau analyses:

1) a classified directory of possible person/a-didactical environment interactions

2) a classified directory of the possible environment organization

3) a classified directory of the possible ways of working of knowledge

4) a classified directory of a spontaneous evolution of knowledge

Regarding this last point we need to underline that, while analysing the evolution of the student learning process, it is important to take care of the external environment relations and of any possible relation between old and new inner knowledge of the student. In fact this is what rules adjustment and assimilation problem.

The study on epistemological obstacles gets in at this point.

As a conclusion, the theory of the situation has a goal: giving back to mathematical education its growing and educating role, basic in psychic development especially in problem solving ability.

There are so doubts on the effectiveness of the traditional education way where teaching mathematics would mean a one-way-flow-of-information, from teacher to student.

The theory of the situation suggests a new process that shares mathematical knowledge building; getting deeper into the problem, the aim is to let students know mathematical concepts, starting from problems meaningful to them.

In this perspective students get back the responsibility of the learning process and have a first line role in their own knowledge building process.

The intellectual job done by students under this point of view is close to the researchers' one: they both have to face problems, define them through good questions, try to build models and theories to finally find good and valid answers to a specific problem.

Teacher's role is here to give students the tools they need, to provide conditions that reproduce a "scientific micro society" where our fresh researchers can talk about their

knowledge to build new ones, speak about their hypothesis and give a shape to what they have been discovering.

Theory of the Didactical situations, (shared building of knowledge as a dynamic process)
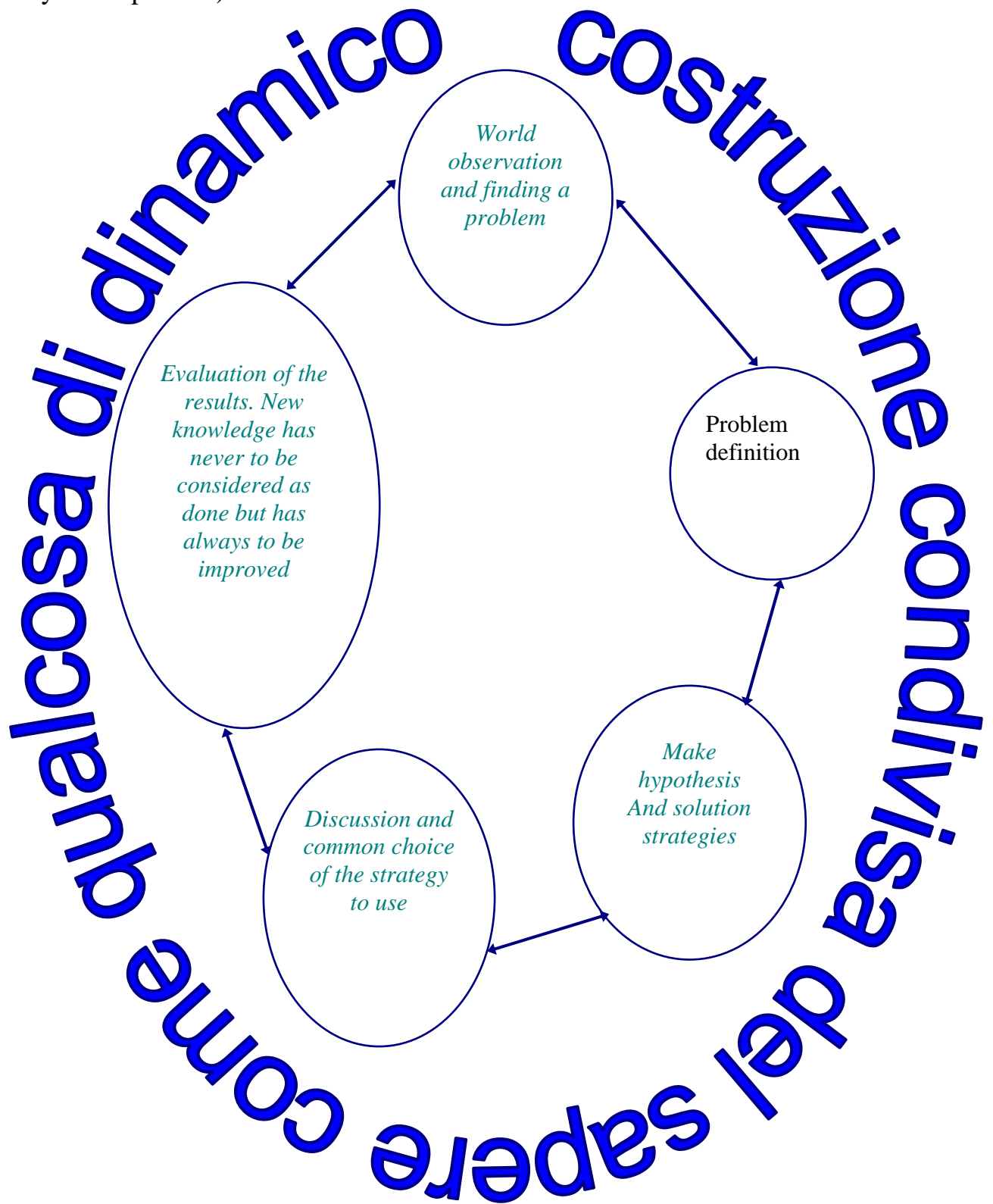
Resuming scheme of an a-didactical situation

In the a-didactical situation the student builds his own knowledge not because he is taught like this, but because the inner logic on the situation pushes him towards knowledge. The didactical purpose of the teacher is not clearly expressed.

I step, the task
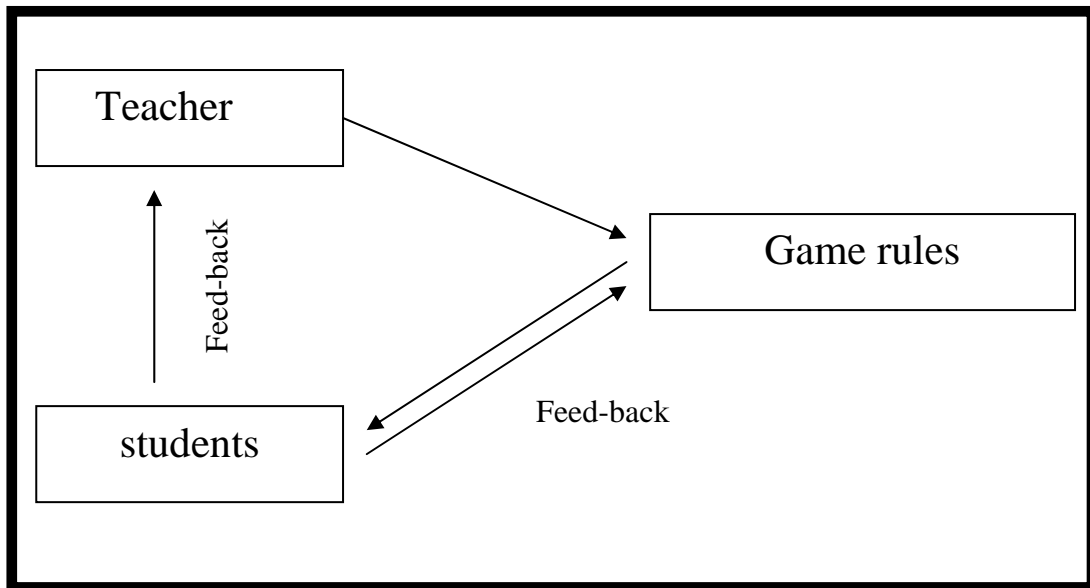
The teacher describes the rules of the play, the problem, the core of the a-didactical situation, showing a practical demonstration. The action in fact reduces the ambiguity of the spoken words. Through the action the teacher can also see the feed back process made by the student that might go back through the situation in order to check out and change something.

Second step: action situation
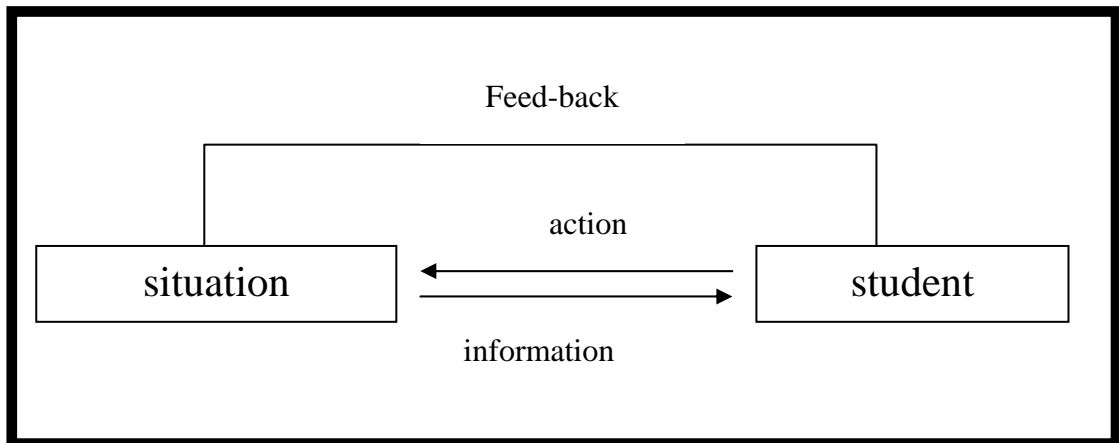
Students start working on the problem and produce new hypothesis and strategies proved by new experiences. The interaction between students and the environment (other students, the problem context, the teacher) is useful to create some first strategies and is called "dialectic of the action".

At this moment students build an implicit model: a set of rules and relations useful to take new decisions without being conscious of it or needing to express them in an explicit way.

Third step: formulation situation
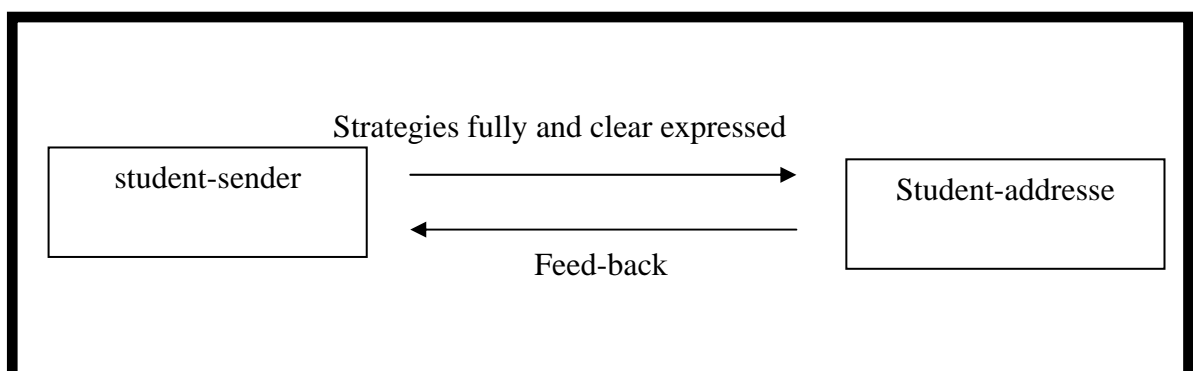
Now the context gives students the chance to create their own implicit model, to express strategies with words, to discuss and preserve them, making other student accept them. To do so every one will have to use a language understood by other students. The communication exchange between students lead them to a keep going strategy creation, we are in the dialect of the formulation.

Fourth step: validation situation

Models that come from the previous steps can be accepted or refused by other students. In their group all students have an equal grade so they can discuss their strategies, the hypothesis they all agree on becomes a theorem.

Students often accept wrong theories, the a-didactical situation lead them to a review of their process to make sure that they use a proper strategy. In this way mistakes are a basic point in the knowledge building process.

With the validation step it is possible to give the mathematical concept a shape that in the traditional way of teaching is a starting and never a ending moment.

1.4 Obstacles in the theory of the situation

If we accept the theoretical idea that students learn from the environment by adjustment, we need to analyse how this adjustment phenomenon can make easy to leave beside old wrong knowledge. Old knowledge often puts obstacles on the way to new knowledge, and we have to analyse how old knowledge can still persist despite the fact that the new one opens new paths to students.

To Brousseau, cognitive breaks are fundamental (the process that changes the implicit models to let new models and knowledge take place of the old ones); every break has to be predictable from the study of the situation and from predicting students' attitude.

Brousseau was the first who talked about obstacles and we are going to analyse this concept as well as the one of "epistemological obstacle".

The science philosopher Bachelard[24] was the first one to use the word "epistemological obstacle" in his book " *La formazione dello spirito scientifico*" (1938); it was the first time

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[24] Spagnolo F., *Insegnare le matematiche nella scuola secondaria*, La Nuova Italia, 2000, pp112-115

that someone used that concept in a scientific work about science history and epistemology.

Brousseau uses the concept of "epistemological obstacle" in a teaching context: the teaching of mathematics, talking about "didactic obstacles".

Bachelard introduces the word to us:

"We have to think about scientific knowledge in terms of obstacles. We are not talking about external obstacles such as the non lasting character of the phenomenon or their complexity, nor to think that it is weakness of meanings or of human spirit 's fault; is the only and simple act of knowing that brings troubles and unbalance within. Is there, where we go slow and back, is there where we find epistemological obstacles".

Bachelard classifies physics' science obstacles:

1) obstacles we know by experience
2) obstacles of general knowledge
3) verbal obstacles
4) obstacles of not authorized use
5) obstacles of familiar images
6) obstacles of unity knowledge
7) obstacles of pragmatic knowledge
8) realistic obstacles
9) soul obstacles
10) quantity knowledge obstacles

An obstacle is an idea that has been effective to solve previous problems when a new concept[25] was born, but it does not have any useful role to solve new problems.

Brousseau (1976) describes some properties of the obstacle.

1) it is not a lack of knowledge; it is knowledge indeed
2) students use it to give answers in a well-known context
3) if they try to use it out of the old context they fail, and realize that they need to have different points of view
4) obstacles create contradictions, but students resist to them, it seems then that they need a more general level of knowledge that embraces the new

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[25] The concept is: a theme of sets C=(S.I,S) where:
✓ S is the referent, the set of situation that give concept a meaning;
✓ I is the meaning, the set of fixed things where schemes are based on and operate;
✓ S is the significant, the set of linguistic and not linguistic shapes that allow to give a symbolic representation of the concept, its procedures, situations and processing ways.

situation of failure. This point must be fully declared and students have to realize this

5) even if overtaken, sometimes obstacles come back again

To better understand this topic, didactical studies have provided a directory of obstacles:

1) ontogenetic
2) didactical
3) epistemological[26]

Ontogenetic obstacles:

They are linked to mental development (often different from the biological age and development) of the person, which sometimes can have some lacks of abilities necessary to cognitive purposes linked to the age. If the lack is only due to a slow mental development (and not to a pathological situation) it will disappear together with growing on.

Didactical obstacles

They are caused by the way a teacher chooses to create a curriculum of studies.

Epistemological obstacles

They are caused by the inner nature of the topic, so that we find them into the concept path itself.

It is possible to understand when a student finds an obstacle in a teaching context or while learning a new knowledge:

1) We are in front of an obstacle when looking at a story of a concept we find a break, a change in it
2) A classic symptom at didactical level is: repeating and persistent mistakes around a concept

If both situations happen at the same time we are in front of an "epistemological obstacle". Brousseau suggests how to find these obstacles:

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[26] Both didactical and epistemological obstacles fall into the category of epigenetic obstacles, or linked to communication

1) Find the repeating ones, and show how they are likely to gather around some ideas

2) Find obstacles in the story of the concept

3) Bring face-to-face obstacles, such a procedure will define they as epistemological obstacle.

The inner nature of an epistemological obstacle is strictly linked to the kind of image that the math teacher chooses to represent a mathematical institutionalised knowledge.

Obstacles properties can so be summarised like this:

1) it is a kind of knowledge suitable to many different situations

2) it is source of mistakes

3) students defend their mistakes

4) obstacles resist to other students criticism on mistakes

5) obstacles are persistent

1.5 Mistakes and conceptions

Brousseau uses Bachelard's idea of epistemological obstacles and changes it into didactical obstacle (in Brousseau thought obstacles are in communication not in mind), this shift gives mistakes a different meaning from the past, they are no longer wrong procedures versus the syntactic rightness of mathematics (e.g. logical rules), but have a positive role of underlining epistemological obstacles.

On the other hand mistakes are not always a symptom of a lack of knowledge, they are often due to previous successful knowledge now useless in a new context. They are not casual mistakes but real obstacles in Bachelard meaning.

The first person to study these casual mistakes was Enriques (1871-1946) in an article, signed with the name Alessandro Giovannini, "Gli Errori in matematica".

Stabler[27] classifies some of the most common thinking mistakes.

Spagnolo and Valenti in a 1984 article classify the most common mistakes that go together with the mathematical thinking development:

1) right concepts but expressed in a wrong mathematical shape

2) right concepts but expressed in a wrong linguistic way

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[27] Spagnolo F., *Insegnare le matematiche nella scuola secondaria*, La Nuova Italia, 2000, p 111

3) right concepts but related to a wrong principle

This summary is not complete and exhaustive but it has certain usefulness when used inside the communication triangle knowledge-teacher-pupil.

We have talked about conceptions; let's try to explain what I mean for conception.

A mathematical conception is a mathematical concept where:

1) the mathematical idea is a statement of a given time
2) the set of meanings related to the context
3) the class of problems where the solution is meaningful
4) tools: theorems and algebraic techniques specific in the concept processing

Regarding students side a conception is about:

1) classes of problematic situation that give their own meaning to the concept
2) the set of meanings that they can use to explain concepts (mental images, symbolic expressions)
3) tools: theorems and algorithms they have, to handle the concepts

Conceptions can be:

1) historical: identified by their mathematical genesis through time
2) collective: known in a mathematical community
3) collective: identified by books, handbooks, and context's interpretation.

1.6 Cognitive conflict in obstacle theory[28]

When a student keeps in touch for the first time with a new mathematical idea he provides an image of it that can be confirmed or denied thanks to his personal previous study experience.

This image will show its ineffectiveness later on studies; the image students think is untouchable, cannot be suitable to new future scenario suggested by teachers. So a cognitive conflict is born.

This paragraph will examine the research now leading about cognitive conflicts and mind images.

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[28] In the thoery of the situation they do not talk about misconceptions but obstacles. Obstacles classification is caused by changes in conceptions. (Spagnolo F., Ferreri M., *L'apprendimento tra emozione ed ostacolo*, lavoro eseguito nell'ambito del contratto C.N.R n. 9001293CT01 p.88-93)

The first thing we have to say is that we are in front of a misconception. A misconception is a wrong concept, something to avoid, even if it is sometimes possible to build a new valid mathematical notion, going through a misconception, that will be adjusted later on.

Let's think about this example:

Triangles are often represented as acute-angles with the base line parallel to the page
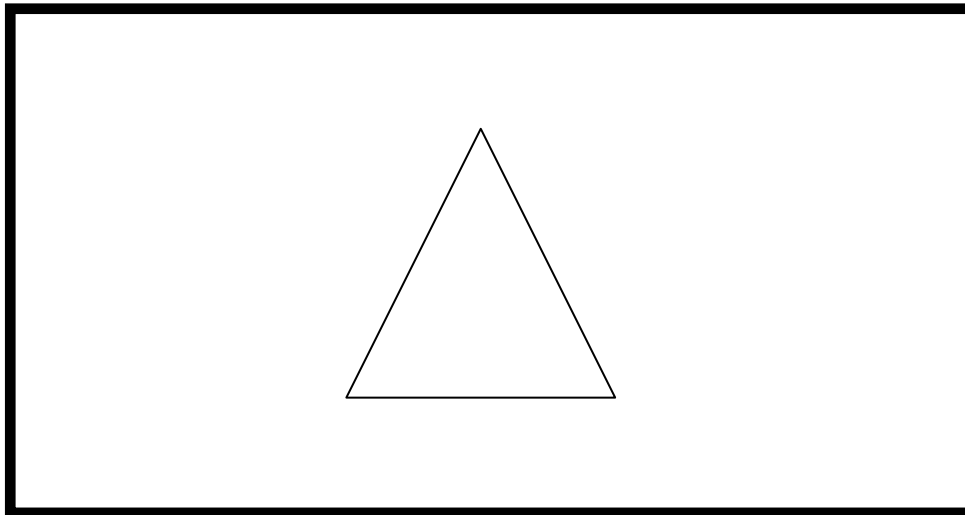
From an experience led in Piazza Armerina[29] (2002) emerged students' problems in drawing heights of obtuse-angles triangles, this did not happen in traditional representations of the triangle.

This example gives the idea of a cognitive conflict between a misconception (standard triangle position) and a new problem situation "draw obtuse-angle heights".

In the experimental side of this job we will analyse another misconception regarding the perpendicularity.

When introducing students to perpendicularity we often ask them to:

"Draw the line perpendicular to line *r* from P"

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[29] G.Manno et alii, "*Le altezze di un triangolo: una proposta didattica*", Quaderno di Ricerca in Didattica, supplemento al numero 10, Palermo 2002.   http://dipmat.math.unipa.it/~grim/quadernosuppl10.htm
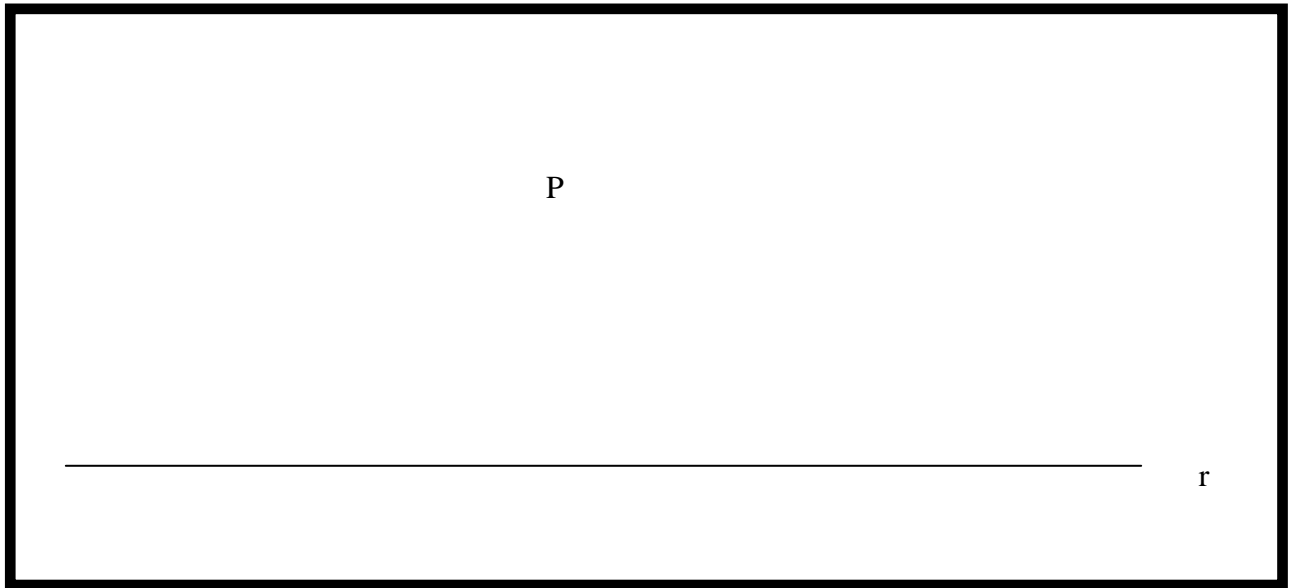
As soon as the reference system is changed, the position of the line *r*, the same problem becomes difficult to solve for students who created a mental image related to the first example (plumb line model), and a new cognitive conflict is born, between the image and a general situation.
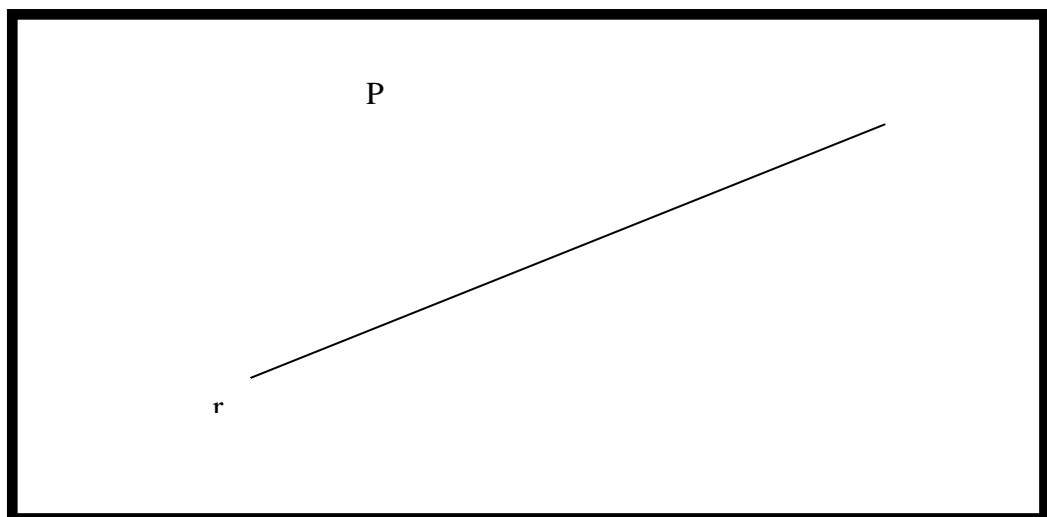
It emerges as a duty to study mental images. Students that build their own mental images believe that they are fixed and unchangeable. But suddenly a new problematic situation or

new information makes inadequate this image and they need to make some adjustments in order to suit to the new requirements of the problem.

Many mathematical concepts are shaped in such a manner, a keep coming mental images series that aims to a new one, fixed and suitable to the context.

When people are able to build and confirm by daily practice such an image of the mathematical concept we are in front of an ultimate image.

In a model shaping procedure it can happen that:

1) The model shapes at the right time, this is the model teachers were looking for and students succeed.

2) The model shapes too early, when an image cannot survive other stimuli, the model is not solid and it is a misconception itself, or a misunderstanding of the given information.

Finally we have to say that a model property is its firmness.

We have seen that to shape a new model students have to get through conflicts. They allow them to adjust the new image to the new cultural context and requirements.

What is a conflict caused by?

We may say that:

A cognitive conflict is an inner conflict caused when two concepts, or two images or an image and a concept, or an intuitive model and a mathematical model do not get along.

If one of the above scenarios happens students tend to preserve a previous image that has been successful before the new cognitive event.

A conflict can also be a social entity; a student can have a misconception shared by the all class (just like the a-didactical situation experimented in this work).

Such a model conflicts with the one suggested by teachers, and now students realize that they were wrong about their previous ideas: it can happen that, while the entire classroom accepts the new model coming from teachers, only one student keeps following his own model refusing the conflicting teacher's one.

Conflicts are misconceptions based, mistakes can then tell teachers about students' misconceptions and allow teachers to be conscious of wrong models held by students.

(Coming back to the perpendicularity experience consider that what we have said up to now is a general theoretical framework, children too have naive mathematical conceptions, mental images, which are processed to build a model. To prove it we chose to operate on children from I and II elementary grade that still have not a formalised perpendicularity

concept, and on III-IV elementary grade and I intermediate grade where teachers have already spoken about perpendicularity).

It is worth to dedicate these last words to that question:

"How can we explain a cognitive development in a student mind?"

Our cultural references are Piaget's studies on balancing and adjustment.

The cognitive process is seen here as a search of organization pushed to an increasing general value; that is to say that all personal experiences, learning, out side interactions, become of general value. We can speak of knowledge building.

Even tools needed to build knowledge through, are more and more general and independent from the object of knowledge. A specific knowledge adjusts itself to a wider use and meaning.

Organizing and adjustment functions seem to have a strong importance, the last one includes two more functions: "assimilation" and "agreement".

Students take in new information and knowledge adapting them to a previous scheme and vice versa.

If this process is unsuccessful, then students have to change their previous schemes in order to gain new knowledge, this happens when the cognitive conflicts pushes enough to learn and change old models.

It comes from this a relational attitude of knowledge:

1) the take in process (assimilation) can be seen as a motion towards the learning-person

2) the agreement process is an out going process that defines the object of knowledge.

Fig.8 How we gain a knowledge

If there is a conflict between two concepts

⬇

People organize and adapt old knowledge to embrace the new ones

⬇

The adjustment process is done by two more functions: The take in and The agreement

"The take in" uses objects and events and put them into already known scheme

The agreement modifies new objects and put them into old scheme

People move on knowledge path
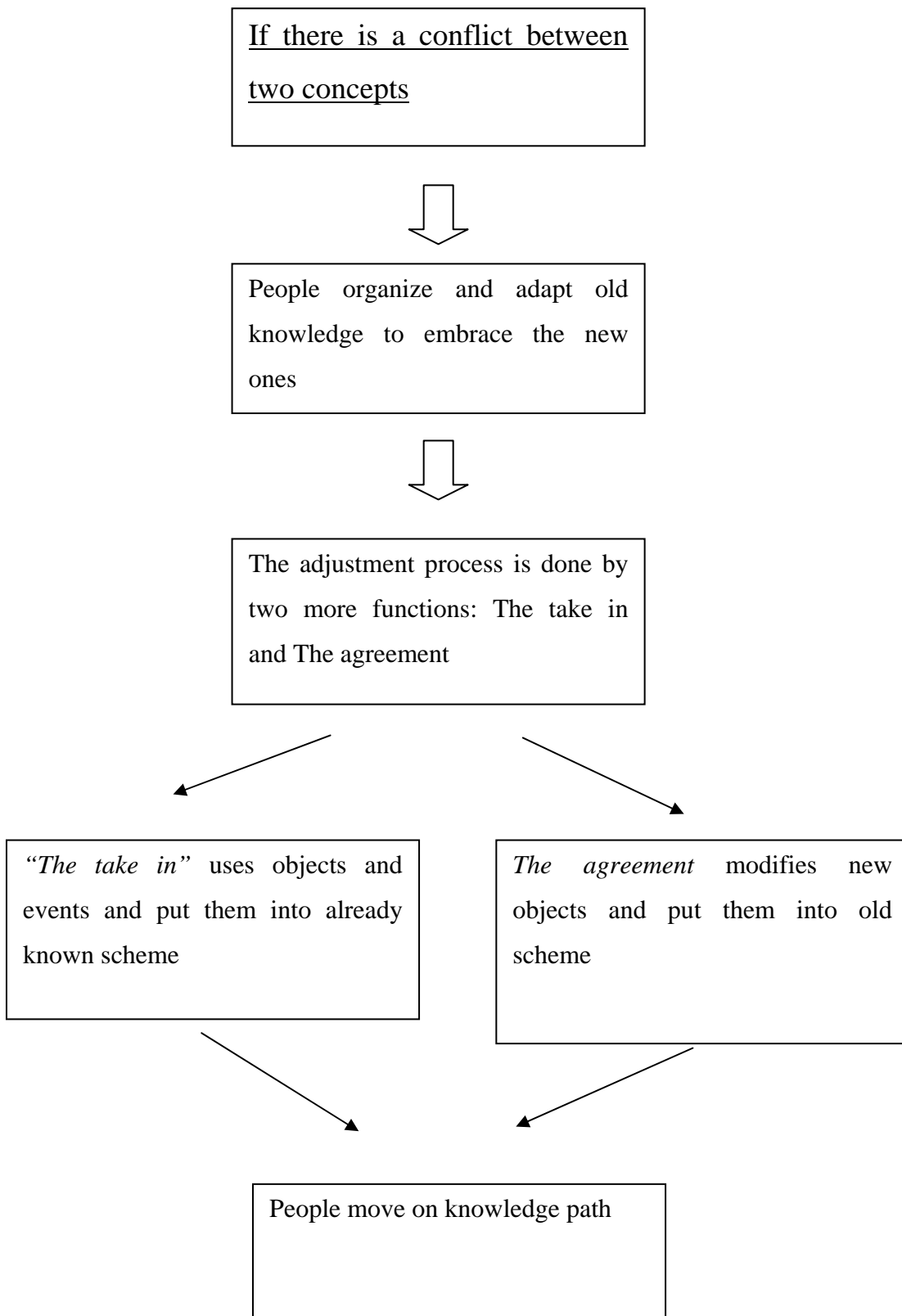
CHAPTER II

Conflict between mathematical language and every day language in students[30]

2.1 Introduction

In this Chapter, I referred to some works by H.Maier, who puts in evidence the possible causes of the concflict between the natural and mathematical language.

In the first chapter we have talked of epistemological and didactical obstacles; we have explained how they are related to the problem of mathematical communication.

This chapter will try to give some answers about mathematics communication in a classroom.

1) Is it mathematics a language itself?
2) Is it strictly necessary to use a formalised mathematical language?
3) What are the properties of mathematical language?
4) Do students experience an inner conflict between mathematical language and every day language?

2.2 Mathematics as a language

Many authors think that mathematics itself is a language or, at least, has to be thought like this. As a language we can use the typical language and communication tools to study it, such as semiotics.

Let us try to define what is communication: a place where a sender and a receiver come together into a new semiotic relation. We think of a language as a set of tools that makes easier to both, sender and receiver, to build this kind of semiotic relation.

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[30] Maier H., (1993), *Il conflitto tra lingua matematica e lingua quotidiana per gli allievi,* La matematica e la sua didattica
Maier H., (1993), *Problemi di lingua e comunicazione durante le lezioni di matematica,* La matematica e la sua didattica

It is better here to recall some concepts about semantic.

A sign is a meaningful unit, which is interpreted as 'standing for' something other than itself. The sense refers to the sense made of the sign.

The semiotic relation (Charles Morris)[31] refers to the relation that links a sign to a sense.

The semiotic act is a set of actions, which users use to establish a semiotic relation. It is better known as semiotic process, it can follow two directions:

When the user is the sender or addresser (the one who produces the semiotic relation) he starts from a sense and then produces a sign; this kind of process is called productive semiotic process.

When the user is the receiver or addressee he starts from a sign elsewhere produced and he has to give a sense to this sign. This semiotic process is said receptive.

This second possibility is the one we find in the triangle pupil-teacher-knowledge, the focus of this relation is, in my opinion, the need for communication into this system.

Students' role into communication is to give sense to a sign, a much harder job than giving sign to a sense.

Students have also to work with these new senses and signs in order to change them into a fresh semiotic relation, themselves becoming the new senders.

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┌─────────────────────────────────────────────────────────────┐
│                                                               │
│                                        ┌──────────────────┐   │
│                                        │ Productive        │   │
│                                        │ process           │   │
│      ┌──────────────────┐              └──────────────────┘   │
│      │                  │  ───────►                           │
│      │ semiotic relation│                                     │
│      │                  │  ───────►  ┌──────────────────┐     │
│      └──────────────────┘            │ Receptive         │    │
│                                      │ process           │    │
│                                      └──────────────────┘     │
│                                                               │
│   ┌──────────────────────┐        ┌──────────────────────┐   │
│   │ Teacher: sender,     │        │ Students: receiver-  │   │
│   │ mediator between     │  ◄───► │ sender of a new      │   │
│   │ students and knowledge│       │ semiotic relation    │   │
│   │ (thanks to the didactical│    │                      │   │
│   └──────────────────────┘        └──────────────────────┘   │
│                                                               │
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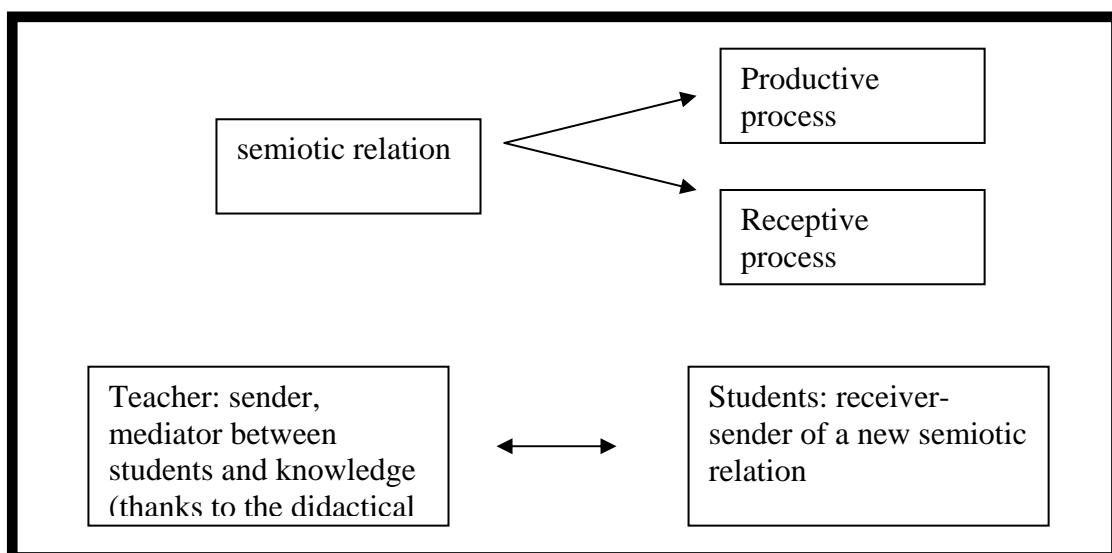
[31] For the definition of semiotic relation by Charles Morris you can see De Mauro T, *Minisemantica,* Editori Laterza, Bari, 2000.

Considering that mathematics do have:

1) a syntax (relations among expressions)

2) a semantic (analysis of expressions and their meanings)

3) a pragmatic (interpreting signs considering the real users of the language)[32]

We can properly say that mathematics is a language. Teaching is so a way of communication and it is necessary to study wider problems of human communication pragmatic, and also specify what we mean by language.

Communication is only a possible way among several, to consider language.

There are at least four different ways to define the word "language":

1) as language, a semiotic system with its own working method

2) as different ways and shapes of conversation used by speakers (a story tell, an explanation etc.)

3) as a general function of communication among people

4) as a code wide spread and known

When talking about language and thinking we refer to Piaget and Vygotskij[33].

Oral spontaneous language:

✓ to Piaget it is a tool to study the logic of the subject, because the use of a language needs a logical work, this is the only link between language and thinking

✓ to Vygotskij it is a stimulus to thinking process because children live into a speaking world where language is the way to relate to adults.

Considering the problem of socializing these two authors have deeply different opinions:

✓ to Piaget it goes together with communication: sending a message to an addressee

✓ to Vygostkij socializing goes with the proximate developing site, in other words the distance between the proper developing degree level and the potential level determined through problem-solving procedures under an adult or better performing mates guidance.

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[32] Syntax, semantic and pragmatics are the three branches in which is articulated Semiotics, the science studying signs and languages.

[33] Vygotskij Lev S., *Il processo cognitivo,* Universale Bollati Boringheri,Torino,1987, Postfazione di Steiner e Souberman, pp175-186

Where the two authors do not agree at all is the concept of self-centeredness.

✓ It is related to meta-discoursive function of language to Piaget. It is the character of someone speaking not considering people listening and their comprehension. It is also related to the ontological function of the inner individual conscience belief or so to say the individual property of conscience and knowing.

✓ Vygotsij considers the meta-discoursive functions of language and develops the concept of external language (communication devoted, intended to make someone understand what we are saying) and inner language (unknown to anyone but ourselves). The self-centred discourse is a way of inner discourse.

2.3 The language of mathematics in a classroom

The difficulty in learning new mathematical concepts is often due to the uniqueness of math language in contrast to every day language.

It is about facing new words, or old words with new meanings. Maths has its own formalised language aimed to transmit a mathematical thought. Thus this language cannot be used as a common communication substratum between teachers and students, it is too formalised to do it, but it is this formalised language that must be taught to students.

We so face a typical didactical paradox[34]: the paradox of specific language.

✓ Teaching is communication and it has to make easier to students to learn, so a communicator should use a language easy to understand, the solution seems easy: just use every day language while teaching maths

✓ But the aim of teachers is to make students not only understand but also gain that new specific language, it is so impossible to keep them far from it.

How to solve this paradox?

Often teachers mix from the very beginning natural language and math language and a new linguistic code (the last one is the one D'Amore calls *"matematichese"*).

That specific language really exists, every one can be conscious of it by simply opening a math hand book where we find some words like: *"passante", "intersecantesi", "dicesi"*

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[34] D'Amore B., *Elementi di Didattica della Matematica,* Pitagora Editrice Bologna, 1999.

(words used only in math language, used to indicate something crossing, or *dicesi* that simply means "we say").

Students reproduce this way of talking in an almost-model of mathematical language but they often pay a price in losing sense.

But in a real classroom situation we find that neither natural language nor mathematical language is used.

There is also the symbolism problem; many researches show that primary school teachers cannot find any difference between the concept, the symbol and the algorithm that represents it.

As an example we can consider the division: students clearly know the meaning of it and the symbol used, but they have problems in handling the algorithm. It would be worth to pass elementary school grade before introducing the problem of the specific language.

2.4 Properties of mathematical language

The theoretical hypothesis we follow in this chapter is that cognitive development and language development go together (Vigotskij).

Colette Laborde[35] gives us a real didactical problem to deal with: it seems impossible that a student can learn by osmosis math language, it is more suitable that a specific didactical action is required to reach this goal. It is also necessary to use formalised language according to students' age and their knowledge level.

And now move on math language analysis and where it is different from every day language.

2.4.1 Mathematics as an ideal reference

This is a property of mathematical thinking and language; they both do not refer to "real" objects, all mathematical concepts are rather "ideal" than linked to existing things.

In math real objects are not a representation of an idea, but some of the ideas can be represented by things or models or real shapes, such as a triangle or a cube, or they can be mentally built starting from real experiences.

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[35] D'Amore B., *Elementi di Didattica della Matematica,* Pitagora Editrice Bologna, 1999

OBJECT $\Longrightarrow$ MATHEMATICAL IDEA

MATHEMATICAL IDEA $\Longrightarrow$ REAL OBJECT

| Model | Sensitive experience | Semiotic representation |
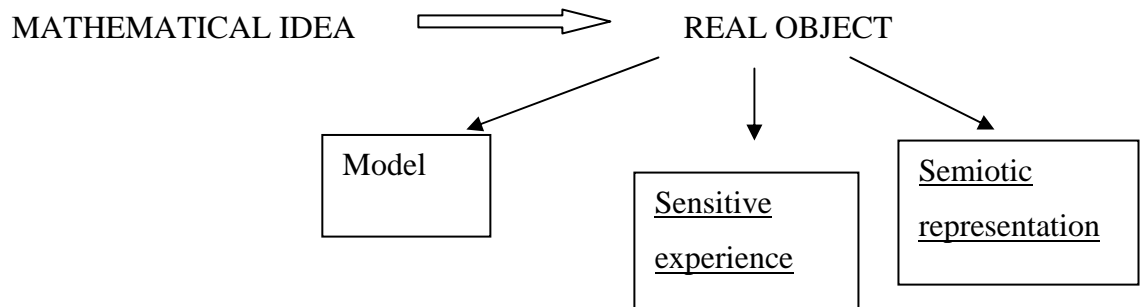
Models are different from the idea itself because of:

- ✓ the abstraction of the idea (moving from an object to a class of them)
- ✓ the idealization (moving from the real object to its ideal image thanks to the intermediate step of the model).

Mathematical language and thinking overcome sensitive experience and they are independent from any link with reality (most of math ideas have no relation to real shapes, nor are based on a relation between objects or sets only formally defined). On the other hand theorems in mathematics are discussed and demonstrated on logical deduction basis not using real objects or models.

This is why every one should not abuse of models and images to make students learn mathematical ideas in an easier way.

Thanks to this property, of being such an ideal discipline with few links to reality, mathematical concepts have to be widely discussed, giving speeches a big role; the most articulated of these speeches is the Definition (nothing can give a wider and more precise description of the mathematical idea and its most suitable use).

2.4.2 The "not ambiguous" mathematical language

The ideal being of mathematics involves another specific issue of the proper language to use: the not ambiguous.

It means that

✓ Every word or symbol meaning is clearly and fully explained and defined, its edges are clear and known

✓ Every word or symbol has only one meaning

✓ Every meaning has only one word or symbol to identify it

2.4.3 Common meaning and the one of that mathematical language

We are used to thinking that mathematical language is so clear because it uses specific words and symbols different from the ones we use in every day language, they have a given and unchangeable meaning; this statement is false for at least two reasons:

1) Math authors can give words different meanings but this never happens in the same job, so we can say that words have their specific and clear meanings.

2) Math authors can use sentences taken from every day language, but the meaning is deeply different. As an example think of the word "similar" (here it is referred to shapes in a proportional relation whose angles have the same degree), "among" (its sense is wider than in every day language, here it refers to a set of real objects and to a relation among close elements; in math definitions every object of the set is thought as "among" every couple of objects coming before and afterwards).

2.4.4 Self-sufficiency, consequentiality, density

Most of the problems found in learning and understanding mathematical language come from the use of every day sentences or words in a different context and with a different meaning. This different meaning is confirmed by the fact that the old words have a new definition that explains their new sense.

In its ideal way mathematical language and its interpretation, should be independent from anything that is not defined, definable or controllable, authors try to prevent readers from moving to a different context in order to understand what it has been written.

Another property of this language has to be the self-sufficiency; it has to contain all information readers need to understand what it is meant.

Another one is the logical consequentiality; a sentence makes the consequence impossible if the consequence is not prescribed by the definition itself.

We can say that a text is not consequential if some of the sentences it contains are in contradiction.

The last property is density; which reaches high level in mathematical language due to the use of symbolism, math language does have a proper semiotic code that explains the huge use of symbols or formulas; this code has mainly two purposes:

1) a fixing function (when we need to name an object) there are some simple fixing functions such as a capital letter or more complicated such as $f(x,y)$ that is a set of single fixing functions in a bigger one with precise syntactic rules:

2) a place function, if we write $[a,b[$ we are not simply defining an interval but we are also saying that it contains a and not b.

What kind of symbolic style do students use? Do they use it? And if yes is that spontaneous?

Researches in this field have shown that they tend not to use it, they prefer natural language and this happens following some strategies:

1) what describes the object is reported word by word

2) students refer to chronological events

3) students use extra-mathematical properties

It would be very interesting to find instruments that focus on the real understanding of a text by students:

1) word problems, i.e. problems where common words and language are used

2) exercises where it is needed to express what is the quantity known and the unknown

3) behaving studies of students coping with words fundamental in understanding the entire sentence, and the solving strategy.

It is all about studying the a-didactical situation specific to a language learning. In order to do this we use the communication process inside the theory of the situation.

To resume what we have said until now just use some words from Laborde that finds these properties of mathematical language:

1) precision
2) coherence
3) universal value

2.5 Mathematical language and communication in a classroom

What we have said shows the deep inside danger in not following the math language rules.

We do not have to follow those rules as a given input, but only as long as this is useful in understanding problems or technical issues, theorems or theories, explanations and comments.

Math language rules have to pursue an increasing communication level in a classroom. Students have to increase their comprehension level according to their cognitive development level and also increase their skills in communicating mathematics.

A possible didactic strategy could be neglecting math language properties; a teacher or an author can use ambiguous way of talking to push students towards a clear understanding. It could be a choice related to an approach that prefers a practical way rather than a theoretical one.

Maier does a very detailed analysis about the use of every day language in a mathematical context showing how the first one interferes in the second one and how students fall into a confused context that hand books often make more complicated. Many of the difficulties found by Maier have to do with communication problems.

CHAPTER III:

A teaching proposal, the perpendicular concept and the vertical concept

3.1 Introduction

New studies[36] carried on maths teaching theories have shown (Maier, "*Conflitto tra lingua matematica e lingua quotidiana negli allievi*" – "A conflict between maths language and daily language among students") that problems students of all classes often find in understanding maths concepts are mainly due to the closeness of the two languages, the mathematics one and the daily life (natural) one.

As an example think of the word "vertical", in maths language it refers to a very specific relation between two lines, perpendicularity, which has to satisfy two essential requirements

1) reference system
2) making four 90° degrees angles

In every day language "vertical" refers to the direction of the centre of the earth.

Students often think of this second meaning when they are in a mathematical context and so change the position of the line regarding what they are asked to draw the perpendicular.

It seems full of interest a deep scan of such a phenomenon to find an answer to some questions:

1) Do pupils have the inner model of the plumb line?
2) Does the misunderstanding of the notion of perpendicular come from a linguistic misunderstanding?

We will try to answer using the "theory of the situation"

The study reference of my job has been the systemic approach knowledge-teacher-pupil (first chapter).

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[36] See Maier H., already quoted.

Studying possible reactions involves the research so far done, about teaching maths, which does have contents, methods and paradigms too.

To know the paradigm of the research (here in a brief abstract I introduce the one I employ, from French professor Brousseau modified by professor Spagnolo) we require some statements:

1) a suitable language
2) methodological tools
3) statistical tools

In teachers' point of view it is important to know any research improvement and tools because they allow to:

1) become acquainted with the positive role of analysing students' mistakes when dealing with a specific demand
2) build a theoretical body of knowledge to refer to in problem solving
3) make students independent when they have to choose in a hypothesis set the best strategy
4) get tools to share processes and research results

Research in teaching theories is so an enriching element in a teacher professional profile. It is so very important to:

1) find out research stages in teaching theories
2) build some kind of a-priori analysis of the problem-situation
3) produce effective research hypothesis related to the a-priori analysis
4) bring out an evidence that people not involved in the research experience can use to get the model procedure used in teaching research.[37]

3.2 Research goals

The aim of this research is to find out different notions or ideas of the concept of perpendicularity that students have, whether if they are spontaneous or coming from teachers.

The real first question my work is based on is:

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[37] Quaderni di Ricerca in didattica N.8 Pa 1999.

Do students have an inner model of perpendicularity between lines regarding whom the notion of perpendicularity is the same as uprightness (vertical)?

From this point of view the reference model would be the plumb line, which is linked to the concept of vertical instead of the notion of perpendicular.

It interesting to read perpendicular and vertical definitions' given in maths dictionaries. These definitions are taken from an Italian dictionary (Rizzoli) and literally translated into English:

✓ **Vertical:** late Latin, *verticalis*, that goes from a top to a bottom, vertical is any line that follows the plumb line direction. Any plane that contains a vertical line is itself a vertical plane.

✓ **Perpendicular:** from Latin, *perpendiculum*, plumb line, it comes from hang, exactly weigh.We can assume from the above words that perpendiculum means plumb line, but its direction does not describe the concept of perpendicular but the vertical one. In every day language indeed perpendicular is an adjective that refers to something that has the direction of the plumb line, a synonymous of vertical. The following analysis takes in great account what we have said until now, and allows to assume that one of the obstacles in learning maths is the closeness of its language and every day language such as the two words perpendicular and vertical. Often students compound different words, they take words from natural language and use them in a maths context without changing their meaning or, vice versa, they use maths words with their daily language meaning.

The hypothesis I start from is:

H1 vertical and perpendicular are synonymous to students

H2 if teachers formalise the concept of perpendicular (in a maths way) students become acquainted to the concept as long as they use in a proper way the concept of the plumb line and are able to adapt it to the new situation. If students do not break this epistemological obstacle (implicit model) they will not be able to solve a given problem where the reference system is not a line crossing the centre of the earth.

H3 To build a particular "*milieu*", in consideration of the Theory of situations, it can be a contribute to create the didactical way more opportune for the correct and persistent

formation of the concept. Is it possible to build a context where the perpendicular straigth lines are conceptualized as minimum distances?

We can easily think that obstacles hard to overtake are born in a very early time of psychic and physic development, and teaching action produces rather an instruction than a conceptualisation.

3.3 Research tools

On the first stage of the research we gave students an open test made of four questions. To be sure of the independence of their answers from teaching or research stimuli, students had been told that their answers would have helped teachers throughout their job and would have been evaluation-free.

That has been a winning strategy because it has made students free from any emotional impact. All the students involved joined the research, a very positive result itself, considering that schools involved were all at risk (so to say, schools were dealing with a difficult social and economic context).

Further more we used cameras to shoot students over the period of research, and this has been fully accepted.

On the second stage we created an a-didactical play situation, called play-path where students had to play Tom and Jerry characters.

Teacher used the gym to build paths and ways as they were shown in the test and asked students to find out where Tom and Jerry should be in order to let Tom catch Jerry.

Students had so been able to experiment that the concept of perpendicular is not linked to the plumb line model but to the concept of minimal distance.

We followed all the stages implied in an a-didactical situation:

1) action situation
2) formulation situation
3) validation situation
4) institutionalisation situation

3.3.1 Tom and Jerry test

Step 1

Give students some questions (paper sheet and pen)

➢   Tom is sleeping on a sloping roof when he sees Jerry walking on the sidewalk. In your opinion where will Jerry be when Tom jumps from the blue point sure of catching him? Please explain why you answered the way you did.

➤ Jerry, orange point, is walking on a roof and Tom, blue point, is hidden waiting for him to arrive, when does the cat jump sure of catching the mouse? Please mark the point with a pencil and explain why you answered like this.
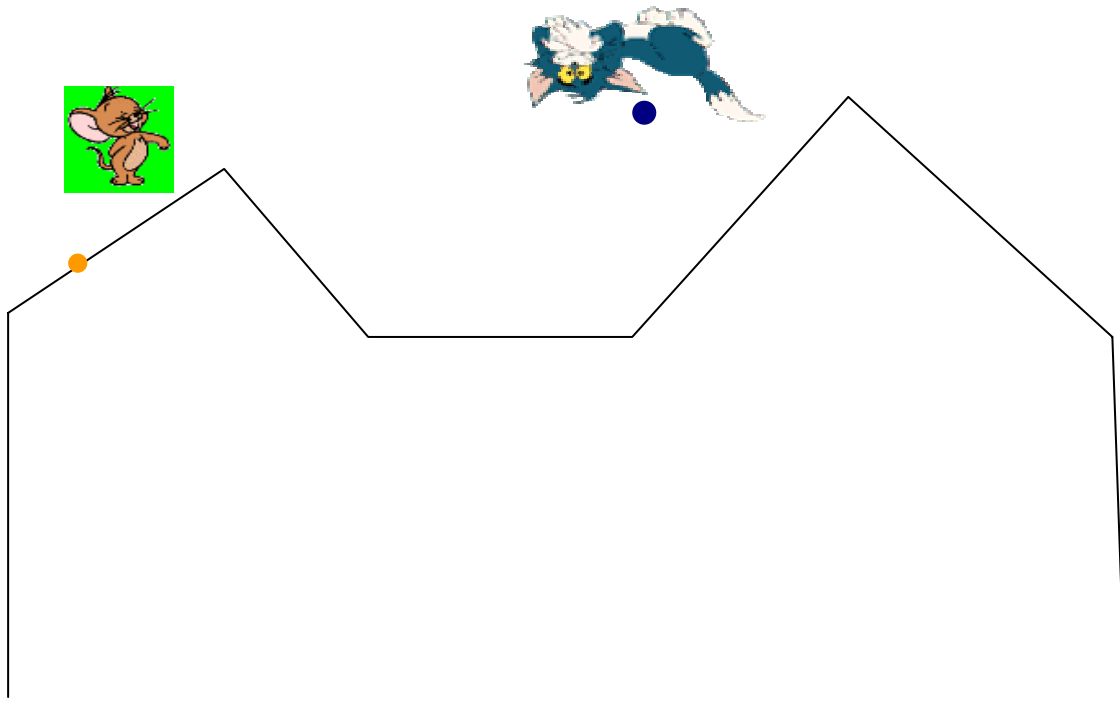
➢ the points you see are Tom (blue one) and Jerry (orange one). Tom is hidden waiting for Jerry, where will Jerry be when Tom jumps sure of catching him? Please mark the place with a pencil and say why.
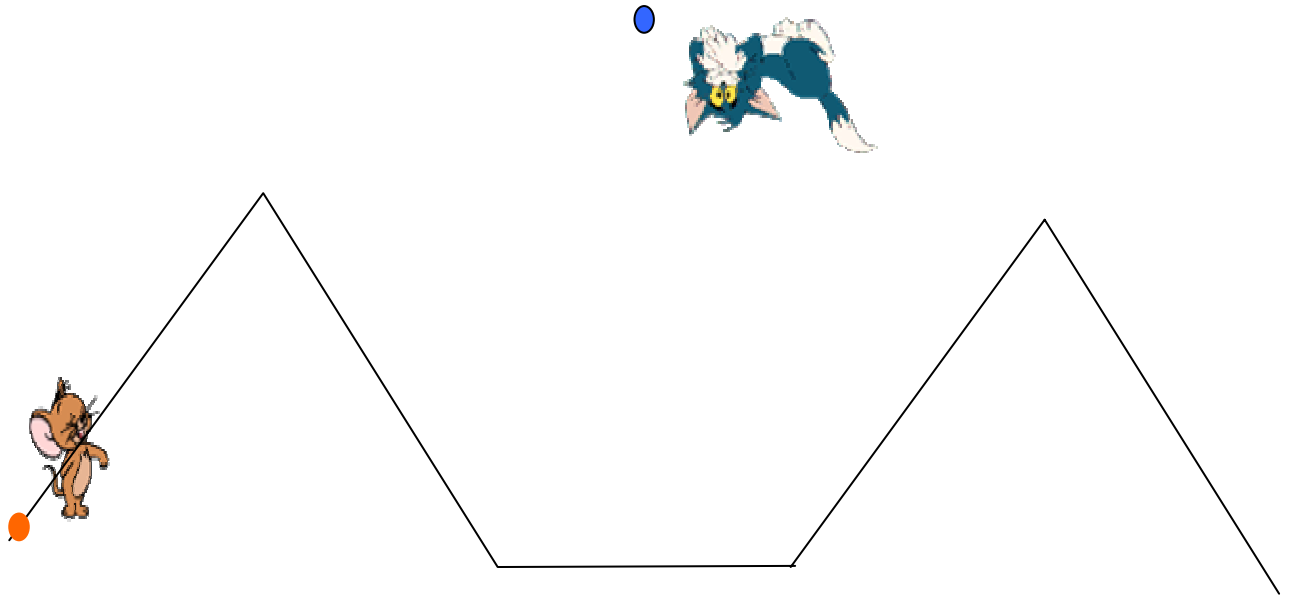
➢ Tom and Jerry are at the sea. Jerry runs on the beach and Tom is relaxing on the baywatch seat. Please mark the position of Jerry when tom jumps off the seat sure of catching him. Say also way you answered like this.
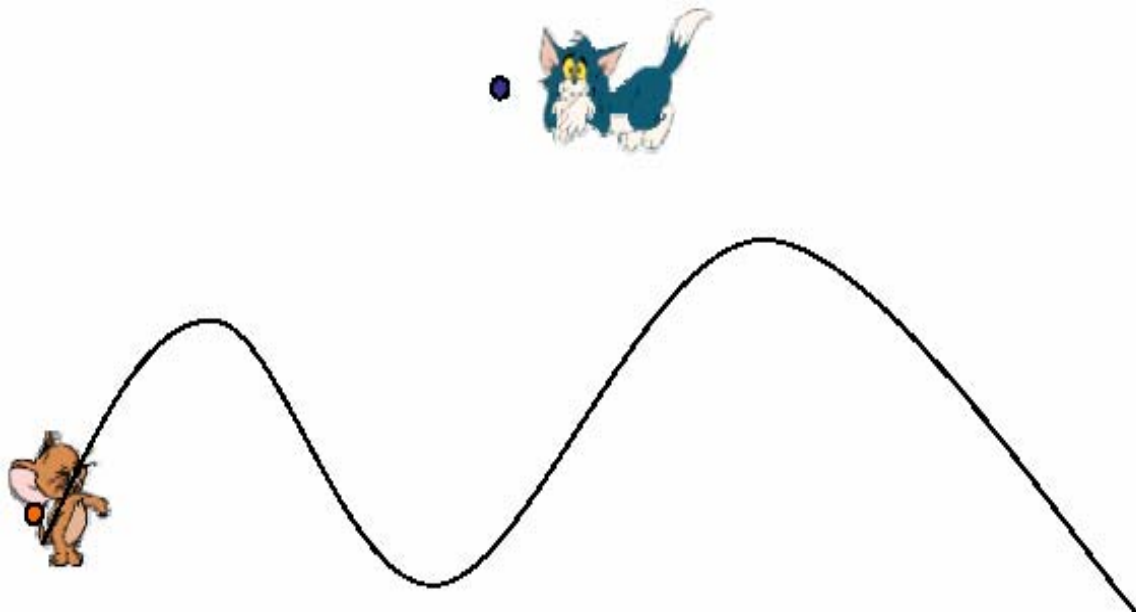
3.3.2 PLAY-PATH

Step 2

In the gym there are paths similar to the ones in the given exercises.

- ✓ matter used: sellotape, paper sheets with the path to go through, rope and string
- ✓ teams: every team has five members, three of them have been given the role of tom and jerry and a time keeper, the two left have an observing role
- ✓ time-in of the play: one of the children keeps the time clapping his hands. The pink child goes a step forward every time he hears a clap. The yellow child waits keeping his position as in fig. (make children repeat the path several times and give them a string telling how to use it, or say that it can be useful to solve the problem).

You can see a scheme of the play-path:
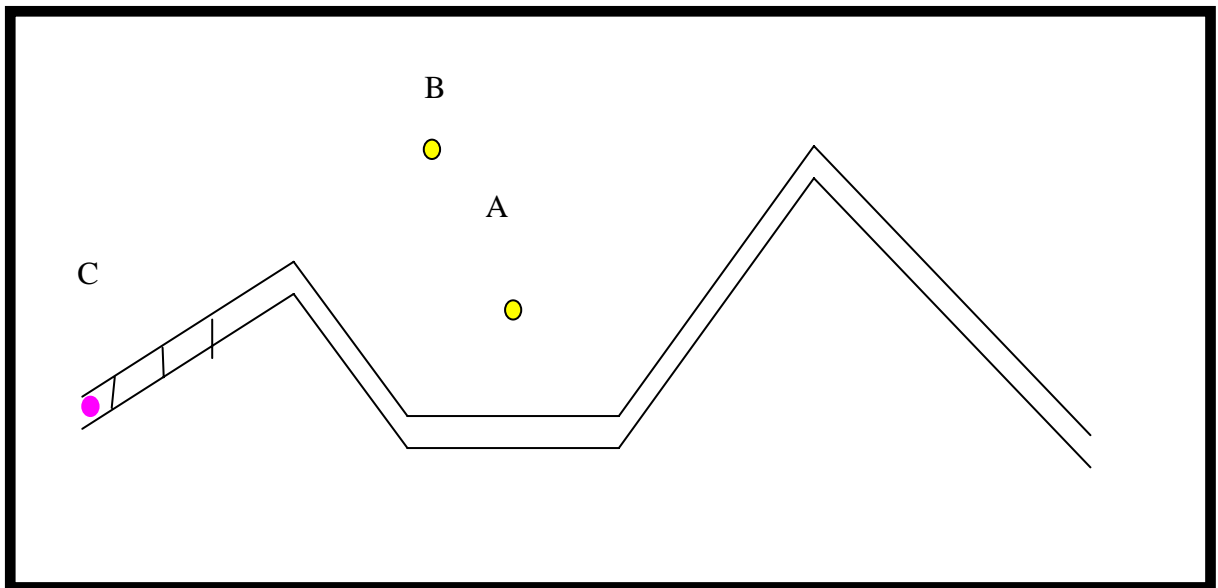
The pink point shows the beginning position of the child playing Jerry, A, B, C indicates different beginning position of Tom-child, as they change every time the play begins.

- stage 1 explanation and demands: teachers explain the game, simulating every step and the rules demand: two of you play Tom and Jerry, Tom sleeps waiting for Jerry to arrive, Jerry walks. In what position are Tom and Jerry the closest? Left members of the team have to draw the game in order to understand what strategy suits the best. Who finds the winning strategy wins the competition. How can Tom realize the best moment to jump and catch Jerry? Find out a method that allows Tom to catch Jerry wherever the mouse is.
- Stage 2 (action situation): different teams draw on a sheet the play situation finding the right strategy
- Stage 3 (formulation) different teams socialize sharing their strategies
- Stage 4 (validation) teachers ask to express the found strategies, some shared conjectures are required in order to become theorems, in this stage it could be useful to analyse the use of the string.
- Stage 5 (institutionalisation) a new theorem is born, as a solution or winning strategy, shared by everybody.

3.4 Students involved in the experience

This experience is part of the S.P.O.R.A. project whose leading school is the D.D. Ferrara. It is worth to shortly analyse the socio-economic context before moving on.

The "centro storico" district of Palermo includes old markets and places such as Ballarò, Vucciria and Capo. Inhabitants of this area often live a deep disease situation. Children's parents often do not have a regular job, and when they eventually find one, it is a low qualification job. The analphabetic level, whether primary or secondary, is high (centres EDA to get primary and intermediate school degree have been running for years in this area).

There is also a consistent percentage of foreign people, from Africa, India and China too.

Schools involved in this research project have shown throughout years the highest level of scholarship lost in Palermo neighbourhoods.

The following scheme resumes students and teachers involved in the project.

surname	name	Teaching subject and place	Grade	Foreign students	origin	Students with special requests	Number of students in the class, section or schedule
Bua	Sonia	Childhood school	3 year		Various	Yes	21
Rao	Margherita	Maths and logical	4°year primary				31
Ruvolo	Francesca	Maths and logical	2° year primary	1	Bangladesh	Many	24 23
Troia	Maria	Linguistico /espressivo	1° year primary		/	/	/
Gallina	Rosetta	Linguistic	3°/4°year primary		/	10	35
Tabbi	Antonella	Maths and logical	3°/4° year primary		/	10	35(scheduledteaching)
Caliò	Giuseppina	scienze mat	1°/2°/3° year intermediate level		India latin america/Africa	10	22

3.5 Test analysis

The test consists of four questions that require drawing a line that joins Tom and Jerry position when Tom is sure to catch Jerry.

That line is the one that crosses the point where Tom is and is perpendicular to the line where Jerry walks in.

The first question of the four is quite easy because in this case vertical and perpendicular are the same.

Second question requires finding out the perpendicular line when Jerry walks on a slanting line (changing the reference system from the first question).

Third question is similar to the second one but it is open to two different solutions; and this is quite interesting because children think (didactical contract) that teachers cannot give twice the same problem and tend to answer in different ways.

The last question is an open problem because it is about finding the perpendicular to a curve crossing it in a point.

3.6 A-priori analysis

To have an a-priori analysis we thought of different representations of the concepts got by students, we considered all the possible answers and of course we did also an analysis of possible epistemological pathways related to the concept of perpendicular.

A1	Finds out the perpendicular
A2	Starts when Jerry is at the beginning of the path
A3	Draws the shortest path without thinking of Tom and Jerry position.
A4	Guesses, draws a line that is not perpendicular because it does not make 90°angles.
A5	No answer
B1	Draws the vertical
B2	Finds the perpendicular
B3	Joins Tom point to the top
B4	Draws a parabolic line that connects Tom point to a top.
B5	Traces a slanting line
B6	Draws a line that connects the cat to the highest and closest to Jerry point.
B7	Draws an horizontal line
C1	Finds a perpendicular
C2	Finds two perpendicular
C3	Draws a vertical
C4	Connects the blue point to the tops
C5	Draws a slanting line from the cat to the slanting line
C6	Draws a slanting line from the cat to the horizontal segment
C7	Draws a line from the cat to highest and closest to the mouse top
C8	Draws a line parabolic to the slanting segment
D1	Draws the vertical
D2	Finds the perpendicular
D3	Connects Tom point to the highest and closest point to Jerry
D4	Connects Tom point having an extreme in a point in the concave side of the hill
D5	Draws a segment that joins Tom to the highest point of the hill e further from Jerry
D6	Draws any segment

3.7 Evaluation criteria for the quantative analysis

We prepared an evaluation scheme (see a-priori analysis) where every correct answer has been marked 1 and every wrong 0. We have done this to better handle data. Test results have been written in an excel sheet, after processing data we did not consider answers containing only 0.

3.7.1 The Qualitative and quantitative analysis

Demand: children generally have been able to answer and give explanations, the analysis of the entire body of tests, so to say their percentage, would let us think that finding the minimal distance between two points is the same that finding a vertical line crossing one of the two given points.

After a first glance on children explanations we can assume that the concept of vertical is seen through the lens of daily experience of gravitation.

Children know that if they let an object falling it goes towards the floor, thus if Tom jumps when Jerry is under him he will catch him. Students consider the vertical position between Tom and Jerry a winning strategy. To enhance our opinion we can look at some children explanations to what they have done:

"Because the mouse walks and at a point all of a sudden the cat jumps on him"

"Because it is easier"

"Because Tom was ready and jumped at the right time"

"Because tom jumps and catches him"

"Because the cat is in line with the mouse and it is easy to catch him";

There have been children that understood that uprightness (vertical) was not the right answers for all the questions, and that was better to make Tom jump when Jerry was the closest *"the cat waits for the mouse to be closer"*. Also very interesting are children explanations about the position choice of Tom and Jerry, they have not been deceived by the idea that if Tom jumps following the vertical line he will go faster and *all of a sudden* he will catch Jerry, children rather have an idea (they are not yet presented the concept of

perpendicular) of the concept of minimal distance. Here we have some of the children explanations:

"because he is closer"

"because the cat waits for the best time"

"because Tom has been waiting for Jerry to come closer and he caught him"

"because Tom wants to catch Jerry and when he is closer he catches him"

Other children did not follow the two ways described above, but thought that, being Jerry on the top of the roof, he was closer to Tom, or they found a hiding place for Jerry (answers where they connects the blue point to the horizontal segment C6). Many children showed an emotional commitment during the test, thing that was not foreseen, being close to the Jerry point of view or to Tom *"because he was hungry"*. Other students misunderstood the demand, explaining that the essential condition to get Jerry is to be well hidden, to succeed in catching the mouse Jerry does not have to be conscious of Tom being there. Other children answered that Tom would have caught Jerry only if *"the mouse had been slow on his run"*.

Quantative analysis of 1-2 grade primary school (age 5 to 7) tests

The quantative analysis has been done with Chic software on a sample of 33 students, not statistically meaningful, but useful in order to suggest future research hypothesis. After an accurate a-priori analysis we chose to put together right and wrong ideas.
The analysis of the similarity reveals that children ideas are split into three directions:
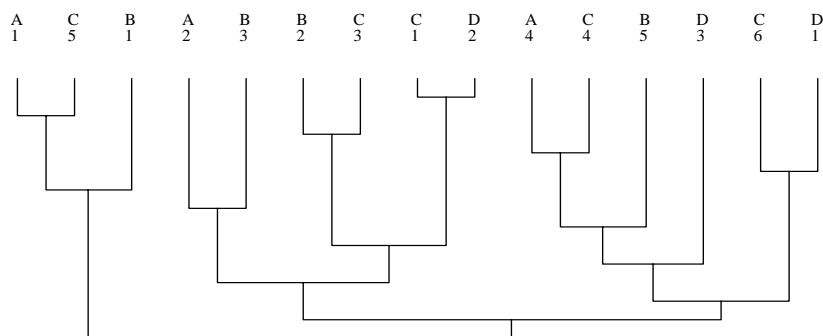
Arbre de similarité : C:\WINDOWS\Desktop\analisi quantitativa SPORA\fra e ma ultima.csv

A1, C5, B1

A2, B3, B2, C3, C1, D2

A4, C4, B5, D3, C6, D1.

Strong similarities are:

A1, C5: able to find out the vertical line in exercise 1 and to guess vertical in exercise 3

A2, B3: wrong ideas and conception in exercises 1 and 2; C1 and D2 correct strategies on question 3 and 4.

The last similarity group shows that when children have fragile correct ideas they are not able to make them general and to use them in different contexts.

Form the above graph we can also imply that:
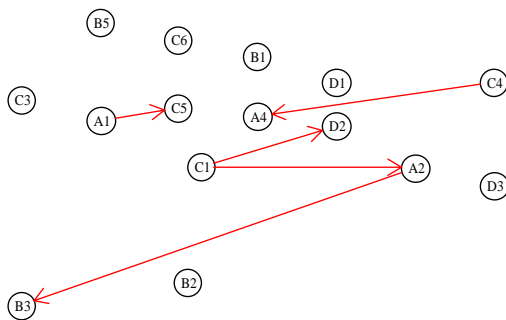
Graphe implicatif : C:\WINDOWS\Desktop\analisi quantitativa SPORA\1
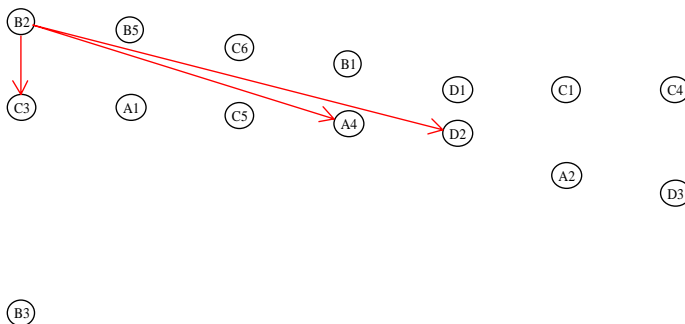
Graphe implicatif : C:\WINDOWS\Desktop\analisi quantitativa SPORA\fra e ma ultima.csv 99 95 90 85

1) if a child finds out the perpendicular line in exercise 1 he will have some correct ideas in left exercises but not always he will follow the correct strategy (A1, C5)

2) (C4, A4, D3) if a child follows once a wrong strategy he will follow it also in a different context

3) (C1, D2) if a child finds out the perpendicular line in exercise 3 he can also understand the minimal distance between Tom and Jerry in exercise 4. If we consider answers children gave in exercises 3 and 2 we can assume that even if they are in front of similar questions they give different answers (C3, B3). To explain this we could say that the problem is in the number of solutions, only one possible in exercise 2 and two possibilities in exercise 3. Children follow the didactical contract and think that it is not possible that two exercises have the same solution, especially if they do not understand that there are two possible choices to solve exercise 3. Why should teachers give two exercises that are almost the same? No student gave answer C2 to exercise 3.

Quantiative analysis of 3-4 grade primary school (age range: 8-9 years) and 1 intermediate (10-11) tests
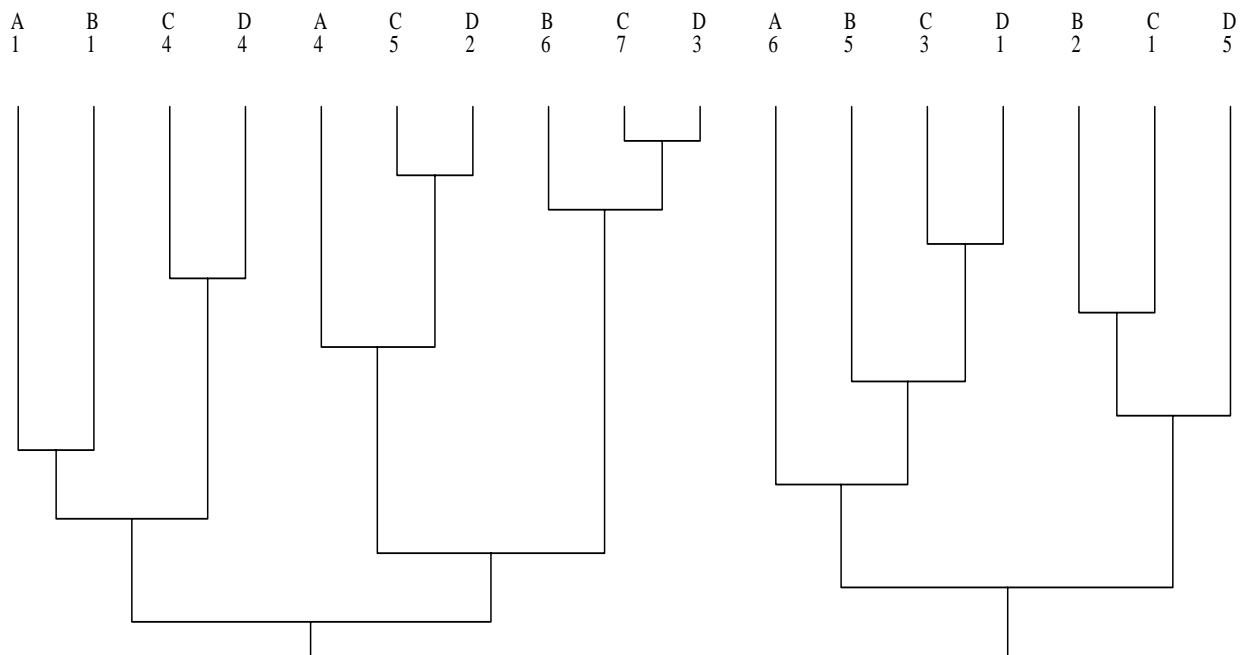
Similarity : D:\corso al Ferrara\Analisi quantitativa\III,IV,Roset IV,Marg I media.csv

The data analysis has been done on a 73 students sample using Chic software. The different social-cultural context has made possible to cling together data from primary grades and first intermediate.

The similarity analysis reveals that children ideas can be shared in five groups:

A1, B1: able to find out the perpendicular line and to draw the vertical line

A4, C5, D2: able to guess the perpendicular line, draws a slanting line from the cat to the slanting segment, has an idea of perpendicularity.

B6, C7, D3: this set of answers is the one where children draw a line that connects the cat to the highest, and closest to them, mouse point.

C3, D1: in both these answers children draw the vertical.

B2, C1: here children guess an idea of perpendicularity.

From the graph we can imply that:
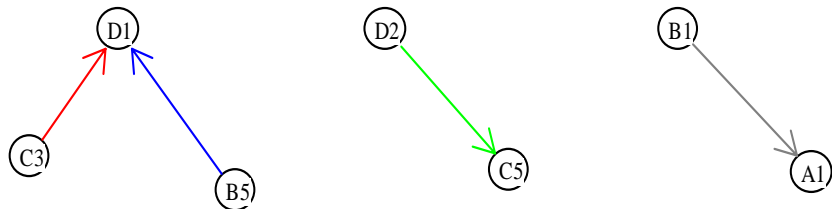
III,IV,Roset IV,Marg I media.csv   91 90 84 75

1) children that draw a vertical line in exercise 3 (C3) and children that draw a slanting line in exercise 2 (B5),   draw a vertical line also in exercise 4. They so have the same model of vertical and use it in different exercises.

2) children that guess perpendicularity in exercise 4 (D2)   do not get an implicit and inner model of vertical in exercise 3 (C5).

3) if children in exercise 2 draw the vertical line (B1) they found out perpendicular line in exercise 1 (A1). This maybe happens because in the first exercise the perpendicular line coincides with the vertical one.

3.8 Play-path analysis

During the first path teams have found the "right" point thanks to the concept of vertical. They in fact used the following expressions:

"*is under Tom*"

"it is straight"

This can make us think that they still do not have a dominant concept of closeness between their point and Tom, but they rather have a strong idea of vertical; although a student tried to talk about it, the rest of the team-mates did not pay very much attention to it. It was still a rough guess.

While doing the second path one team kept following the concept of vertical repeating the word *under*. Other teams, also not forgetting the vertical concept, have found as more relevant the "in front" position of cat and mouse, thing that makes easier to see each other (here more easily seen than in a written test): to prove that, they often use the word *"he sees better"* and criticise the curve on other teams paths.

The idea of closeness, even if in a certain way guessed, was still not fully expressed and understood in its importance.

During the third path, the idea of closeness as the most important criteria to solve the problem starts to reveal its importance, also thanks to some stimulus-questions.

In order to help students to find the winning strategy they had been given a string without being told of its usefulness. Students repeated their ways using this string. One edge of the string had been given to "Tom" the other one to "Jerry", they had also been told to keep this string always straight, thus "Jerry" had to wind up and down the string to keep it always straight. This has been essential to make them realize that the string was shorter in the closest points and longer in points very far from each other.

Students have had group discussions about winning strategies (validation) and ended up on an agreement: the "right" point to catch Jerry was the closest and this point was the perpendicular concept itself.

The most successful theorem was: *"the perfect time for Tom to catch Jerry is when the mouse is the closest to the cat not when he is under the cat"*

Finally the team that followed the correct strategy in both the form and the game paths won the "competition".

3.9 Conclusions of the experience

The hypothesis I start from is:

H1 Vertical and perpendicular are synonymous to students (language misconcept)

H2 If teachers formalise the concept of perpendicular (in a maths way) students become acquainted to the concept as long as they use in a proper way the concept of the plumb line and are able to adapt it to the new situation. If students do not break this epistemological obstacle (implicit model) they will not be able to solve a given problem where the reference system is not a line crossing the centre of the earth.

H3 To build a particular "*milieu*", in consideration of the Theory of situations, it can be a contribute to create the didactical way more opportune for the correct and persistent formation of the concept. Is it possible to build a context where the perpendicular straigth lines are conceptualizated as minimum distances?

We can easily think that obstacles hard to overtake are born in a very early time physic development, and teaching action produces rather an instruction than a conceptualisation.

A resume of the principal results put in evidence that the first hypothesis will be validated; in fact there is a strong connection for the pupils between the two words perpedicularity and verticality, so there is a language misconception.

In particular I try to resume the principals results:

Qualitative analysis of the text:

For the first question, children generally have been able to answer and give explanations, the analysis of the entire body of tests, so to say their percentage, would let us think that finding the minimal distance between two points is the same thing that finding a vertical line crossing one of the two given points.

After a first glance on childrens' explanations we can assume that the concept of vertical is seen through the lens of their daily experience of gravitation.

The children know that if they let an object it falls towards the floor.

But it is possible to observe in the childrens' answers that there is an embryonic idea of the concept of the perpendicularity, in fact some children understooded that uprightness (vertical) was not the right answer for all the questions, and that it was better to make Tom jumping when Jerry was the closest to the path.

Also childrens' explanations about the choice of the position of Tom and Jerry are very interesting, In fact, they have not been deceived by the idea that if Tom is jumping following the vertical line, he will go faster and *all of a sudden* he will catch Jerry. Rather, the children have an idea (they have not yet the concept of perpendicular) of the concept of minimal distance.

Quantity analysis of 1-2 grade primary school (age 5 to 7) tests

The analysis of the similarity reveals that children ideas are split into three directions:
Strong similarities are:

1) Able to find out the vertical line in exercise 1 and to guess vertical in exercise 3

2) Reveals that the correct strategies uses in the 3-4 exercice is not a general compatence infact the student uses a worong strategies in 1-2 exesice.

3) The last similarity group shows that when children have fragile correct ideas they are not able to make them general and to use them in different contexts.

The analysis of the implicative graph reveals that children ideas are split into three directions

1) If a child finds out the perpendicular line in exercise 1 he will have some correct ideas in left exercises but not always he will follow the correct strategy;

2) If a child follows once a wrong strategy he will follow it also in a different context

3) If a child finds out the perpendicular line in exercise 3 he can also understand the minimal distance between Tom and Jerry in exercise 4. If we consider answers children gave in exercises 3 and 2 we can assume that even if they are in front of similar questions they give different answers. To explain this we could say that the problem is in the number of solutions, only one possible in exercise 2 and two possibilities in exercise 3. Children follow the didactical contract and think that it is not possible that two exercises have the same solution, especially if they do not understand that there are two possible choices to solve exercise 3. Why should teachers give two exercises that are almost the same?

Quantitative analysis of 3-4 grade primary school (age range: 8-9 years) and 1 intermediate (10-11) tests

The similarity analysis reveals that children ideas can be shared into three groups:

1) Many student consider perpendicular straight lines and verticality as the same concept, they stress the verticality for indicated the pependicularity.

2) Some students have the correxct idea of the perpendicular line, but it is only an idea no a mathematical concept;

3) Some answers given by the students reveals that children draw a line that connects the cat to the highest, and closest to them, mouse point.

From the graph we can imply that:

1) children have the same model of vertical and use it in different exercises. (The model of plumb line is very strong and childrens uses it in different context);

2) children that guess perpendicularity in exercise 4 do not get an implicit and inner model of vertical.

To remuse, it is possible to think that the implicit model of verticality is very strong in the pupils, and that those who have not this inner model are not able to identify the mathematical concept of perpendicularity. They possible know that perpedicular straigt lines are nor vericality lines but they have not the formalization of the concepts.

The analysis of this principals results reveals that there is in the mind of pupils a sort of *Natural* classification between the two concepts, in the sense that perpendicularity is a sort of a *special case* of verticality. In effect we must considered perpendicularity as a substrate of verticality. The explanation of this situation is found in the body esperience and in the activity in an early stage of psyco-phisical development of the subject. From the point of view of mathematical learning this is translated into the connection between the *concrete* and the *abstract* in mathematics. It is possible to suppose that the obstacles for the learning of the concept of perpendicularity are related with the fact that verticality is a concrete fact, as I commented previous, and perpendicularity is an abstract fact that is formalized by mathematics.

So I think that he first and the second hypothesis will be valued in the first phases of the experience.

The third hypothesis concerns about the possibility of creating a didactical context adapt to solve:

1) the linguistic misconception

2) the passage from a concrete idea derived from the personal experience to a correct mathematical concept.

So I created the play-path. Here I resume the principals results.

<u>Play-path analysis</u>

During the first path teams have found the "right" point thanks to the concept of vertical.

This can make us think that they still do not have a dominant concept of closeness between their point and Tom, but they rather have a strong idea of vertical; although a student tried

to talk about it, the rest of the team-mates did not pay very much attention to it. It was still a rough guess.

While doing the second path one team kept following the concept of vertical repeating the word *under*. Other teams, also not forgetting the vertical concept, have found as more relevant the "in front" position of cat and mouse.

The idea of closeness, even if in a certain way guessed, was still not fully expressed and understood in its importance.

During the third path, the idea of closeness as the most important criteria to solve the problem starts to reveal its importance, also thanks to some stimulus-questions.

In order to help students to find the winning strategy they had been given a string without being told of its usefulness. Students repeated their ways using this string. One edge of the string had been given to "Tom" the other one to "Jerry", they had also been told to keep this string always straight, thus "Jerry" had to wind up and down the string to keep it always straight. This has been essential to make them realize that the string was shorter in the closest points and longer in points very far from each other.

Students have had group discussions about winning strategies (validation) and ended up on an agreement: the "right" point to catch Jerry was the closest and this point was the perpendicular concept itself.

The most successful theorem was: *"the perfect time for Tom to catch Jerry is when the mouse is the closest to the cat not when he is under the cat"*

The conclusion of the activity reveals that pupils connected the perpendicularity to the concept of minimum distance. There is a strong modification of conceptions that derived from a new body experience suggested by the theachers involved in the project.

In this case I speaks about *no falsification* of the third hypothesis because to speak of validation in my opinion it is necessary to create others *mileu* confirming the results

obtained and it will be interesting to create an adapt context using other mathematical concepts.

In referement to third hypothesis it would be interesting to create another a-didactic situation where the pupil has the opportunity to experiment phisically the difference betweeen perpedicularity and verticality.

One example would be the "stair-experience". After having rested a stair on a wall, we let a pupil to go up it and we ask to him:

       1) what is your position respect to the floor?

       2)   what is your position respect to the stair?

CHAPTER IV:
The Embodiment Theory

4.1 Introduction

The fourth chapter has been about brain structures involved in learning and tools and knowledge coming from neuro-physiology studies.

This analysis requires a last issue talking about: the connection between body and brain; or rather, how a mathematical knowledge (concrete-conceptual) can be born of a daily experience. This question is moreover the focus point of the didactical experience we have built.

1) What are the elements to be considered to a well performing teaching result of an a-didactical situation?

2) Is there any sign that indicates for sure that pupils can learn a mathematical concept from a game built to make them familiar to the concept itself?

3) Is the plumb line model, together with the minimal distance concept, metaphors of the perpendicular and vertical notion?

4.2 The philosophic point of view of the embodiment theory

[…]...the only mathematical ideas human beings can have are the ones allowed by human brain...[…][38].

Before getting into the embodiment theory we need to talk about the philosophic and theoretical assumption that stands backwards; the embodiment theory has to deal with brain and mind idea of Cognitive Sciences and Neurosciences.

The approach of Cognitive science considers three questions:

1) What are the mathematical ideas on the side of cognitive sciences? Which cognitive procedure do they use?

2) Given some inborn capacities (such as being able to identify three objects without counting them, *subitizing*), which is the cognitive mechanism that magnifies this capability to generate new complex mathematical ideas?

3) What are the mathematical ideas that we can get from our experiences?

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[38] From Lakoff G., Nùñez E.R., 2000, *Where Mathematics comes from*. Basic Books, pg 4

4) What mathematical ideas are metaphorical and what others are conceptual blends?

The cognitive approach of cognitive sciences has also to deal with that maths notion Lakoff and Nunez call "*the romance of mathematics*"[39] according to what:

✓ Maths is an inner and proper character of the universe; mathematical objects are real; mathematics truth is universal, sure and unconditional.

✓ What people think of maths does not affect the subject, maths would be equal to itself also without human beings, this is why maths is said *disembodied*

✓ Mathematicians discover absolute truths

✓ As long as logic ability can be formalized as maths logic, maths itself defines and embeds the property of rational thinking, so maths abilities are the top of human intelligence

✓ The mathematics of physics is into the physic phenomenon: "the book of nature is written in maths language"- Maths is the leading science among sciences.

According to them mathematical science is in a way transcendent and alive, and it embeds what human beings have discovered and proved due to the mathematical demonstrations that overcome human beings limits. Mathematical demonstrations allow discovering maths truths. But is there any scientific evidence to empower this statement? The answer is no. Another point of view near the romance of mathematics is the one that sees maths as a division of the physical world, based on huge successes and achievements of mathematicians on physics issues, where maths have been very useful to describe events and make correct forecasts.

"The universe works according to maths rules" or so to say, firstly come laws and rules and then occurrences follow. Considering that events and rules are out of living beings control, maths also has a proper and separated existence that is part of the physical universe.

But nobody sees these rules, it is only possible to see what regularly happens in the universe we are in, and maths laws are statements that encode it. Physicists can understand this constancy using the conceptual system provided by body and mind, and they also understand maths thanks to this system.

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[39] Lakoff G., Nùñez E.R., 2000, *Where Mathematics comes from*. Basic Books, pp338-341

If only one of these two approaches was true - that maths is out of human mind - cognitive approach and science would be of no interest because:

✓ Cognitive sciences study the way human beings conceptualise and understand maths, the way they represent maths into their own mind, they way they can learn it, how new discovers happen, but it does not say anything about maths itself

✓ If the hypothesis of maths as self-alive would be true studying neural nets and cognitive processes would not allow getting closer to maths true nature.

What is the true nature of maths we aim to study?

Human beings get into maths concepts thanks to mind, body and brain that create ideas that are physically structured into neural system; to us maths is *embodied mathematics*. The only maths we can know is the one that our body and mind let us know. The *embodied mathematics* is the theory of the only maths we can know, of what really maths is. The embodied mathematics theory is an empiric theory on embodied mind, which is part of the embodied cognition and uses the methodological approach of the embodied cognitive science. Its main elements are:

1) *Images schemas*

2) *Aspectual concepts*

3) *Basic-level concepts*

4) *Semantic-frame*

5) *Conceptual metaphors*

6) *Conceptual blends.*

Going through an analysis of cognitive mechanism of our conceptual system the embodied mathematics theory aims to answer such questions: *which is the cognitive mechanism used in definitions, theorems and demonstrations and in all concepts related to their learning and understanding?*

All the given answers have also to consider neural processes of human conceptual system.

4.3 Strong and weak aspects of the embodiment theory

The cognitive approach to maths necessary admits some weak requirements to the embodiment theory: knowledge, thoughts and concepts use brain's neural structure.

In a weak sense embodiment means that every concept has to be influenced by our neural structure, every thought comes out from a neural process, learning too is possible due to a neural mechanism.

What we know and what is going to be known stays in our neural structures. In a cognitive perspective the fact that maths has been made and learnt means that there must be a biological description of its creation, learning, representation and use.

Even language and meanings have to be explained using a neural and cognitive mechanism.

In regard to the strong requirements (see Nunez 1999), cognitive science has to clear up how is possible to think in a non-figurative way, how is possible to understand concepts that are not expressed by senses (we cannot smell or touch justice). Cognitive science tries to explain, in body and mind terms, how we understand and use these concepts, the same thing cognitive maths wants to do.

4.4 World interaction defines maths properties

According to the idea of maths, described in par. 5.2, it does not have a proper and lonely existence. Maths properties come from our folk theories about objects and daily experiences:

- ✓ Containers (metaphor of objects held together in a set)
- ✓ Keep going walks (metaphor of number lines)
- ✓ Separated object (metaphor of counting)
- ✓ Subitizing
- ✓ Objects collection (metaphor of sets)
- ✓ Measure (metaphor of cardinal numbers)

Here we have some properties of objects we daily experience and come from maths:

- ✓ Universality
- ✓ Precision
- ✓ Consistence
- ✓ Stability
- ✓ Capability of generating
- ✓ Capability to be discovered

Being maths a creation of mind that comes from the study of the objects in the world, it inherits its properties from the world itself.

The point of view regarding what maths is a product of *embodied cognition-mind,* as it comes from world interaction, clears why maths gets these properties.

4.5 Embodied mathematics properties

New studies carried out on neuro-physiology, cognitive science and history of maths all agree on *embodied mathematics* theories. Here we have their properties

- ✓ Maths is a product of human beings. It uses human biology limited resources; it comes from our brain structure and work, from our bodies, from our conceptual system and it concerns human culture and society.
- ✓ Human cognitive side generates advanced maths, thanks to average cognitive skills such as the ability to think with conceptual metaphors. This is why everyone could possibly do maths.
- ✓ The subitizing capacity is an embodied one
- ✓ Geometry, arithmetic, calculus and probability are born of human activities. Maths is born too of human activities.
- ✓ One of the characters of maths is its precision, that is made possible due to the ability that human beings have to distinguish between objects, to classify things according to criteria and to recall them (such as non figurative entities like numbers or shapes)
- ✓ Precision is improved by human skill to handle symbols, letting calculating getting much easier.
- ✓ The conceptual metaphor is an embodied neural and cognitive mechanism of huge importance; which allows making inferences in a frame and transferring them to another frame.
- ✓ Once established by a scientific community, calculus and maths inferences do not change in time or according to culture changes. The stability of embodied maths is due to the equality of brain and body human structure and the nature of the relation between people and environment that determines the birth of maths ideas. Maths inferences tend to be fixed among people because of the use of categories, spatial-temporal concepts, conceptual metaphors and subitizing. Some abilities are inborn, others develop during teen age, and others develop only after a specific cognitive training. When we learn a basic cognitive mechanism it keeps unchanged for the rest

of life. Another reason of stability is spatial-temporal relations, the simplest are common to almost every language: the containment concept, the walk, the centre, the limited portion of a plane or the space in a space relation system is easily the same in different languages or cultures. Once that a mathematicians community established a metaphorical map - numbers as spots on a line - every member of the community shares the same inference regardless to specific culture. If that community keeps over the time inferences will be always the same.

✓ Maths is not unique

✓ Maths describes and predicts some aspects of reality as we know and prove this reality.

Resuming all this properties we can say that embodied maths is:

1) Universal
2) Consistent
3) General
4) Precise
5) Symbolic
6) Stable
7) Effective

4.6 Relation between culture and *Embodied mathematics*

Culture has no role according to the "*romance of mathematics*" where maths has its own life. According to the embodied point of view maths in some way depends on culture in some ways does not. Let's try to summarize them:

1) Maths is independent of culture: once a scientific community discovers and sets some maths ideas they are true and valid regardless to any cultural environment.

2) Maths is dependent of culture when specific cultural ideas, historically important, bringing out of a proper maths context, find in it an application field. Such ideas can change maths contents themselves, such as the Greek idea of essence, the attitude to think of human thinking activity as a way of mathematical thought called logic, the willing to establish solid bases for a given mathematical issue; these are all general ideas coming out from a maths context in a specific cultural environment and historic time. Every one then gets into maths field as a special case.

The third idea deals with the need that every theory, just like buildings, has to have strong bases, in maths this approach is as old as Euclide times and has been enriched in the XIXth century by Freghe.

The foundation motion of maths at the beginning of the XIXth century implied the following notions:

1) The essence of an issue is given by a set of statements

2) The mathematical thinking is a way of calculus that allows mathematician to find out the truth (using maths logic)

3) Every issue has to have a sure basis where to build everything on.

Such an approach collapsed on itself leaving still open a question: where do maths ideas come from?

They are from human culture and history.

The idea that there must be an "*absolute basis*" to maths is by itself a property culturally descending from maths. The paradox is in the fact that people thinking that maths is independent of culture are, at the same time, the supporters of the "foundation motion", that is itself a cultural characteristic of the subject.

A concept clearly stands up: many of the most important maths ideas are born of general aspects of culture rather than maths itself, that is a part of the bigger cultural environment. Maths is so keeping developing, taking shape in bodies and minds, due to human cognitive skills and daily activities.

Maths has an absolutely natural cultural dimension according to the embodied perspective: if maths ideas come from human beings living in a cultural environment with cognitive skills, it is natural that culture works on specific issues too.

From the embodied perspective there are other effects of history and culture on maths. Mathematicians live in a defined historical context and base their job on older ones job, so maths develops (think of the concept of number into centuries: zero, negative numbers, complexes integer numbers, rational numbers, quaternions).

This development is not linear due to the historical dimension of embodied maths, because events and motions out of maths often affect contents. Just think of the information theory that pushed a new birth of maths themes.

What we have said does not imply that maths is only a cultural matter. The embodiment theory recognises the action of history and culture on maths but refuses any cultural relativism, rejecting any statement declaring that maths is only born of history and culture,

moreover realizing the important role of cognitive universe and experience. It is the not arbitrary property of the embodiment that makes it different from the post-modern approach.

In fact embodiment considers real properties of maths:

1) *Conceptual stability;*

2) *Stability of inference;*

3) *Precision;*

4) *Consistency;*

5) *Generalize;*

6) *Discoverability;*

7) *Calculability;*

8) *Real utility in describing the world*

4.7 Cognitive science in studying maths learning

Maths is a human product, limited and structured by brain and human skills. Neuro-physiology and cognitive science have helped to better know human mind. Brain and body develop so that the brain makes the body work better. Most of the brain surface is addicted to sight, motion, space coordinating, interpersonal interaction, emotions, language and thinking. Human concepts and language are not arbitrary but structured and limited by brain, body and world.

Such sentences give space to these questions:

✓ Which is the brain way of working that allows human beings to thinking mathematically?

✓ Is all maths based on brain and mind? (Embodiment's denial of platonic maths notion)

To answer the first question we have to think in a cognitive perspective, as a multi-subject mind science. As an empiric issue on brain and mind this question needs more than maths study, it is rather required a deeper understanding of cognitive processes that makes possible to answer also this question: *how are cognitive procedures implied in creation and understanding of maths ideas.*

The second question is the heart of maths philosophy we talked about in previous paragraphs. As a short summary just remind that: maths we know is human maths, made by human brains. Where does maths come from? From us, not as well as a result of historic times nor social relations because maths uses the structured conceptual procedures of human mind as it develops in real world. Maths is a product of neural skill of our brain, of our bodies, of our evolution, of our behaving, of our cultural and social history.

4.8 New discovers on mind nature

New items in cognitive sciences are:
1) *The embodiment of mind:* special nature of our bodies, brains, daily behaviour structure concepts and mind and even mathematical thinking
2) *The cognitive unconscious:* some of our thoughts are unconscious, not in a Freud way, but simply not accessible to a conscious sight. It is not possible to enter our conceptual system nor high thinking level processes even mathematical.
3) *Metaphorical thought:* human beings conceptualise non figurative items thanks to figurative ones, using ideas and models built through senses and perceptions. Such a system is called conceptual metaphors; maths uses it in a large rate.

Despite the fact that maths is one of the most beautiful human outputs, it often appears difficult for everyone to understand it, this might be due to the not full development of dedicated cognitive structures. A step towards a maths for everybody could end, according to Nunez and Lakoff, in discovering the embodied metaphors the subject is made of. In fact the most difficult thing about maths is that conceptual metaphors are not considered as metaphors but as the figurative meaning they have. Conceptual metaphors have a focus role in mathematical thinking, not only a way to clear and understand concepts. As examples think of the metaphor of numbers as spots on a line, this is not the unique metaphor about numbers but is the most relevant, without it there would not be any analytic geometry.

4.8.1 The conceptual metaphors

Metaphors are not simply a language entity but a cognitive process that lets human beings thinking of a matter, as it was another thing; it owns to the thinking world. New tendencies

on neurosciences think of mind as deeply rooted in bodies (cfr. J Lakoff and Mark Johnson 1998).

According to the old idea mind was out of any body relation, freely moving out of the body and by chance into a brain. According to Lakoff even simple and low concepts are into the gestaltic perception of motion programmes. Categories related to colours are based on colour vision neuro-physiology. Space relations seem to be embedded in the perception system.

Conceptual metaphors are those imagination processes that, being related to solid matter, develop into non-figurative concepts.

This approach implies an idea of individual far from the traditional psychology one. Motion - body language - gets a first role in learning verbal language. The conceptual metaphor has a specific meaning: it is a neural procedure that, using the inference set of a given subject (geometry), makes possible to think of another subject (arithmetic); it makes possible to use the same set of knowledge in different maths divisions; it improves maths but causes some difficulties if metaphors are not clear or limited to their figurative meaning not going to the higher level, the non-figurative.

Zero (0) can be a spot on a line or an empty set, both of them or neither of them, but every choice is a metaphors' choice.

Maths is done by metaphors over metaphors. It is cognitive science duty to reveal the undergoing cognitive structure. This is particularly important in teaching activity, showing maths cognitive structure makes it easier to understand. Because metaphors are based on daily experiences, pupils can understand maths ideas that use them simply trough their conceptual functions.

4.9 Brain areas dedicated to mathematical thinking

Low parietal cortex works on numbers and symbols, as can be assumed looking at individuals with lack of this area. A patient of L. Cohen and S. Dehaene suffering from low parietal cortex damages could not answer what number was between three and five, but could eventually say which letter was between A and C. He could give names to numeral numbers but could not do simple arithmetical calculus. His rote memory was not damaged: he could recall number tables, could multiply 3 x 9, but was not able to add numbers whose result was over ten. He could do algorithms by memory but he did not understand their meaning.

Low parietal cortex is an associational area located where sight, taste and hear neural connections join; a suitable place for number activities because they involve all senses. Damages on this area cause problems in writing, in using fingers to represent things and in differentiating left from right. Not all patients suffering from area damages have all symptoms together, probably because there are two associated sub areas. The pre-frontal cortex is the brain area involved in more complex calculus not using the rote memory. Patients damaged there are not able to proper do sequential calculation. Basal ganglia, brain sub cortical places, have a role in rote abilities.

Even algebraic skill are separated from the arithmetical ones, Dehane (1997) tells of a patient able to do algebraic calculus but not arithmetical.

At this point neurosciences can tell what are brain's areas dedicated to maths skills, what are inner and inborn and what are not. But this amount of knowledge is still not enough to explain not only where but also how it happens; we also have to add the analysis of the cognitive process that leads to mathematical knowledge.

4.10 Daily conceptual procedures determining in maths learning

The conceptual system we use in thinking, in communication activities, in inferring is mostly unconscious, so to say, not open to straight conscious analysis. This might let us guess of an unconscious memory, even if this idea seems to contradict the belief of a conscious access to memory; there are evidences that prove this guess of an unconscious memory. Maths cognitive science aims to broaden this hypothesis to maths understanding and learning.

The unconscious thinking is automatic, immediate and implicit rather than explicit; it gives maths a sense without knowing how this happens. We understand daily mathematics events not with demonstrations nor given instructions but without being able to explain how and why. The word understanding here gets a different and proper meaning: it is unconscious and automatic. The focus point is so:

"Which of the maths understanding uses areas and cognitive procedures used to understand ideas other than maths? Do we use same procedures to handle normal ideas and mathematics one?"

According to Lakoff and Nunez the most of maths understanding procedures are common to other subjects and ideas, they are:

1) Elementary spatial relations

2) In grouping

3) Small quantities

4) Motion

5) Spacing things

6) Changes

7) Body orientation

8) Objects handling (rotation, lengthen)

9) Repeated actions

10) Other

Especially:

11) The maths notion of "classify" uses the daily notion of collecting objects in a limited place

12) The concept of repetition uses the daily idea of repeated action

13) The derivata notion recalls the idea of motion, going to the edge

From a non-technical point of view this may seem obvious, but in cognitive perspective the determining side is what are the cognitive procedures and concepts related to daily experience that we use and how we unconsciously conceptualise maths ideas, considering that maths often is the result of an interaction between the world and every day experiences.

Let's try to have a look on main cognitive procedures that allow building maths knowledge, developing specific skills from inborn knowledge (subitizing, handling arithmetically short lists of objects).

4.10.1 Spatial Relation Concepts and Image Schemas

Every language has a spatial relation system; this system changes according to the used language. Cognitive science displays that language-specific spatial relations are structured into primal concepts called "image schemas", which are universal.

Think of an example: "the ash-tray is on the table", the term "on" can be decomposed in:

✓ Above schema (the ash-tray is on the table)

✓ Contact schema (the ash-tray touches the table)

✓ Support schema (the ash-tray is supported by the table)

The above schema expresses orientation; it specifies the direction in the space following a falling direction

The contact schema is a topological one, indicating the lack of holes.

The support schema is a dynamic force in nature; it is about nature force and direction.

In maths the container schema is very important and it makes clear the meaning of in/out terms, it is made of:

1) *Interior*

2) *Boundary*

3) *Exterior*

Such a structure comes from Gestalt, intending that only a piece of it has no meaning without relating itself to the whole.

To give a scheme the in/out terms we need to have in mind another distinction, the one between figure/ground; e.g. "the car is in the garage", the garage is ground, the landmark regarding what the car (figure) is located.

Cognitive science calls the ground *landmark* and the figure *trajectory.*

Summarizing:

1) *Container schemas:* with interior, exterior, Boundary;

2) *Profiled:* the interior;

3) *Landmark:* the interior.

The image scheme has a peculiar cognitive function, being conceptual and perceptual at the same time. It is a bridge between language and thinking on one side and sight on the other side. Images schemes can fit in visual perceptions or create visual scenes when there is not a physical container.

In any given language spatial relations terms are made of complex image schemes that work as a link between language and spatial relations.

Images schemes build a logic in the spaces that comes from their structure itself
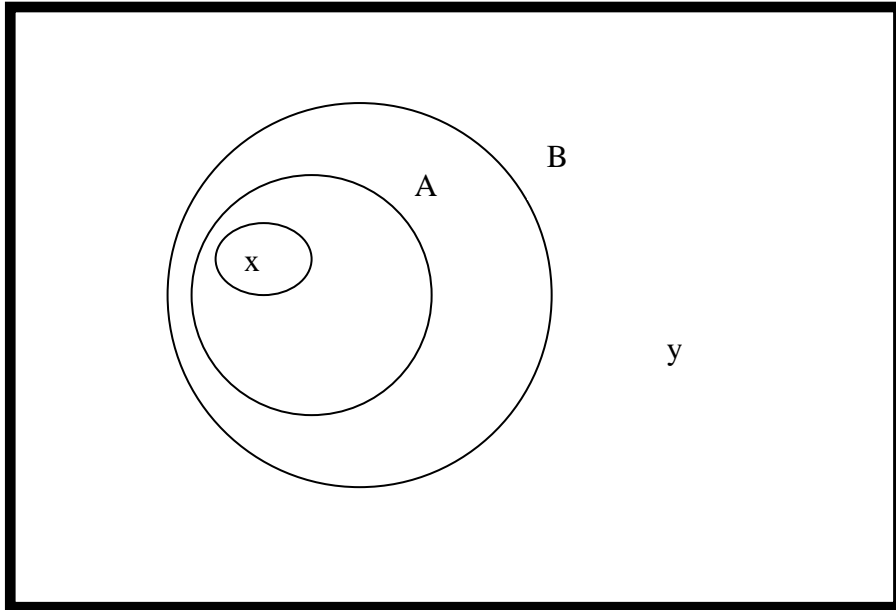
Consider the example above:

1) Given two *container schemas* A and B and an object x, if A is in B and x is in A⇒xd is in B;

2) Given two containers schemes A and B and an object y, if A is in B and y is out B ⇒y is out of A

We do not need any deduction to say this; it is enough to look at the drawing. Image schemes have their own spatial logic; they work as spatial concepts and are used in spatial thinking. All thinking about space is done in spatial terms, using image schemes rather than symbols as it is in maths demonstrations or in symbolic logic deductions.

Ideas do not circulate abstractly into the world, but are born of brains and of proper neural structures. Images schemes are ideas; they do not have a specific nature or structure but emerge from special devoted neural circuits.

The concepts of orientation and containing are decisive to maths, they own to thinking and language, and come from neural structures not always dedicated to maths activities. This suggests that the embodied mathematics do not live alone, independent from other embodied concepts that we daily experience and use. Maths uses our adaptable skills, our skills to suit different procedures and structure to new maths goals.

The brain vision system where the neural structures, called "orientation cell" assemblies, stay is also devoted to mental imagery.

Experiments on vision system show that primary vision frontal cortex is active when we create some mental imagery without any visual input.

The visual system is linked to the motion one thanks to the prefrontal cortex. Motion schemes trough this link can be used to design image schemes with the hands or other parts of the body.

4.10.2 Motor Control[40] and Mathematical Ideas

New studies (David Bailey,1997, Srini Narayanan, 1997)[41] on relations between the motor control and human conceptual system tell that the motor control system can be involved in mathematical thinking.

It would be easy to think that the motor control system has nothing to do with concepts, especially the non-figurative ones, but Narayanan says,

"The same neural structures used in motor control can be used in thinking of events and actions".

That is to say that the motor control scheme has the same structure of what linguists call aspect, the general structure of events.

All we think of an action or an event, as it is conceptualised, has such a structure. We think of events and actions thanks to this structure: the aspect scheme.

The neural motion control shares the same properties.

4.10.3 The source –path-goal schema

The source –path-goal schema is the main image schema concerned with motion, it has the following elements:

1) A trajectory that moves;

2) A source location (the starting point);

3) A goal that is, an intended destination of the trajectory;

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[40] Motor Control: cognitive mechanism that rules our body motion
[41] See Lakoff G., Nùñez E.R., 2000, *Where Mathematics comes from*. Basic Books, pp 34-37

4) A route from the source to the goal;

5) The actual trajectory of motion;

6) The position of the trajectory at a given time;

7) The direction of the trajectory at that time;

8) The actual final location of the trajectory, which may or may not to be the intended destination. It is possible to represent this schema:
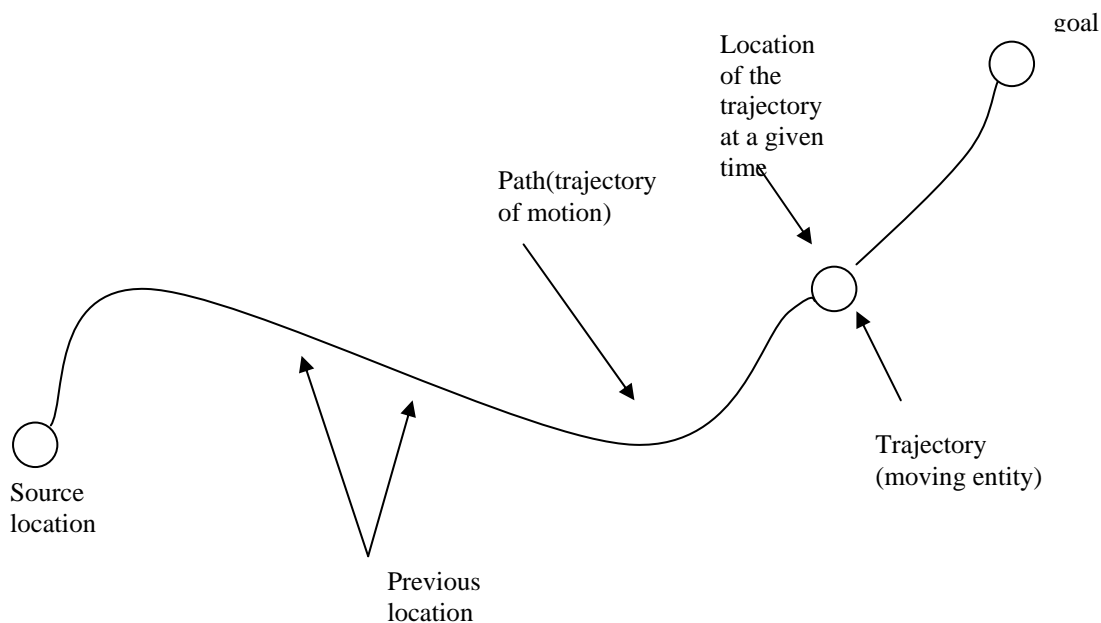
We conceptualise linear motion using a conceptual schema in which there is a moving entity (the trajectory), a source of motion, a trajectory of motion (the path), and a goal with an unrealised approaching that goal. There is inherent logic in the structure of the schema, as in the case of container schema. The source –path-goal schema is ubiquitous in mathematical thought, for example in the concept of graph or function.

4.11 The structure of the Conceptual Metaphor

Metaphor is a central process in everyday thoughts; it is the basic process by what abstract thought is made possible. One of the most important results in cognitive science is that abstract concepts are typically understood, via metaphor, in terms of more concrete concepts.

Hundreds of conceptual metaphors have been studied in detail. They are extremely common in everyday thought and language. They are used unconsciously, effortlessly, and automatically in everyday discourses; that is, they are part of the cognitive unconscious. Many arise naturally from correlation in our commonplace experience, especially our experience as children.

Metaphor has a leading role in building processes of meanings of maths concepts[42]; it is not only a simple tool to allow an easier comprehension of concepts, but is a thinking tool itself, a focus point in mathematical thinking, using a metaphor requires:

1) Being able to realize analogies and differences in different situation

2) To use familiar contexts properties in new contexts (Boyd, Kuhn, Angela Pesci).

Such correlations in experience are special cases of the phenomenon of *conflation*[43]. *Conflation* is part of embodied cognition. It is the simultaneous activation of two distinct areas of our brains, each concerned with distinct aspects of our experience. In a conflation, the two kinds of experiences occur inseparably. It is via such conflations that neural links across domains are conceptualised in terms of the other.

Metaphors have the same structure. Each is a unidirectional mapping from entities in one conceptual domain to corresponding entities in another conceptual domain. Their primary function is to allow us to reason about relatively abstract domains (*Target Domain*) using the inferential structure of relatively concrete domains (*Source Domain*). The structure of Image Schemas is preserved by conceptual metaphorical mappings.

To see how the inferential structure of a concrete source domain gives structure to an abstract target domain, consider the common conceptual metaphor: "Categories are containers", through which we understand a category as being a bounded region in space and members of the category as being objects inside that bounded region. The metaphorical mapping is stated as follows:

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[42] many researches in maths teaching are focused on metaphor use as thinking tool in maths concepts building, see works of Pimm(1981), Sierpinska (1994), Boero, Bazzini , Garuti (2001), Arzarello, Robutti (2001) and Bazzini (2002).

[43] Johnson C. (1997). Methaphors vs conflaction in the acquisition of polisemy: The case of SEE. In M.K.Hiraga, C. Sinha, & S.Wilcox (eds.), *Cultural, Typological and Psycological Issues in Cognitive Linguistics.* Current Issues in Linguistic Theory. Amsterdam: John Benjains

Source Domain	Target Domain
Containers	Categories
Bounded region in space	Categories
Objects inside the bounded region	Category members
One bounded region inside another	A subcategory of larger category

The metaphors preserve the inferential structure of the source domain, so, in this case, the logic of Container schemas is an embodied spatial logic that arises from the neural characterization of Container schemas.

Other examples of metaphors in maths are:

1) Numbers as collection of things or spots on a walk
2) Zero as an empty box or start of a split
3) Addition as things put together or make steps of a given length in a direction
4) Multiply as a repeated addition or as counting spots in a rectangular scheme
5) Equation as two different balanced collections of things
6) The two principles of equivalence in equations as two operations on a scales
7) Function as an machine that process a number changing it into something different

It is now going on a debate on using metaphors to build maths concepts; in detail there are some doubts on specific metaphors and why one should be better than another in specific contexts. There is no doubt on their usefulness, but about the best to choose, it seems that the best is the most suitable to the specific context, always knowing that:

"*all metaphors are inadequate and some way out of a maths context, and this is good because if there was a metaphor suitable to all contexts then it would be enough, while maths concepts is richer or would be a surplus; and a simple word would be enough instead of symbols; it is proper of a maths concepts not to identify itself in one single metaphor but to be common to a metaphors family*".[44]

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[44] Pesci A.,(2004) Tesi Master, pp228, see www.dipmath.math.unipa.it

There are anyway lots of works that show how important metaphors are in education process, to ease the meaning building process and to give things maths sense. Students can get benefits from daily experience metaphors used in maths; they would help them in overcoming prejudices and emotional obstacles in maths concepts learning. The metaphorical discourse would make easier to reach the natural mathematical spirit that we all have:

"the language disposition is what you need to do maths. So I will have to explain why so many people cannot use this skill everyone has. Maybe it is because most of the people do not know what is maths; it is not only a matter of numbers and arithmetic. Once you will have learnt what it is really about, the idea that maths thinking is not more than a special use of language skills will not be a surprise any more".

4.12 The theory of the situation and the building of an environment to create metaphors

In conclusion to this chapter we quote K. Devlin[45]:

"If maths is a product of mind, whose this mind is? The answer is that there is only one mathematical world. Although the needed creativity to create a new concept or to do a demonstration stays in only one mind, the maths world structures itself and is determined by the human brain, it is so the same in every human being. The maths world is the result of the meeting between human mind and physics. So maths is determined by both our mind and our brains structure"

Third chapter tells us how the experiment bears pupils to succeed in finding the right theorem; when pupils understand that the concept of "being perpendicular" has to be related to the distance one, they find the way to solve the problem, overcoming the obstacle of the similar meaning of the words "*perpendicolarity*" and "*verticality*".

The hypothesis we start from is the one about the uniqueness of mathematics and its language; otherwise fourth chapter describes us the subjective vertical and suggests that difficulties we find in defining the concept of perpendicularity are strictly linked to our body and brain (we do have gravity receptors in our abdomen). We can easily assume that pupils' problems can have different reasons.

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[45] Devlin K., 2002, *"Il gene della matematica",* Longanesi &C., Milano, pg. 173

It has been really interesting evaluating the results coming from the embodiment experience; this experience has made clear that there are different set of metaphors, some of them are related to the perpendicular concept and some others are related to the vertical concept. The play-path has helped a lot in this achievement.

•*Source Domain, established through the physical experience of a mathematical concept.*
•*The plumb-line experience fixes a map for the abstract concept of uprightness.*

⟷

Abstract concept (Target Domain): the uprightness.

•*Source Domain: trough the play-path pupils experiment the concept of minimal distance.*
Perpendicularity is mapped and lived trough as a real experience.

⟷

Abstract concept (Target Domain): perpendicularity

When using the embodiment theory we also receive a positive feed-back on the effectiveness of the theory of the situation, but these two theories are fully valid alone, they do not need each other positive feed-back to show that they are both true.

Pupils have built a mental image that links the concept of perpendicularity to the plumb-line model, there are several reasons for this: the similar meaning of the two words, as we said above, the inner knowledge of pupils, a personal experience of the gravity concept, the way our body works; but at the end this strategy is successful in some particular cases.

The play-path undermines this mental image and requires a new one that is finally successful in every case, not only in some specific cases.

We have seen that after this experience pupils link the concept of perpendicular to the one of minimal distance.

The embodiment theory proves this result and explains it (new metaphors are born), and it is the experience itself that let pupils link the right metaphor to the right mathematical concept.

In my opinion it has been determining the environment we provided in order to experiment these mathematical concepts, it would be really interesting to repeat a similar experience with other mathematical concepts.

CHAPTER V:

Learning: a neuronal approach

This last section is an analysis of the principal structures involved in learning and in particular in the learning of the concept of perpendicularity. It constitutes a research of new work hypothesis.

5.1. Learning and memory for neuroscience

Before we analyse the basis of learning we will try to give a definition of it. It is necessary to distinguish between memory and learning, without the first the second would be impossible. There is also a difference in "level" among the two, memory is a property of existing structures, learning is superior living creatures complex function, as a result of their interaction with the environment.

Learning is the only way to a life-through continuous change, specific property of the nervous system, which allows enduring changes.

Tablet 1:

```
                        ┌──────────────┐
                        │    Memory    │
                        └──────────────┘
                                ⇕
┌──────────────┐                                  ┌──────────────┐
│  Sensorial   │                                  │  Language    │
│  system      │    ┌──────────────────────────┐  └──────────────┘
└──────────────┘    │ Thalamus, Libic System,   │
                    │ Primary and Secondary      │
                    │ Sensorial Cortex,          │
                    │ Associative Areas          │
┌──────────────┐    └──────────────────────────┘  ┌──────────────┐
│ Sensorial and│                                  │  Motor       │
│ Motivational │                                  │  Systems     │
│ Internal     │                                  └──────────────┘
│ System       │
│              │
└──────────────┘
```
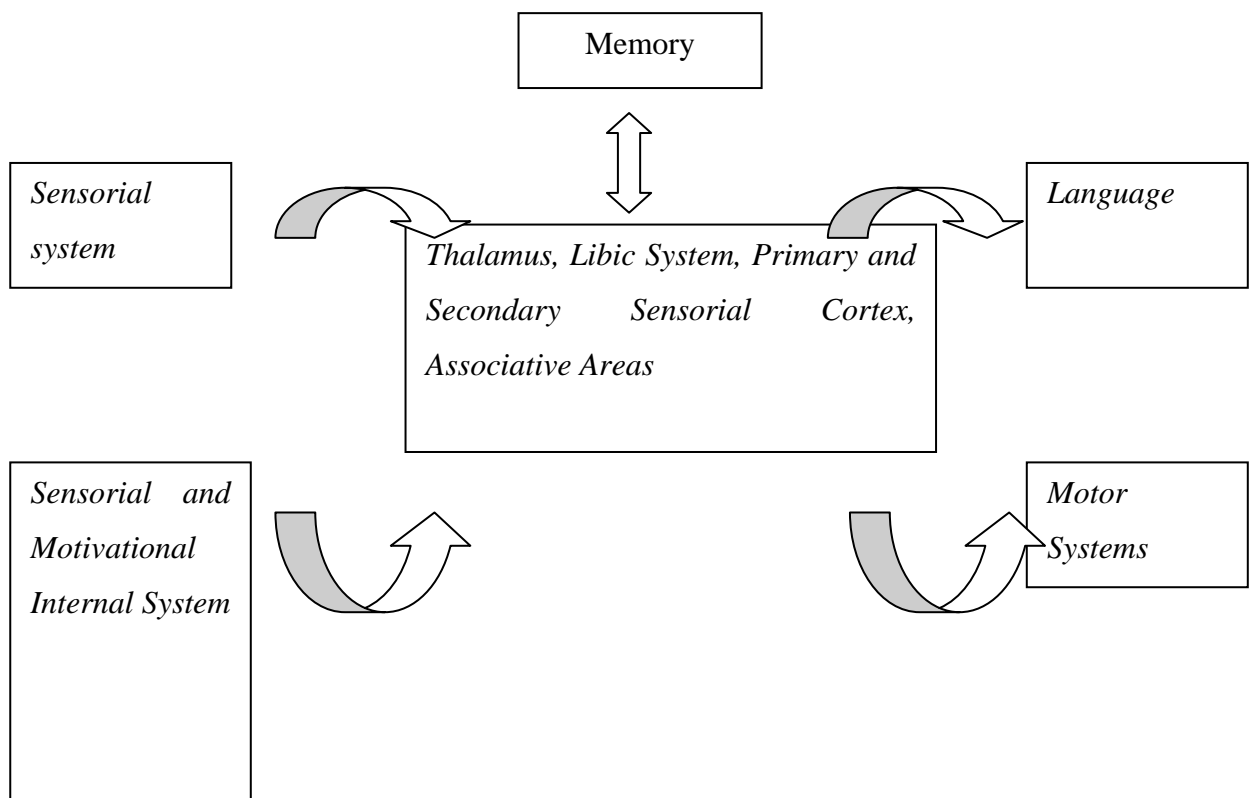
The thalamus is central in focusing attention; it is an elaborating structure, when possible it handles alone the information coming from the perceptive system. It is linked via the cingulate gyrus to frontal and prefrontal areas. Memory is divided into two types: short-term memory (primary memory) and long-term memory (secondary memory).

Short-term memory recalls little amount of information for a short time, while long-term memory increases the amount of information and their lasting time (William James)[46].

According to psychologists, to keep information for a long time we need an enforcing time, when information is still easy to be lost, after this time the memory is fixed into the brain.

The Nobel Prize winner G. Endelman[47] says every mnemonic event acts on categorized structures, not on specific ones, there is not a memory storage space, but we recall images or information from the past via a complex process of working and re-working on information throughout the whole brain.

Memory is so a complex elaborating and classifying process kept together in a multi-level maps system.

We above tried to explain what is memory, let's try now to clear how brain uses incoming inputs (touch, vision, smell, taste) and changes them into memories.

5.2 From perception to memory

There are still debates over the question; it is certain that the amygdala, the hippocampus and some subcortical structures (striatum) are involved. Two hypotheses have been suggested:

1) hippocampus-amygdala path, based on emotion-learning

2) subcortical structures (striatum) path, based on iteration

S. Freud[48] links emotion and context to memory; memories not emotionally based are not memories at all. Emotions are essential to a memory generating process because they organise it and establish a hierarchy in events. Time and order are important to make a

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[46] See Spagnolo Filippo e Ferreri Mario, *l'apprendimento tra emozione ed ostacolo,* lavoro eseguito nell'ambito del contratto C.N.R n. 9001293CT01, pp68-75
[47] Edelman G.M., (1997), *Neural Darwinism*, Basic Books, New York
[48] Freud S., (1973), *L'interpretazione dei sogni*. Boringheri, Torino

memory a real memory connected to the past, not a simple thought. Memory always has an emotional content that makes it different from an unseen event.

An aspiring memory has to be put into a context to become a real memory; Freud thinks that no-sense memories are the ones out of a context.

Human brain has the skill to reorder past events, thoughts and impressions; it can give abstract dreams and memories a concrete meaning and existence. Emotion and context are necessary to memory and consequentially to learning.

Emotion is here intended as motivation (and games can be a way to get it) and context as spacing and timing (we will go to it later).

Mishkin[49] uses Freud's theories and give them a neurophysiological base thanks to the possible amygdala-hippocampus link.

According to Tayler and Discenna (1985) hippocampus is an operating memory system, a kind of neural map of every possible cortical route that can be generated by perceptive experiences. The hippocampus holds a timing map of cortical activations and provides a short-term memory of the chronological index of them.

If the same stimulus occurs more than once to the cortex the hippocampus is able to recognise it. The iteration of the stimuli and the neural activity of the cortex enforce the index and makes easier to recognise similar situations.

Hippocampus so plays a main role in ordering information and creating neural indexes; then it keeps the index, losing the single experience; it also produces a spatial context, having a spacing and timing role.

TABLET 2

External Imput → Thalamus →

Hippocampus

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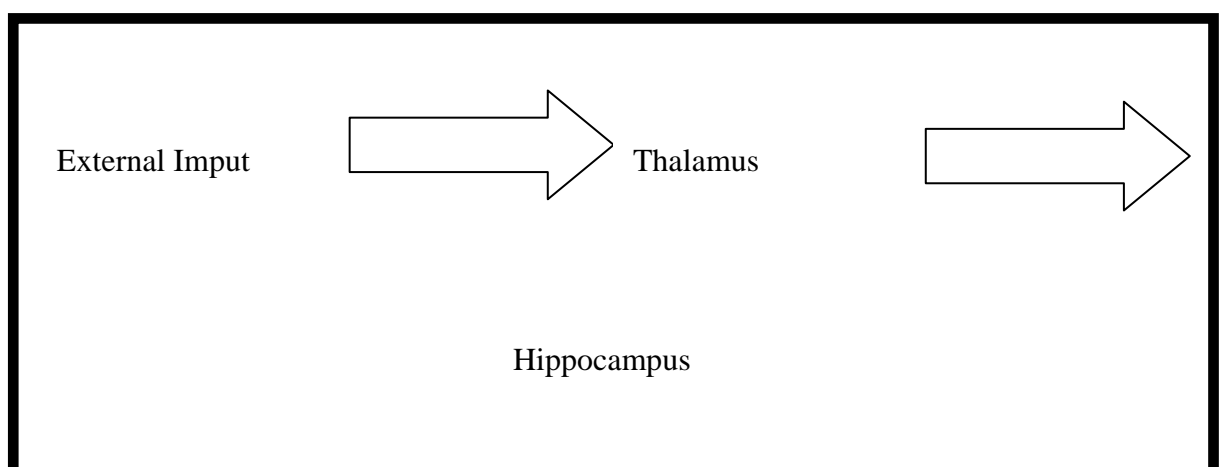
[49] Mishkin., Appenzeller T., *L'anatomia della memoria*, Le Scienze n.228, 1987

Fig.1

5.3 The brain-body link in learning

The learning experience we led needs playing situation and body learning. Neurosciences think of games as a need for learning directly coming from the pupils; in our experience the only chance to be successful was by learning the concepts of minimal distance and perpendicularity.

The basic idea is to make any new knowledge crucial to survive

The need for learning underlines that new approach: pupils are not given knowledge, but they feel and experience the need for learning, what in Latin is "Appetitus Noscendi".

Thus a motivation is born; let's try to explain what a motivation is:

   Def: the mental process that arouses an organism to action, driven by impulses
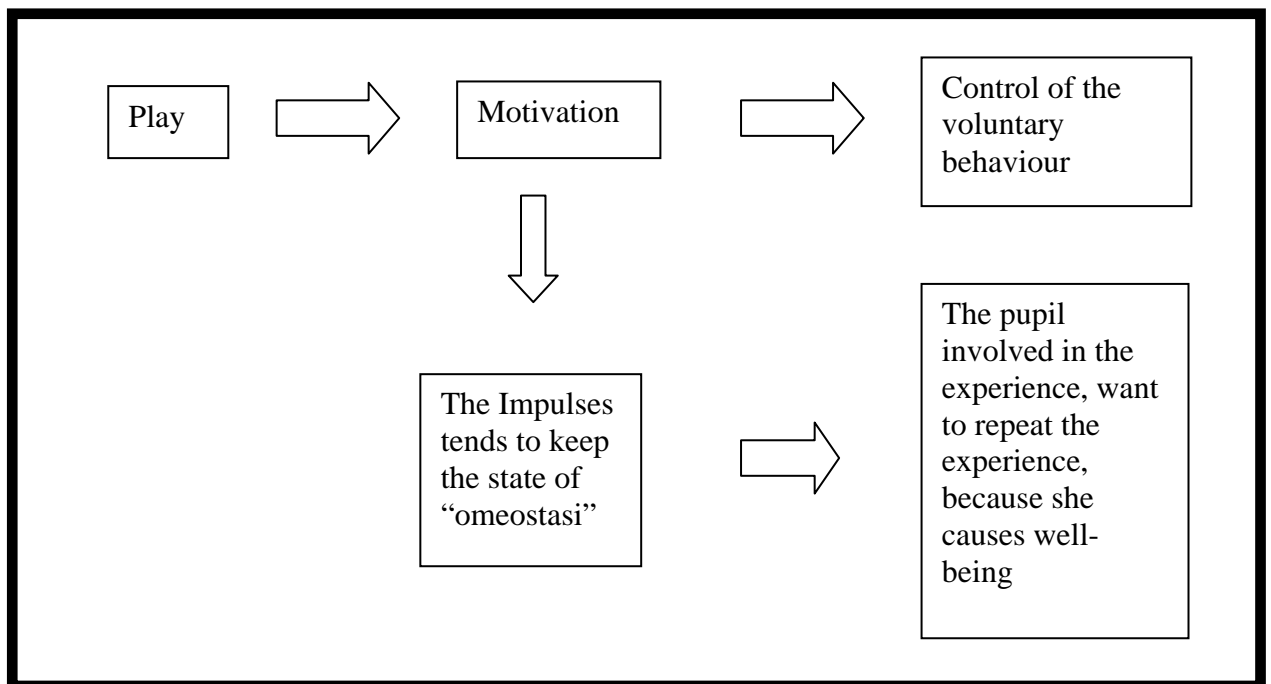
   Def: impulses are mental forces, which simply and directly urge to action, they are specific motivations (temperature, hunger, sex)

Impulses prevent changes in homeostasis, preserving brain's balance.

Behaviours can have several motives: hedonistic ones (endorphin level linked), anticipating ones or also ecological limits.

There it is another element to include: the endorphin level, which plays an important role in an enlarged motivational system. From this point of view games and laughs become stimuli that rise up endorphin level and consequently make the body feeling good.

TABLET 3

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┌─────────────────────────────────────────────────────────────────────┐
│                                                                       │
│   ┌────────┐   ⇨   ┌────────────┐   ⇨    ┌──────────────┐            │
│   │  Play  │       │ Motivation │        │ Control of the│           │
│   └────────┘       └────────────┘        │ voluntary     │           │
│                          ⇩                │ behaviour     │           │
│                                           └──────────────┘            │
│                    ┌─────────────┐        ┌──────────────┐            │
│                    │ The Impulses│   ⇨    │ The pupil     │           │
│                    │ tends to keep│       │ involved in the│          │
│                    │ the state of│        │ experience, want│         │
│                    │ "omeostasi" │        │ to repeat the  │          │
│                    └─────────────┘        │ experience,    │          │
│                                           │ because she    │          │
│                                           │ causes well-   │          │
│                                           │ being          │          │
│                                           └──────────────┘            │
└─────────────────────────────────────────────────────────────────────┘
```

There is another neurotransmitter we have to value when talking of motivational system: the dopaminergic one. While endorphin has mainly an inhibiting action, dopamine is an exciting one (it forces to action).

Here it is another definition of motivation: *inner body property whose existence has been suggested in order to explain various reactive behaviours.*

Different behaviours are due to:

- ✓ needs

- ✓ wishes

Inner homeostatic processes control the motivational balance such as breath or food regulation. Inner needs arouse people to action and give a specific direction too in order to

get defined goals. Most of the activities are pleasant so there is a natural push in repeating them as often as possible.

Motivational stages increase vigilance levels and decrease the behaving threshold[50] making ready to action.

We can make an example: a thirsty animal is watchful (due to its motivational stage) and ready to find water (its threshold decreasing) even if this requires a greater amount of energy to waste.

Inner needs require a defined and organized sequence of events to reach some goals, find some water when sleepy is hard because the behaving threshold is high, but the need for water is stronger than everything else and force to stand up and satisfy the need.

TABLET 4
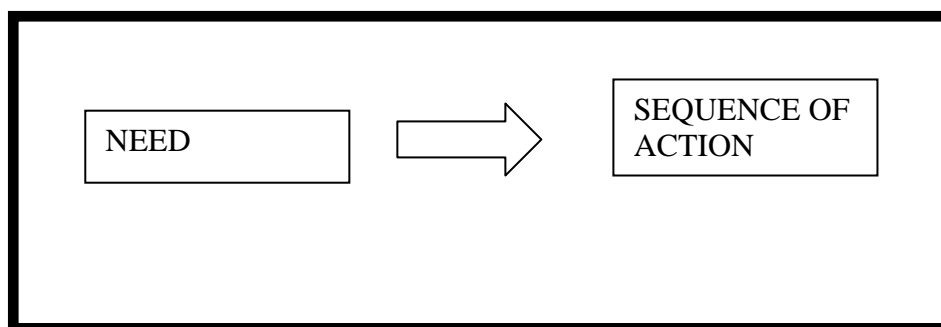
These actions can happen automatically when necessary to survive, that is the purpose of the motivational system. When a goal is reached the linked motivational stage decreases, some events of the sequence of actions disappear and finally the increasing behaving threshold makes the all sequence of response vanishing.

Motivational stages have mainly three purposes:

- ✓ guiding role: goal reaching behaviour
- ✓ switching up role: making people ready to action. The reticular activating system (S.R.A.) send impulses to the amygdala, the limbic system and the cortex alerting them

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[50] The higher the behaving threshold the lower the beginning of the action and the motivational stage

✓ organizing role: putting the needed actions into a coherent and efficient sequence of actions

Impulses are more complex events; they are not related to changes in homeostasis but more to sexual curiosity. They don't come from a "missing" situation but from a search for pleasure (music, literature or anything else).

The idea of motivation can be simply reported as a complex reflex[51], whether exciting or inhibiting, caused by several stimuli of different nature, some coming from the body and some coming from the environment.

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[51] When temperature changes veins and arteries change their size, shaking hands is a derived behaviour

5.4 THE SUBJECTIVE VERTICAL[52]

5.4.1 The vestibular system

Muscles and joints have receptors (muscular, Golgi receptors) that measure limbs motion. They allow what we can call "self-perception", ourselves perception in space.

Skin receptors measure pressure and motions caused by the contact of limbs with themselves or the world around.

Sight measures the world through an image on the retina, the location of objects as well as their shape, colour etc.

The motion of the head can be guessed by information coming from sight and self-perception, but it is also perceived by vestibular receptors. They are located into the inner ear, close to the cochlea, and are made of three semicircular canals (horizontal, front vertical, back vertical) placed on three perpendicular planes, and from otoliths (utricle in the horizontal canal and saccule in the front vertical).

The self-perception receptors are perceptible to within body mass motion but they are not enough to more complex motions such as jumping or running where the brain has to know also head's and body's motion in space; these information are provided by vestibular receptors, that are said "inertial" because they measure inertial strengths.

The semicircular canals are ring shaped and filled with a fluid: endolymph. Each canal ends up with a chamber where there are sensorial receptors. They are sensible to endolymph pressure changes on special hair cells. We will not talk in detail about how this pressure becomes a nerve impulse but it is important to clear what are the mechanical strengths that are able to make receptors responsive: they are the angle accelerations of the head, otherwise the changes in velocity (derived from the angular moving).

The semicircular canal is so responsive to head's angular accelerations. Indications from receptors allow predicting next positions of the head because they are responsive to the derivata of head motion (velocity, acceleration, shake).

One of the main roles of the vestibular system is to assess head's motion in a Euclidean referring system. Such human system is made of:

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[52] For this section see, Berthoz A., *Il senso del movimento*, Mc Graw Hill, Milano, 1998, pp87-103

- ✓ The horizontal level of the head; to recognize it just look at a person on the side and trace a line from the ear to the external side of the eye. The horizontal canal is on a plane of 20° angle with this imaginary line.
- ✓ The other two planes are at a 45° angle to the horizontal and vertical body plane.

This anatomic scheme is very important because the three planes we have found establish a referring system based on human body (self-centred).

The geometry of these canals rules the brain analysis of visual motion and maybe also our motion (it could be the basis of our Euclidean geometry).

A second order of vestibular receptors, the otoliths, figures out the direction of gravity and so the head penchant.

When the head inclines, gravity force acts on receptors and it is possible to have an idea of the penchant of the head relating to the earth gravity system. Animals have soon figured out that gravity is very useful; it is always true and stable in motion. The otoliths system is made of the utricle (whose plane is close to the horizontal semicircular one, it is sensible to linear acceleration when a car moves or suddenly stops) and the saccule (whose plane is close to the front semicircular one and answer to vertical linear accelerations, such as a feet jump).

The two classes of receptors (canals and otoliths) measure head's motion and use the gravity system as a reference to provide brain information on head position, but they also measure stillness: they are essential to evaluate the "subjective vertical" concept.

Vestibular information reaches brain's cortex where it has many roles: conscious perception of direction, eyes motion, posture, and coordination.

This information is useful to build a whole perception of relations between body and space around. The role of the vestibular system in high cognitive functions had been unknown until short time ago.

Vestibular information is important to the body scheme perception and to its spatial representation. We recently proved vestibular existence by tomography techniques; people suffering from diseases of this area have problems in subjective vertical perception. They see the world sloping to the side opposite of their damage.

Neuro-physiology showed the role of vestibular system on cortex areas, here we have two examples of relation between sight and vestibular receptors:

- ✓ Neurons of primal sight areas respond to changes in light and darkness and their direction. Newborn monkeys and cats sense the light but do not realise its direction, while a little percentage responds to vertical and horizontal direction. Especially cats

develop very early this skill. Receptors in ocular muscles, activated by eye's motion, send information to visual cortex activating some cells sensible to direction. Self-perceiving is so important to perceive direction, but it is not the only actor in this process. Some authors recently discovered that vestibular areas have a role in proper sight-related areas, such as V2; 40% of neurons of V2 area react to visual shapes as well as to head's motion. Every receptor is sensible to motion opposite the vertical line; this would clear how we steady perceive external world.

✓ Vestibular system and gravity force: we all know how difficult is to recognise a face when it is 90° bended, just like watching a bending painting. Gravity is very important when talking of shapes and symmetry, as studies on spaceflights show. If we have to figure out whether a shape is symmetric or not we perform better when the main axis is vertical or horizontal, we spend more time if it is oblique. But astronauts lose the vertical leading role. We can experiment the same thing when lying on the back. The otoliths system has a big role in visual perceiving of the environment.

5.4.2 Distance perception

Distance perception is another example of multi-sensorial nature of perceiving. It is the result of information on visual messages and eyes convergence. It joins motion and visual receptions. In visual area V1 cell receptors react to different information coming from the eyes and to the distance itself.

Not all philosophers agree with Helmholtz who tells that unconscious inferences bring distance to acknowledge; Merleau-Ponty, whose thoughts had often been proved by facts, has been wrong on that issue: " *how it is possible to think that the size, the images on the retina and the convergence of the eyes can produce the perceiving of distance? And the difference between the image of the right and left eye cannot generate a relief because none of these signs is strictly conscious, and there is not thinking where there is not introduction*".

5.4.3 Reference systems

The reference system is strictly linked to space concept. We all act inside a space, what Grusser[53] calls "personal" space, "extra-personal" space and "far" space. Each one of them is made of many sub spaces, experienced in different ways that provide different references.

The first one to show a neural basis on personal and extra personal space was Hyvarinen[54]; he discovered that the parietal cortex of monkeys had a neural activity every time some one brought some grape inside the monkeys' "keeping" space. Although monkeys like grape this did not appear the only reason for such neural activity.

The personal space is made of the space of "self" (self-centred space); internal senses perceive it and place it inside body edge, body itself can be an external object: the hand I see is not necessary mine; we need a perceptive act to perceive an object as a body element.

The keeping space is divided into: "intra-oral" and "peri-oral" space (the last is important during the first six months of life), touch, sight and smell define both.

But there are tools that can put throughout our body perception like stilts or wheels that make pilots feel the ground sensation when landing.

The atypical sensations have a big role in such perceptions. The ability to join external tools to our body changes the reference system; in fact our brain sees the tools as something in the external space, as something that enlarge in the space our bodies.

5.4.4 "Allocentric" and "egocentric" reference

We are able to build a spatial relation system having our body as the centre of the whole; we call such a perception "egocentric", we can place objects using a Cartesian coordinates system made of the two axes of the body or the perpendicular plane of semicircular canals; it still is an egocentric reference system.

But we could relate what we are surrounded by to everything but our bodies, an "allocentric" reference system, based on something out of our body.

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[53] Grüsser and Landis,1991

[54] Hyvarinen,1982

It seems that animals basically work with an egocentric perception; human beings and other primates seem able to use both of them.

This skill gives a remarkable advantage because it makes possible to discuss over distance and different sizes among objects and making geometry possible and true.

The last point is that the "allocentric" system is independent from body motion, so that it allows simulating mental motion.

5.4.5 Natural reference systems: gravity

Semicircular canals are a Euclid reference system that makes possible our geometric perception of the space.

As we noticed before it is only an "egocentric" reference system, it gives information only to our body motion.

There is another system linked to external space that is **gravity**:

- ✓ It is always constant
- ✓ It is related to the tangent plane to the planet, it does not change
- ✓ Otoliths react to it; it is a plumb line similar force to refer to body motion in a geometric reference system.

To show how important gravity is just think of normal actions such as walking and jumping, the head is always stable in rotation, in an angle that depends on eyes and sight direction.

The vestibular system makes it possible because it feels the angle of the head according to gravity force; there is an imaginary line that connects the centre of the eye to the ear meatus lining the horizontal semicircular canal.

The head is so stable regarding the vertical line. There are probably otoliths that perceive head's sloping and gravity force, just like if brain created a stable platform to coordinate arms and legs motion.

When walking or jumping feet do not touch ground, which cannot be used to give references, brain uses so the vestibular system and the vertical guide it gives, to generate a mobile referring platform.

Gravity comes into development in very specific times; if we change them we see that mice develop a lack of locomotion. This specific time is about ten days after birth when nervous system needs gravity to organize motion coordination.

5.4.6 The subjective vertical as a multisensorial result

A clear example of multi senses perception is the vertical one: closing our eyes we still feel the direction of gravity, on the other hand if we are in a bright room sloped to the earth vertical axis we can misperceive the vertical axis due to sight information (the "visual vertical"). Sight so let us make inferences on the vertical direction.

There are lots of experiments showing sight role in vertical perception, such the one where people in a sloped room slope their bodies according to what they perceive as vertical (the subjective vertical).

The Aubert effect is another evidence of how senses modify the perception: there is a dark room with only a lighting sloping bar, people put their vertical perception in the middle of the gravity vertical axis and the visual one.

The Muller effect: it happens when people perceive the vertical more sloped than the visual line, thing that happens when the angles are big and wide. The vertical perception is so a multisensorial task.

Brain uses data from vestibular receptors, sight and self-perception and contrasts them with body direction; it also uses the environment visual information like astronauts or people suffering from vestibular diseases. Astronauts have some problems in finding the vertical in gravity absence, like keeping up perpendicular to the floor, to help them they have received a lightened box placed in front of them and fixed to their heads; in such a manner astronauts would see the context steady.

They have not had any information on their body's position, nor they could use pressure on feet to find the position regarding the floor. The first day of flight astronauts sloped forward, but when asked they said they were straight and perpendicular, information coming from self-receptors was not useful to measure the body angle. These people were able to find the perpendicular attitude when thrown away the lightened box and looking at the spatial station. We can assume that sight is determining to adjust self-perception.

Some days after the same astronauts were able to keep a perpendicular attitude showing that brain has succeeded in adjusting self-perception. Brain is so able to analyse data coming from different senses and extract useful information from them.

5.4.7 Vestibular receptors in the stomach

The vestibular system is a very important "egocentric" reference tool that allows the perception of the plumb line model. Some experiments show the existence of another inner reference system, the "idiotrophic vector": an inner human perception of the longitudinal body axis.

If we put someone on an armchair slightly sloped to the ground vertical axis and ask her/him to close the eyes and find out the vertical ground axis the answer will be a line in the midst of the true vertical axis and the body axis. This experiment suggests that the subjective vertical comes out as a result of two vertical axes: the one of the otoliths and the body axis (the "idiotrophic vector").

Mittelstaedt (1995/96) would say that we needed a new sense to add to the ones involved in gravity vertical perception, he discovered some neural receptors placed in the stomach that react to gravity (he recently ended up saying that these structure were rather placed on kidneys or blood system).

He has given contributes to a wider consideration range over the multisensorial nature of perception. If we have so many receptors involved in vertical perception how do we come to a unique perception? There are two ways of thinking:

The first one is a summing up of different vectors, and it is the most spread hypothesis.

The second one says that there are few brain structures able to build an inner model of the body vertical axis; this body scheme would change according to data coming from receptors.

5.4.8 The turn-spit experience

Sight is not the only sense that has a role in helping vestibular receptors to perceive spatial orientation; touch has a role too. Skin can perceive bad control of posture, enabling the Berlioz thesis of brain using receptor data and places to perceive and act.

This is a held experiment: a person is in front of a turning circle holding on vertical plane while a touch on the shoulder causes a touch asymmetry, enlarging the split between the

subjective vertical and the perceived one. In space astronauts using only the teeth to hold the body have the feeling that the vertical turn together with the body.

The turn-spit experience: when in bed there is not doubt on vertical direction, Lackner (1979) built a turn-spit like engine to study the work of otoliths in perceiving the vertical axis. Subjects lie down and turn according to the rotation axis of the earth. If they are open eyed they understand the direction they are moving through, but when they cannot see they only have touch and vestibular canals to get information from.

During the first acceleration stage semicircular canals react to angular acceleration, and otoliths to gravity, considering that the velocity is unvarying and angle acceleration none, semicircular canals stop working after twenty seconds. In turning only otoliths and skin can determine the direction of the body regarding the vertical axis.

The subjects perceive the rotation axis in the darkness, but if we change the touch information it consequently changes the perceived body orientation. Touch data build the reference system and the direction of the rotation.

This shows up how brain can make some perceptive decisions based on several receptors called "receptors configuration".

5.4.9 Conclusions

What we have said up to now stands out how brain uses several reference systems, according to what it has to do and what kind of information it gets.

It seems likely that reference systems do exist for action or for a purpose.... the infinite number of possible representations would let the brain build one to one reference systems.

This statement is now a hypothesis but if true would explain our skill toward geometry intended as the ability to change our mental point of view on different objects.

Some neuro scientists believe that spatial relations like back and forth, up and down (that stable relations between objects) are due to different processes than the distance setting system.

Here we quote some of Poincarè (*La Valeur de la science*) words on Euclid geometry birth [our translation]:

"Semicircular canals give information on motion like muscular receptors, sense apparatus provides information on changing in environment. It does not appear why the creator

would have given us such apparatus that keeps advising: remember that space has three dimensions because this number does not change at all".

He asks another question: why Euclidean motion is perceived as true?:

"Thanks to natural selection our spirit has got used to the external context, it uses the most suitable geometry, geometry is not true but useful. There is not geometry truer than another; there is only one more useful and suitable. Because it is the simplest, and it goes along well with natural solid properties"

What are relations between geometric and representative space?:

"We often hear that the images of things are placed in an outer space, where they can only exist. This space is equal to the one of geometry and shares its properties with it"

The geometric space is continuous and never-ending, three dimensioned and homogeneous. The representative space (mostly visual) has two dimensions, and gets its third dimension thanks to convergence and adjustment; it is not homogeneous because the most susceptible portion of the retina, the fovea, is not homogeneous to the peripheral area. Poincarè ends up saying that the representative space is an image of the geometric one shaped by our perception system's rules:

"We work on outer shapes but we handle them as if they were into the geometrical space".

"Placing an object simply means to find a motion sequence to reach it, it is not a matter of seeing every single act but feeling the muscular sensation that goes with it".

If the space is only the set of motion to move through it, how was the idea of geometric space born?:

"We need to scan how we realise the changes in things, it can happen by a change of place or state. Every change goes with a change in impressions".

Or simply the changes in receptors' activities, brain distinguishes between changes of state and place because the last can be corrected:

"If there has been a change in place we can get back to the first set of impressions by moving to the old place. Someone not able to correct things by changing place would never been able to distinguish between state and place".

Sight and touch are nothing without the muscles' sense. All these procedures require firm objects; there would not be any geometry without them. Geometry is so the description of

changing in places, otherwise, external changes balanced by our bodies moving, felt and measured by vestibular receptors.
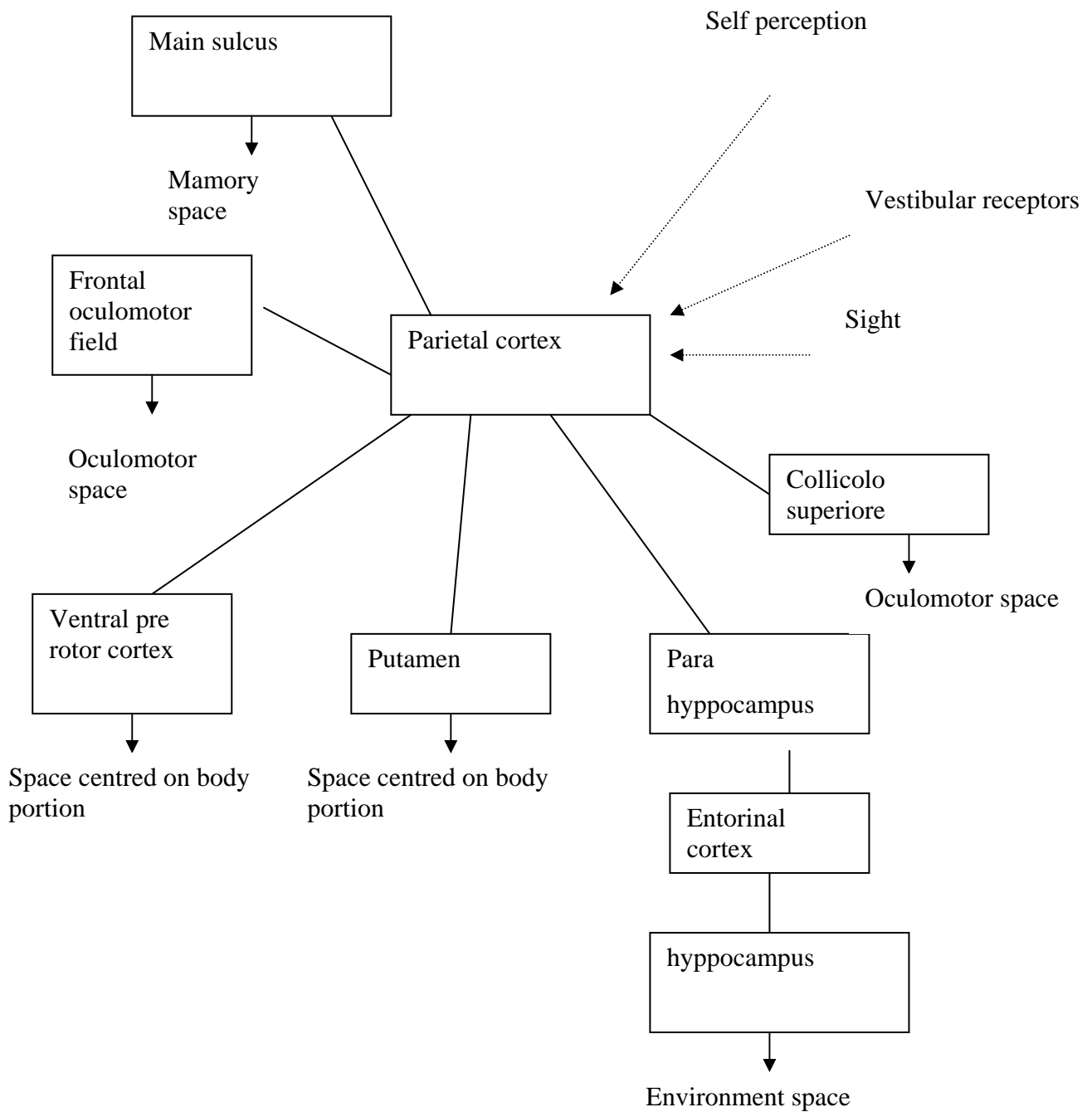
Tablet .6 Data from receptors all go to parietal cortex where they mix with other data from motion and action. Actions are afterwards set into different reference systems that match to body parts or inner space of memory. (from Goss and Graziano, 1995)

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