Chapter 1. A basic critical historical approach to infinity

Before introducing the didactical aspect pertaining to this work, we would like to provide readers with a brief historical-critical excursus on the main phases that the delicate and complex concept of mathematical infinity underwent. The aim of this chapter is therefore to shed light on the origins of the epistemological obstacles related to this subject matter (see paragraph 2.5). These obstacles "justify" teachers' and students' convictions on mathematical infinity, which will be pointed out in chapters 3 and 4.

For the treatment of this chapter we refer to: Arrigo and D'Amore, 1993; Boyer, 1982; D'Amore, 1994; D'Amore and Matteuzzi, 1975, 1976; Geymonat, 1970; Lolli, 1977; Rucker, 1991; Zellini, 1993 and other names quoted throughout the text. The present research work has been primarily influenced by the singular work and personal interpretations of D'Amore (1994), which we are obliged to.

1.1 Prehistory: from 600 B.C. to 1800

> *«There is a concept corrupting and altering all the others. I am not talking of the Evil whose restricted realm is ethics; I am talking of infinity.»* (our translation)
>
> [Borges J.L., 1985]

1.1.1 From the Ancient Times to the Middle Ages

Thales of Miletos (624 B.C. – 548 B.C.). He identifies the *origin of all things* (*arché*) in water as according to him everything is featured by a primordial state of humidity to which all things return.

Anaximander of Miletos (610 B.C. – 547 B.C.). Pupil of Thales, he defines *arché* as something qualitatively undefined (recalling the idea of indeterminate), divine, immortal, imperishable, without any boundaries (recalling the idea of unlimited) but not consequently chaotic. He calls it *ápeiron* (infinity). According to Marchini (2001), it is reasonable to think that in Anaximander's times the concepts of infinity, unlimited and indeterminate were considered synonyms.

Anaximenes (586 B.C. - 528 B.C.). He suggests that the origin of things lies in the infinite air, since air is the substance that better represents unlimitedness and omnipresence, which are typical of the primordial principles.

Two main currents are thus created. The first considers infinity in a negative way: incomplete, imperfect, without boundaries, indeterminate and source of confusion and complications (i.e. Pythagoras' followers and Aristotle). The second holds infinity positively, as it is a concept that embraces all qualities [**Epicurus (341 B.C. -270 B.C.)**].[1]

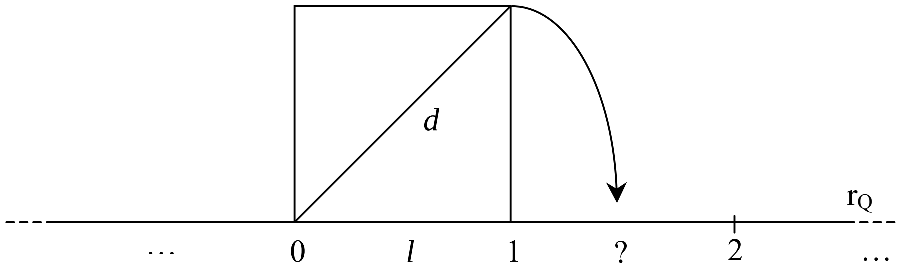
Pythagoras of Samos (580 B.C. - 504 B.C.). Mathematics is the key to the understanding of the whole universe. Everything can be described through natural numbers and their ratios which are aggregates of *monads*, which in turn are unitary corpuscles provided with size, though being so small not to be further divisible and not without dimension in any case.

Every single body is composed of monads, not randomly arranged, but on the contrary positioned according to a given geometrical-arithmetical order. Pythagoras is therefore a finitist as well as Plato.

His School is faced with the problem of incommensurability, originated from the conception of expressing everything with a natural number of monads or better said

[1] According to Epicurus, infinity is the positive principle in the becoming of bodies, whereas void represents the negative principle. This statement was drawn on also in religion and mysticism, which attached an ontological meaning to infinity.

with natural number ratios.[2] There are cases where it is not possible to express with a rational number the ratio between the lengths of two segments:[3]



The discovery of incommensurability of the square's diagonal and side (Kuyk, 1982) is to be traced back to Hippasus of Metapontum (V century B.C.) who lost his life because of his outrage to the Pythagorean School.

The dispute was between intuition and reason and represents the first case where the latter goes in the opposite direction of the former. Mathematical entities stopped being sensible and became purely intelligible, thus opening doors to infinity. This could perhaps be considered as the first step towards the conception of mathematics as belonging to the world of ideas, a conception that would dominate Greek philosophy.

Parmenides of Elea (504 B.C.).[4] In his Poem: *Perì Physeos* (*On Nature*) Parmenides introduces a dichotomy between two different ways of interpreting truth: a truth of sensible origin (*doxa*) and an opposite Truth of rational nature (*Alétheia*). The human being can use *doxa* only for the supreme goal of reaching *Alétheia*. In order to avoid paradoxes, *doxa* excludes the concept of infinity (i.e.: shooting an arrow a few steps before the end of the universe). Whereas *Alétheia* represents the spiritual height, the highest knowledge, the single immutable *being*, indivisible, eternal, immobile. Infinity

[2] In this respect, the following sentence, quoted from Plato's Theatetus, is quite remarkable: *«The ignorance of those who believe that all pairs of magnitudes are commensurable is disgraceful»* [our translation].

[3] It was Archita (430 B.C. - 360 B.C.), who first managed to demonstrate that the ratio of these two segments could not be expressed as a ratio of two natural numbers.

[4] Dates concerning Parmedides' life are quite difficult to assess. We therefore indicated 504 B.C., the date of the LIX Olympics, which according to Diogene Laertius corresponds to the most significant period of the work of Parmenides.

is thus conceived as totalising, all embracing, though *limited («The universe is limited because without limits it would be missing everything»*).

Zenon of Elea (born in 489 B.C. ca.). Parmenides' pupil inherits the knowledge of his master reinforcing the idea of immobility and immunity of the being that had raised harsh criticisms. Zenon's famous paradoxes confute the ideas of plurality and movement (i.e.: Dichotomy, Achilles and the Tortoise, The Arrow, The Stadium). As Marchini states (2001): *«To Zenon considering infinity an attribute of the being, due to inexhaustibility of infinity itself, brings about the irrationality and impossibility of the being. He is actually against this vision of infinity in act»*. All the paradoxes linked to the concept of infinity caused such confusion that Aristotle forbade the use of it, in order to avoid this *«scandal»*. It is therefore thanks to Parmenides' abstract position and to Zenon's paradoxical creations that Greek mathematicians had seriously to face the problem of infinity, though desperately trying to avoid it.[5]

Melissos of Samos (end of VI century B.C. - beginning of V century B.C.). In order to demonstrate Parmenidean theses concerning the idea of *single being*, Melissos elaborates his master's thought denying the concept that the determinate nature of being implies its finite character too. He conceives a spatially infinite being that admits nothing outside itself.

The rebellion to Parmenides started with the Pluralists and among the others **Anaxagoras of Clazomenae (500 B.C. - 428 B.C.)**. This philosopher devoted all his life to reflecting on the matter and its components creating the term *homeomeries* to indicate infinitesimal elements, not further divisible and characterised by different qualities. Interesting to the aim of present research are the following statements written in his book *On Nature*: *«In the large as well as in the small there is an equal number of parts (...) with regard to the small there is no smallest, but always an even smaller, because the existent cannot be annihilated (by division). Thus, with regard to the large there is*

[5] From a didactical point of view, a number of research studies deal with the debate on the truth of rational nature as opposed to the truth of sensible origin: Hauchart and Rouche, 1987; Nuñez, 1994; Bernardi 1992a,b.

always a larger, and this larger is like to the small in plurality, and in itself everything considered as the sum of infinite infinitesimal parts is at the same time large and small» (in modern language, it is obvious that a shorter segment is included in a longer one, but if we think of both entities as sets of points, we will observe that in a longer segment as well as in a shorter one there is the same number of points). Mathematicians would often return to this concept during the course of history, but it will be only thanks to the German scientists of the XIX Century that the above-mentioned notion will find a rigorous systematisation. In Anaxagora's statement the ideas of infinity and infinitesimal are strictly related to one another. In some parts, it seems that the infinite subdivision is to be understood as potential, whereas at times Anaxagora seems to refer to actual infinity.

The rebellion went on with the Sophists such as: **Protagoras of Abdera (485 B.C. - 410 B.C.)** and **Gorgias of Leontini (483 B.C. - 375 B.C.).** They claimed the superiority of sensible experience towards rational truth, thus influencing the mathematical thought and the issue that is the topic of our research as well. As a matter of fact, according to the sensible experience the circumference does not touch the tangent at a point but along a segment of a certain length.

Among the Atomists we recall **Leucippus of Abdera (460 B.C.)** master of **Democritus of Abdera (460 B.C. - 360 B.C. ca.).** According to their thought the void exists and is the place in within the atoms move. Democritus, in particular, drew a distinction between two different aspects of infinite divisibility: from an abstract mathematical point of view, every entity is infinitely divisible into parts (especially segments and solids); from a physical point of view things change: there is a material limit to divisibility and the limit is a unitary indivisible material corpuscle called atom. There seem to be even more kinds of atoms with different dimensions.

Aristotle of Stagira (384 B.C. - 322 B.C.). As Plato did, and Socrates even before, Aristotle accepts the Parmenidean idea of a limited universe according to the nature of the Greek philosophy that despises disorder caused by the matter in its chaotic form. These limits surrounding the universe and arranging it at a rational level, make it acceptable to the human logic: *«…Since no sensible magnitude is infinite, it is not possible that a given magnitude could be overcome as in that case there would be something greater than the sky»*.

As to infinity, Greek philosophy and mathematics felt great embarrassment towards this subject because it was full of contradictions and paradoxes, profoundly influencing Aristotle's thought. He was the first to reveal a double nature of infinity: "in act" and "in power". "In act" means that infinity appears as a whole, given as a matter of fact, all in one go. "In power" means that infinity is referred to a situation which is finite at the moment we are talking about it, but with the certainty that the set limit could be overcome all the time (thus the limit is not definitive): *«A thing comes from another with no end and each thing is finite but of these things there are always new »*.

In short: *«[the actual infinity is] that beyond which there is nothing else; … [the potential infinity is] that beyond which there is always something else»* (*Physics*).[6]

Aristotle forbade the use of *actual* infinity to mathematicians solely allowing the use of potential infinity: *«Therefore infinity is in power and not in act»*. In Aristotle's opinion a segment is not composed of infinite parts (in act) but is divisible by infinite times (in power).

> *«In any case our debate is not intended to suppress mathematicians' research due to the fact that it excludes that the infinity by progressive growth is such that it cannot be taken in act. As a matter of fact, at present state, mathematicians themselves do not feel the need for infinity (and they do not even use it) but they only need a quantity as large as they please, though finite in any case. (…). Thus for their demonstrations' sake they will not care about the presence of infinity in real magnitudes»* [our translation] (Physics, III, ch. 7).

For a long time this prohibition was conceived as a real dogma: many scholars from the Middle Ages and the Renaissance, as well as from more recent times, were almost about

[6] There are many "Aristotelian" studies on the potential and actual use of the term infinity both in the subject form (infinity) and in the adjective form (infinite): Moreno and Waldegg, 1991; Tsamir and Tirosh, 1992.

to “master” the concept of infinity including its paradoxes, but Aristotle’s legacy was ever somewhat binding.

Aristotle also pointed out the distinction between infinity by addition and infinity by division (*Physics*) as explained in Zellini (1993): *«If you consider a length unit and you add it to itself infinite times, the result will be for sure an unlimited distance not coverable in a finite time. But if you envisage the unlimited by means of a somewhat opposite procedure, dividing by dichotomy the length unit into infinite intervals, infinity could be considered in some way exhaustible within a limited time interval»* [our translation].

Euclid (300 B.C. ca.). In his immortal and famous work *Elements* Euclid accepts Aristotle’s point of view. In other words, Euclid is well aware of the problem of infinity and strenuously tries to avoid it.

- In his definition XIV of the book I he states that figures are all finite.
- In the postulate II of the book I he does not use the term *straight line* but he talks of a geometrical entity called *eutheia grammé* (*terminated line*) which by means of a postulate can be *«continuously prolonged straight ahead»*.
- The V and most known postulate still refers to *eutheia grammé* and not to *straight line*. In particular, it explicitly requires the unlimited prolongability of two terminated lines and therefore it would be as much avoided as possible by Euclid himself in his future treatment.
- In the proposition I of the book VII he applies the following procedure: *«If you take two unequal numbers and you successively subtract the minor from the major, the difference from the minor and so on, the remainder never divides the immediately preceding number until unity is obtained. The initially given numbers will be primes to each other»* [our translation]. Taking into account any two numbers the procedure always ends after a finite reiteration of operations.
- In the proposition XX of the book IX Euclid does not demonstrate that *«There exist infinite prime numbers»*. On the contrary, *«Prime numbers are more than any other previously suggested total number of primes»*, in accordance with the position of

Eudoxus of Cnidos (408 B.C. - 355 B.C.)[7] who deals with infinity never calling it by its name.

- One of the most famous common notions (*coinaì énnoiai*) subscribed by Euclid is: *«The whole is greater than its parts»* which is in contrast with Anaxagoras' intuition.
- The problematic nature of infinity is not always revealed by the aspect of prolongability or, as in Aristotle, by the infinity by growth. Euclid's viewpoint includes also the infinitesimal with the demonstration that *the contingency angle* is minor than any *rectilinear angle*. This denies the Eudoxus' postulate today called Eudoxus-Archimedes postulate. As a matter of fact, in the book V of Elements Euclid states: *«Two magnitudes are set into relation if each of them, multiplied by a certain appropriate number [natural], overcomes the other»*, cleverly excluding in one go mixtilinear angles from the set of rectilinear ones (thus avoiding to talk about "actual infinitesimals"), (D'Amore, 1985).

Euclid's work with regard to infinity is all based on Aristotle's philosophic choice: he completely rejects the actual infinity and accepts and uses only the potential one. By sharing this position, he is extremely rigorous and strict.

Archimedes of Syracuse (287 B.C. - 212 B.C.). He was committed with the method of exhaustion based on the division of geometrical figures (plane or solid) into infinitesimals (actual) and infinite sections. Archimedes dealt nonchalantly with very delicate matters showing to be not particularly prone to remote philosophies. He obtained significant and courageous results. At this point, it is reasonable to wonder if Archimedes knew the issue of infinity or not. Evidence of this is given in his work *The Sand-reckoner*. In this text Archimedes calculated how many grains of sand are contained in a sphere whose radius is given by the distance of the Earth from the Sun. The answer is approximately 10^{63} and Archimedes had to invent a numerical system

[7] Eudoxus of Cnidos managed to elaborate a theory on proportions, which allowed to operate on ratios without using actual infinity. We also owe him the method of exhaustion, also aiming at eliminating actual infinity. Both these methods do not abolish infinity, but they tend to prefer the potential infinity to the actual.

that goes beyond *myriads.*[8] The greatest number ever reached by Archimedes is *a myriad of myriads of unities of the myriadesimal order of the myriadesimal cycle,* i.e. $M^{(M^2)} = (10^8)^{(10^{16})}$, far larger than the "only" 10^{63} grains of sand he needed. This proves the necessity of ever-increasing numbers than the ten thousand possible in the ancient Greek language. At the same time, it has to be noted that he feared to "exaggerate", i.e. to run the risk of "involving" infinity. The need for a well-defined limit is very strongly felt.

Lucretius (10 B.C. – 55 A.C.). Known for *De Rerum Natura* (*On the Nature of the Universe*): *«Suppose for a moment that space is limited and that somebody goes up to its ultimate border and shoots an arrow…»,* this sentence embraces the idea of an unlimited universe (Book I, 968-973).

Clemens of Alexandria (150 - 215). Infinity is considered as a divine attribute. It is applied with a positive connotation to divinity and with a negative one to our unableness to understand divinity in its ineffability.

Diophantus of Alexandria (250 ca.). He introduces numerical variables using an advanced symbolism subsequently adopted and studied by his "pupil" Fermat in the XVII century. In the algebraic use of numerical variables the concept of infinity is concealed.

St. Basil the Great (330 - 379). Infinity becomes synonym of the completeness of divine perfection. From this moment onwards infinity will be always quoted in relation to divine attributes. Therefore philosophers will try, in different ways, to prove such a quality of the Supreme Being.

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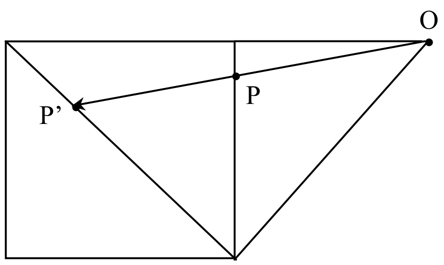
[8] Rucker reports (1991): *«The Greeks did not know the notation of exponentiation, they just used that of multiplication, moreover the maximum number they could name was a myriad, which is equal to 10,000 that is to say* 10^4*».*

St. Augustine of Tagaste (345 - 430). In his work *De Civitate Dei* he admits the actual infinity of natural numbers: *«God knows all numbers in an actual way. Actual infinity is in mente Dei»*.[9]

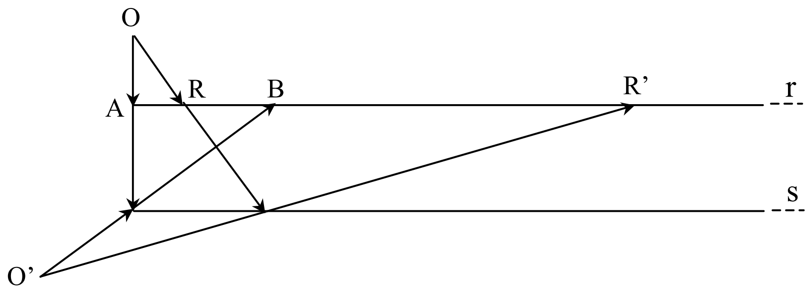
Proclus (410 - 485). Infinity is still connoted as potential when it gradually expands starting from the intelligibles, whereas Proclus seems to stand for actual infinity when he tries to convey finite and infinite principles into the One: *«Every existing thing is somewhat finite and infinite because of the first Being…* (*since*) *it is clear that the first being communicates to all things the limit as well as infinity, being itself made of these elements»* [our translation] (*Elementa teologica*).

Therefore is ever growing the importance of the distinction between philosophic and mathematical infinity.

Roger Bacon (1214 - 1292). In his work *Opus Maius* (1233) he claims that we can establish a biunivocal correspondence (as we would say today) between the points of a square side and those of the same square diagonal, although they have different lengths (the idea will be further developed by Galileo). Moreover, such a biunivocal correspondence could be established (by translation or double projection) between two half-lines (one with A origin and one with B origin) positioned on the same straight line *r*.



[9] In the second half of the 19th century, Cantor acknowledged Augustine as one of his sources of inspiration to support the set theory.



He concludes stating that mathematical infinity in act is not possible according to logic: the whole would be not greater than its parts; this principle would be anti-Euclid and therefore also anti-Aristotle, an attitude still perceived as forbidden.

St. Thomas Aquinas (1225 - 1274). In *Summa Theologiae* we find evidence of the idea of actual infinity conceived to be in *mente Dei*. In this text, Thomas admits the possibility of the existence of different levels of infinities in the infinity, but he also claims in some other passages that the only actual infinity is God. With regard to things, he talks about the infinity in power and consequently he attributes to mathematical infinity the solely potential aspect: «*… it stands out clearly that God is infinite and perfect… So even if He is God and He has an infinite power, He cannot create something un-created* (*this would be a contradiction*)*, He thus cannot create any thing that is absolutely infinite*» [our translation].

William of Occam (1290 - 1350). He writes in *Questiones in quator libros sententiarum*: *«It is not incompatible that the part is equal or not minor than its whole; this is what happens every time that a part of the whole is Infinite. This is verifiable also in a discrete quantity or in any multiplicity whose part has units not minor than those contained in the whole. So in the whole Universe there are not more points than in a bean, as a bean is made of infinite parts. So the principle that the whole is greater than its parts is valid only for the things composed of finite integral parts»* [our translation].
William is accused of heresy in 1324 and is held for being questioned for 4 years in Avignon, then he escapes to take refuge first in Pisa and then in Munich. So far, it is still too dangerous to contradict Aristotle’s thought.

Nicholas Oresme (1323 - 1382). He has an intuition about the coordinates to which nowadays we refer to as Cartesian. He sets the value of the following “sum” *s* implying an absolutely modern use of infinity:

$$s = \underbrace{1+\frac{1}{2}}_{>1}+\underbrace{\underbrace{\frac{1}{3}+\frac{1}{4}}_{>1/2}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>1/2}}_{>1}+\ldots \quad \ldots$$

Given any natural number M (large in any case), after a certain number of addenda, *s*>M. So: *s* is greater than any natural number though large.

Nicholas of Cusa (1400 or 1401 - 1464). He considers mathematics as an ideal of perfection and therefore he feels the need for a cosmos ordered according to “weight, number and size”. He refers to infinity only from a mathematical point of view, dealing with the infinitely large and the infinitely small. Nicholas of Cusa is the last medieval neo-Platonist. Infinity is almost absent as cardinal and is considered as ordinal or as a not well-identified “vastness”. In accordance with the medieval spirit, Nicholas of Cusa confuses infinity with unlimited or at times even indefinite (this confusion will last till the XIX century and even further, see teachers’ statements reported in 3.7.1).
In his major work, *Docta Ignorantia* (*Learned Ignorance*), one of his most famous and beloved analogies is to be found: *«Intellect is to truth as the polygon of n sides is to the circle. When n tends to infinity, the polygon tends to the circle; the truth is therefore the limit of the intellect to infinity»* (ch. III, Book I). Moreover in this text, a paradox concerning the actual infinity, similar to those treated by Galileo and Bolzano is to be traced: *«If a line is formed by an infinite N number of one foot long segments, whereas another line is formed by an infinite M number of two feet long segments, these two lines are equally long and this length is infinite; therefore it can be concluded that “in the infinite line one foot is not less long than two feet”»* [our translation] (ch. XVI, Book I). In addition, in the *Conjectures*, an improperly carried out argumentation of Zenonian nature is to be found, it aims at demonstrating that any two lines have the same number of points (ch. IV, Book I). As already mentioned, this topic was a matter over-debated for millennia, e.g. Anaxagoras and Roger Bacon had already dealt with it

and the question will be settled only at the end of the XIX century thanks to the work of Cantor. On the idea of maximum, Nicholas of Cusa claims: *«No infinite number is known and no given maximum either»* (*Conjectures*, ch. XI, Book I).

In conclusion, a proper and solid conscience of infinity is still to be achieved, and the history of mathematical thought has still to wait till the Renaissance when, thanks to the research of major artists in the field of perspective and Galileo's brilliant reflections, the accomplishment of such a miracle could be witnessed: Bonaventura Cavalieri and Evangelista Torricelli could finally "see" what scholars from the Middle Ages could not clearly and thoroughly "see".

1.1.2 Infinity in the Renaissance

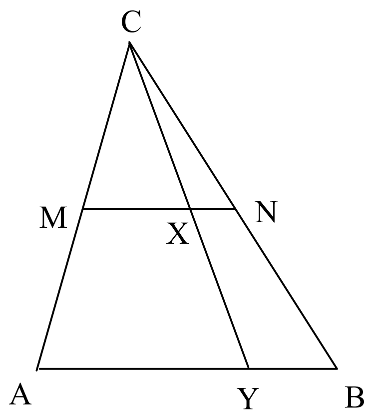
Infinity is extremely present in the Renaissance, not in the "numerical Universe" but in the world of geometry and fine arts (which happened to coincide at that time): **Piero della Francesca (1406 - 1492)** writes *De Prospectiva Pingendi*, a mathematical-pictorial work of great value; **Girolamo Cardano (1501 - 1576)** writes the treatise *De Subtilitate* (1582) on subtlety, i.e. on something that we can also call "infinitesimal magnitudes". In particular this work deals with the contingency angle.

Moreover, in the Renaissance "the method of indivisibles", already dealt with by Archimedes, is further developed thanks to: **Leonardo da Vinci (1452 - 1519)**, **Luca Valerio (1552 - 1618)**, **Galileo Galilei (1564 - 1642), Paul Guldin (1577 - 1643)**, **Bonaventura Cavalieri (1598 - 1647)**, **Evangelista Torricelli (1608 - 1647)**.

Galileo Galilei (1564 - 1642). He firstly based his work on Democritus' reflections, but he widened the scope of applicability from geometry to more extended classes of analytical problems. In his last work: *Mathematical Discourses and Demonstrations on two new Sciences* (1958) dated 1638, Galileo collected most of his major considerations on the infinity paradoxes.

Actual infinity is mentioned in several occasions. According to Galileo, lines as well as concrete objects to be found in nature are all formed by a continuum (actual infinity) of parts small as we please though measurable (hence divisible). *«Each part (if one can* *still call it a part) of infinity is infinite; since, even if a line one hundred span long is major than that of only one span of length, there are no more points in the longer than in the shorter but the points of both lines are infinite»* [our translation].

Therefore his geometrical considerations envisage a concept of infinity that can collide with the VIII Euclid's common notion: «*The whole is greater than its parts*». It may suffice to draw a triangle to see that between the AB side and the MN segment that joins the midpoints of the other two sides, there is a biunivocal correspondence obtained joining the points of AB with C. This is clearly in contrast with the common intuition that being AB twice as long as MN it should be formed by a greater number of points.



> «*These are the difficulties deriving from the reasoning of our finite intellect on infinities giving to them those attributes that we assign to terminate and finite things. I consider it as inconvenient as I believe that those majority, minority and equality attributes are not suitable to infinities, about which it is not possible to say if one is major, minor or equal to the other»* [our translation] (Galileo, 1958).

In a non-geometrical field, be:

0 1 2 3 4 … the sequence of natural numbers (N)

0 1 4 9 16 … the sequence of perfect squares (Q_N)

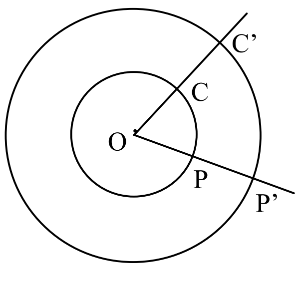
Q_N is strictly contained in N, this according to Euclid would mean that N contains more elements than Q_N, but for each natural number there is its square, that is to say a well determined element of Q_N (and vice-versa). With an obvious intuition, it can be deduced that there are as many elements in N as in Q_N (*Galileo's paradox*).[10] The present treatment appeared also in *Dialogue on the two Greatest Systems*, where the acceleration of a falling body was mentioned.

> *«I cannot come to any other decision than saying that, infinite are all numbers [natural], infinite their squares, infinite their roots, and the multiplicity of squares is not minor than that of all numbers [natural], neither the latter is major than the former, and lastly attributes such as equal, major and minor are not appropriate for infinities but only for terminate quantities»* [our translation] (Galileo, 1958).

Galileo outlined a first definition of infinite set later developed by Dedekind.

The history of infinity has come again to a delicate phase. The mechanism created by Aristotle, to protect from mathematicians one of the possible uses of infinity, has been demolished, though scholars should still go a long way before full conscience of infinity and thus the consequent ability of "dominating" it with technical means, even not extremely sophisticated ones, are reached.

Evangelista Torricelli (1608 - 1647). A pupil of Galileo's, he got in touch with the Geometry of Indivisibles of Cavalieri thanks to his master. He developed hazardous conceptions of infinity and infinitesimals (considered from the actual perspective) and he could intuitively envisage the hyperbole's improper points and consider the finite area not only strictly related to limited figures, as it was believed in his time but it is not actually. In addition, Torricelli recognised that two concentric circumferences (of different lengths) are formed of the same number of points; it is sufficient to consider the common centre as the origin of a projection.



[10] Many studies in the field of didactics concern Galileo's considerations: Duval, 1983; Tsamir and Tirosh, 1994; Waldegg, 1993.

René Descartes (1596 - 1650) and **Pierre de Fermat (1601 - 1665)**. They both deal with "infinitesimals" in order to solve the problem of the determination of the tangents to a curve.

Notable is that Descartes was able to see geometry from a totally new perspective: all geometrical entities and related properties were expressed through an algebraic language. He also dealt with the debate on infinity, but…: «… *we will never get uselessly involved in discussions on infinity. De facto, we are finite and it would therefore be absurd if we established anything on such a matter and tried to render it finite and possess it…»*. Descartes introduces a distinction between *infinity*, attribute proper to God and *indefinite* used to indicate unlimited magnitudes in quantity or in possibility.

Fermat, on the other hand, seems to make no mention of infinity.

Nevertheless the development of analytical geometry deeply influenced the issue of infinity, since it forced towards a comparison between the number infinity and the infinity of geometrical entities giving an enormous contribution to the passage from prehistory to history of the debated subject on the basis of two main reasons:

1. Mathematical Analysis is finally founded (and also infinity can find a rational systematisation);
2. Proper answers are given to the questions: How many are the points of a square and those of its side? How many are the straight lines of the plane? …

On this last point mathematicians still could not find a definite solution; Cantor and Dedekind will finally and eventually shed light on this aspect.

Blaise Pascal (1623 - 1662). He seems to stand for the actual infinity: *«The unit added to infinity does not make it any larger… Finite is annihilated by infinity and it becomes a pure nothing… We know there exists an infinity but we ignore its nature. Since we know that it is false that numbers are finite, it is therefore true that there is infinity of number… We therefore know the existence and nature of finite because we ourselves are also extended and finite in the same way. We know the existence of infinity but ignore its nature, because it has the same extension as we have, but it has no boundaries as we have. We do not know either the existence or the nature of God* *because it has neither extension nor boundaries. But it is only through faith that we know of His existence»* [our translation] (*Infinity. Nothing*).

Gottfried Wilhelm Leibniz (1646 - 1716). He suggests three kinds of infinity: *infimus*, in quantity; *medium*, as the totality of space and time; *maximum*, representing only God, as the fusion of all things into one. As in Kuyk's (1982): *«To Leibniz, each monad had an actual infinity of perceptions and each body was made of an actual infinity of monads»*. Notwithstanding his confidence in dealing with infinitesimals thanks to the efforts of scholars from the Middle Ages and the Renaissance, he showed himself somewhat worried and reluctant when dealing with the above-mentioned magnitudes. Prove of this can be found in a letter addressed to Fouchet: *«Je suis tellement pour l'infini actuel, qu'au lieu d'admettre que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte partout, pour mieux marquer les perfections de son Auteur»*.

We are indebted to **Isaac Newton (1642 - 1727)** for the explicit development of Mathematical Analysis which will be widely spread and developed among the others also by the great mathematician **Carl Friedrich Gauss (1777 - 1855)** who is still convinced that: *«…I protest against the use of an infinite magnitude seen as a fully accomplished whole, as this never happened in mathematics…»*. Infinity is present though still not explicitly investigated.
So deeply rooted are the prehistorical convictions on infinity that they can be still traced back in present times, as we shall see in chapters 3 and 4.

Immanuel Kant (1724 - 1804). He was one of the first who "wiped out" the risk of misunderstandings deriving from the hazardous approach to the notions of infinity (actual) and infinitesimal (actual) adopted by the XVII and XVIII century mathematicians. Kant discovered antinomies in the *constitutive*[11] sense of infinity (see first and second antinomies of the Pure Reason) e.g.: when the world or anything in it contained is considered as finite, the mind can think of it as an extension; when the world or anything in it contained is considered as actually infinite, the mind cannot

[11] According to Newton and Leibniz, infinity had a *constitutive* meaning.

think of it at all. In both cases the mind is not consistent with the world: to reason, finite is too small and infinity (actual) too large (Kant, 1967). As stated by Kuyk (1982): *«Kant's solution was to consider infinity not in a constitutive but in the regulative sense. (…). By this shift of meaning, the notion of infinity goes from ontology to epistemology»*.

The dispute between actual and potential infinity continues. On the occasion of a competition promoted by the Berlin Academy [presided over by Lagrange (1736-1813)] and whose goal was to clarify the concept of infinity, the winner **S. L'Huilier (1750 - 1840)** advocated a return to classical infinity, the Aristotle's one, against the acceptance of actual infinity supported by Leibniz.

1.2 From prehistory to history of the concept of mathematical infinity

From the second half of the XIX century up to these days, the concept pertaining to actual infinity has profoundly influenced mathematical thought.

1.2.1 Bernard Bolzano (1781- 1848)

Between 1842 and 1848 Bolzano wrote *The Paradoxes of Infinity,* only posthumously published in 1851 (Bolzano, 1965). The book is a collection of 70 short paragraphs. Extracts of some of them are reported hereafter:

§13: The set of propositions and "truths in themselves" is infinite.

Wissenschaftslehre (proposition in itself): *«By W. I mean any proposition stating that a thing is or is not, without taking into account if the statement is true or false or if it has been verbally expressed or not by anyone»* [our translation]. When a W. is true is a Wahrheit an sich (truth in itself).

Be A_0 a Wahrheit an sich; be A_1 the new W. an sich: «A_0 is true»; be A_2 the new W. an sich: «A_1 is true»; …

Be $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$. $\mathcal{A}$ is "greater" than any finite set therefore is infinite. Moreover, the elements of $\mathcal{A}$ can be set in biunivocal correspondence with the elements belonging to the set N of natural numbers (by establishing a correspondence between A_i and i).

(It has to be noted that the infinite set $\mathcal{A}$ is built on the language, or better explained, on various metalinguistic levels).

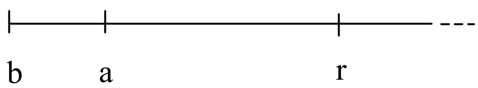
§20: A remarkable relation between the two infinite sets is the possibility to form pairs joining each object of a set with another belonging to the counter set, so that for each object of one set there is always its correspondent and no object happens to appear in two or even more pairs (biunivocal correspondence of two infinite sets).

§21: Notwithstanding their property of being of equal number, two infinite sets can be in a inequality relation as their multitudes are concerned, so that one set is a proper part of the other. (Using modern language: one set is infinite if and only if it can be put in biunivocal correspondence with one of its proper parts. This intuition was developed before Dedekind; but it was no definition and maybe there was still no fully-fledged awareness).

Bolzano was known not only for the significant results we mentioned but also for some famous errors and uncertainties. Some examples are provided here as follows:

§18: If A is a set and some elements have been subtracted from it, then A contains fewer elements than before.

§19: There are some infinite sets that are larger or smaller than other infinite sets. The half-line br is major than the half-line ar, then it can be deduced that there are infinities of different magnitudes.



§29: There is confusion between the cardinality of the {1, 2, 3, …, n, …} set and the value 1 + 2 + 3 + … + n + …

§32: Guido Grandi (1671 - 1742) raised the problem of calculating the “sum” of infinite addenda: $s = a – a + a – a + a – a + \dots$ obtaining several answers:

$s = (a – a) + (a – a) +(a – a) + ... = 0 + 0 + 0 + ... = 0$

$s = a – [(a – a) + (a – a) + (a – a) + ...] = a – [0 + 0 + 0 + \dots]= a – 0 = a$[12]

[12] In 1703 Grandi wrote: *«If we differently position the parentheses in the expression 1 – 1 + 1 – 1 + 1 …we can obtain either 0 or 1. Therefore the principle of creation ex nihilo is perfectly plausible».*

$s = a - (a - a + a - a + a - a + \ldots) = a - s = \Rightarrow 2s = a \Rightarrow s = a/2$ (this solution proposed by Grandi[13] himself was particularly appreciated by Leibniz who defended it).

In Bolzano's time the question was still open and debated. As for this latter paradox, as Bolzano himself reported, in 1830 a writer known as M.R.S tried to provide a demonstration of the third solution publishing it on the *Annales de Mathématique de Gergonne*, 20, 12, to which Bolzano reacted in the following way: *«The series within parentheses has clearly not the same set of numbers of that originally indicated with x* (s in this case), as *the first term a is missing».*

§ 33: Precautions to be observed by the calculus of infinity in order to avoid "mistakes":

Be S_1 the sequence of numbers 1, 2, 3, …

Be S_2 the sequence of their squares $1^2, 2^2, 3^2, \ldots$

Now: since all terms in S_2 appear also in S_1 and there are terms of S_1 that do not appear in S_2, this would imply that the sum of S1 terms is major than the sum of the terms of S_2, whereas the sum of the terms of S2 is major than that of S_1, as both S_1 and S_2 can be set in biunivocal correspondence and each term of S_2 is major than (with the exception of the first term) its correspondent in S_1.

§ 40: Paradoxes on the concept of space: two segments of different lengths are formed of different number of points.

§ 48: A volume contains more points than its lateral surface and the latter more points than the curve enclosing it.

According to Cantor (1932), Bolzano's problems are due to the fact that the idea of a cardinality of a set[14] was at his time missing. There is still a long way to go, we are just at the beginning of our path.

We cannot leave **Karl Weierstrass (1815 - 1897)** out, considered by many to be the one who provided a rigorous systematisation of Mathematical Analysis. He is important for

[13] D.J. Struik (1948) wrote: *«He* (Grandi) *obtained the value ½ on the basis of the anecdote of a father who hands down a precious stone to his two sons. Each of them has to keep it alternately for one year, so that in the end, each son will turn out to own half of the stone».*

[14] As for didactics, many recent studies aim at analysing similarities between students' "naive" remarks and some of Bolzano's statements. On this subject see for example the work by Moreno and Waldegg (1991).

our research because he conscientiously investigated the subject of infinity. Some considered the work of Analysis systematisation initiated by Cauchy (1789 – 1857) according to the modern definition of limit and continuous function (the so-called ε-δ Weierstrass' definition), as the ultimate abandonment of infinity in act in favour of the infinity in power (Marchini, 2001). Others believe that Weierstrass' work was a contribution, also from a formal perspective, to the evolution of the potential infinitesimal towards the actual infinitesimal (Arrigo and D'Amore, 1993; D'Amore, 1996; Bagni, 2001). Ideally, this evolution continued in the XX century with non-standard analysis (Robinson, 1974).[15]

1.2.2 Richard Dedekind (1831 – 1916)

In his book *Continuity and Irrational Numbers* of 1872, the fourth paragraph has a charming and meaningful title: *Creation of irrational numbers*. Creation… and as a matter of fact, thanks to his famous method of "cuts" or "sections", he creates, starting from Q, the set R adding to Q the irrational numbers.[16]

Real numbers are classes of definite sections in Q. (Q, <) is dense but not continuous (this demonstration is basically attributed to Pythagoreans); (R, <) is dense and continuous[17] (Bottazzini, 1981).

Of particular interest is the correspondence between Dedekind and Cantor that will be dealt with in paragraph 1.2.4. The necessity of defining continuum is to be traced back to that period and the above-mentioned German mathematicians provided the two most probably famous continuity axioms (Bottazzini, 1981; Kuyk, 1982).

According to Rucker (1991), in 1887 Dedekind published one of his most famous works: *Was sind und was sollen die Zahlen* (What are numbers and what should they

[15] In the sixties, Abraham Robinson (1918 - 1974) managed to form a consistent theory, based on important theorems of Mathematical Logic and some of Skolem's (1887 - 1963) ideas, to handle actual infinitesimals and infinities through non-standard analysis.

[16] Among the research studies pointing out the difficulty of the notion of irrational number, we would like to mention: Fischbein, Jehiam and Cohen, 1994, 1995.

[17] On the difficulty of the concept of density for primary school pupils see: Gimenez, 1990. Whereas on the difficulty of the notion of continuum for 16-17 year old students, see: Romero i Chesa and Azcarate Gimenez, 1994.

be), a demonstration of infinity of the *World of thoughts, Gedankenwelt* in his language. Demonstration will be shown as follows:
if *s* is a thought: "*s* is a thought" is a thought;
""*s* is a thought" is a thought" is a thought;
"""*s* is a thought" is a thought" is a thought" is a thought;
…

In a letter dated 1905 Cantor wrote on this "demonstration":

> *«A multiplicity* [set] *could be such that the assumption that all its members "are together" leads on to a contradiction, so that to conceive multiplicity as unit, a "finite thing" is impossible. I would call these multiplicities absolutely infinite or incoherent multiplicities. It is quite evident, for instance, that "the totality of thinkable things" is such a multiplicity…»* [our translation].

(The reason for excluding that the set of all thoughts is a thought is that such a set would therefore be a proper element of itself).

The infinite set definition already envisaged by Galileo is attributed to Dedekind: *"A set is infinite when it can be put in biunivocal correspondence with one of its proper parts"*.

1.2.3 Georg Cantor (1845 - 1918)

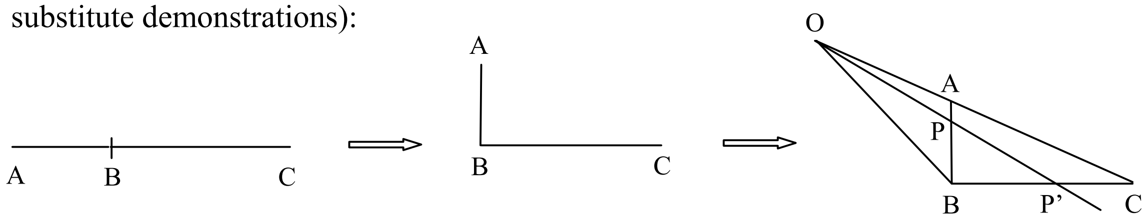
Young and brilliant mathematician, Cantor focuses his research work on those mathematical problems academic senior scholars are interested in: the problem of uniqueness of the decomposition of a real function into a trigonometric series. In 1872 (Cantor is 27 years old) he devotes his study to the infinite set of the points placed in an interval but not coinciding with the interval itself. In so doing, he analyses how the points of a straight line are positioned, the reciprocal positions between different segments; segments and straight lines, ... Everything dealt with in the actual sense with no philosophical embarrassment.
Cantor finally abandons the formal academic mathematics and starts to investigate the infinity by itself. That is the beginning of his adventure.

Some extracts from *Gesammelte Abhandlungen* (1932):

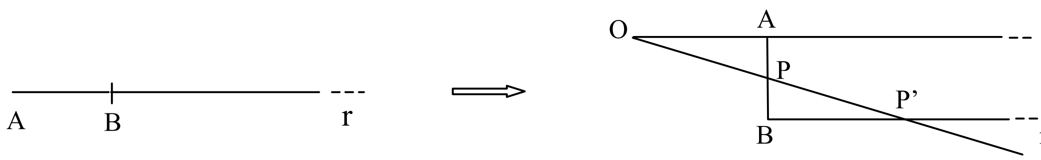
> *«Potential infinity has just one borrowed reality since the concept of potential infinity is to be always reconducted to that of actual infinity that logically precedes the former guaranteeing its existence.*
>
> *Actual infinity manifests itself in three contexts: the first is the most accomplished form, a completely independent being transcending this world, Deo, this is what I call Absolute Infinity; the second has to do with real world, the creation; the third is when the mind grasps infinity in abstracto as a mathematical magnitude, number or type of order.*
>
> *I want to clearly and firmly state the difference between the Absolute and what I call Transfinite, i.e. actual infinity in the last two forms, since it is about objects apparently limited and susceptible of growing process and thus related to finite».*
>
> *«The fear of infinity is a kind of short-sightedness that destroys the possibility of seeing actual infinity, even if infinity in its highest expression created us and sustains us and through its secondary forms of transfinite surrounds us and even dwells in our minds».*
>
> *«Therefore inevitable is the need for the construction of the concept of actual infinite number obtained through the appropriate natural abstraction, as well as the concept of natural number derives from finite sets by means of an abstraction process»* [our translation].

We provide a modern representation of some of Cantor's results.

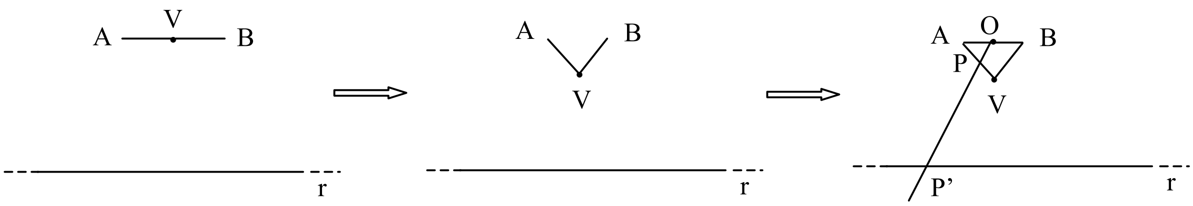
- Two sets are of equal number if a biunivocal correspondence exists between them (notable is the fact that there is no distinction at all between finite and infinite sets).
- The segments AB and AC (conceived as a set of points) are of equal number independently of their length that has no influence at all (here as follows figures substitute demonstrations):



- The set of the points of a segment has the same number of that of the points of a half-line:

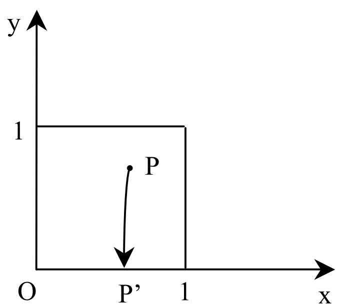


- The set of the points of a segment has the same number of that of the points of a straight line:



- The set of the points of a square and that of the points of one of its side are of equal number.

Let us consider for example the unit square in a system of Cartesian coordinates and thus having its side in abscissa coordinates:



In the duality of the possible representation for the same number, for ex.: 0.40000000… 0.399999999…, let us choose one and eliminate the other (we exclude the period 9 in this case).

Every point internal to square has coordinates such as:

P $(0.a_1a_2a_3\ldots a_n\ldots; 0.b_1b_2b_3\ldots b_n\ldots)$;

To it we make correspond a well determined point on the side (in abscissa coordinates):

P $(0.a_1b_1a_2b_2\ldots a_n b_n\ldots)$

and vice-versa.

The biunivocal correspondence between the points of a square and those of one of its side is established.[18]
(At the beginning Cantor was convinced that the cardinality **c** of the straight line was $\aleph_1$, the plane cardinality $\aleph_2$, the space cardinality $\aleph_3$ and so on. Conversely, this demonstration shows that the cardinalities pertaining to all these continuous point sets are always equal to **c**).

1.2.4 Cantor-Dedekind Correspondence

This is an extract of one of Cantor's letters to Dedekind dated 2 December 1873:

> *«As for the matters I've been occupied with lately, I realise, the following is pertinent to them:*
> *can a surface (a square including its edge for instance) be put in a univocal correspondence* [today we would call it biunivocal correspondence] *with a curve (a straight line segment with end-points included for instance) so that to each point of the surface corresponds one point of the curve and reciprocally to each point of the curve one of the surface?*
> *In this moment to answer this question seems to me very difficult and here there is so great a tendency to give a negative answer that a demonstration would seem superfluous»* [our translation].

Extract from a letter from Dedekind to Cantor dated 18 May 1874:

> *«… I talked to a friend in Berlin about the same problem and he considered the thing somewhat absurd as it goes without saying that two independent variables cannot be handled as one»* [our translation].

Extract from a letter from Cantor to Dedekind dated 20 June1877:

[18] The research work of Arrigo and D'Amore (1999) focuses on the difficulty high school students encounter in accepting this demonstration.

«I would appreciate knowing if you consider the demonstration method I used as strictly rigorous from an arithmetical point of view. It is about proving that surfaces, volumes and continuous varieties of p dimensions can be put in univocal correspondence with continuous curves, thus with only one-dimensional varieties, that surfaces, volumes, varieties of p dimensions have the same power of curves; this opinion seems to contrast with the most generally accepted, especially among founders of the new geometry according to which there are varieties once, twice, three times, … p times infinite; it is as if the infinity of points of a square surface could be obtained elevating it somehow to its square, that of a cube elevating it to the cube, the infinity of points of a line. (…). I want to talk about the hypothesis according to which a continuous multiplicity extended p times necessitates, in order to determine its elements, of p real coordinates independent of each other. This number, for the same multiplicity cannot be increased or decreased. I also came to the conclusion that this hypothesis could be correct but my point of view differed from all the others in one point. I considered this hypothesis like a theorem awaiting a proper demonstration and I expressed my point of view in the form of a question posed to some colleagues also on the occasion of Gauss' Jubilee in Göttingen».

«Can a continuous variety of p dimensions, with p >1, be put in a univocal correspondence with a continuous variety of one dimension so that to each point of one corresponds one and one only point of the other?
The majority of people whom I posed this question were quite surprised by the fact itself that I posed such a question, for they believed as obvious that, in order to determine a point in an extension of p dimensions, p independent coordinates are needed. Those who could, despite all, penetrate the question had at least to admit that the "obvious" answer "no" needed at least to be demonstrated. As I told you, I was among those who held a negative answer as probable, until very recently, when after so complex and strenuous reasoning, I came to the conclusion that the answer is affirmative and with

> *no restrictions. After a while, I found the demonstration that you will see hereafter»* [our translation].

(Cantor showed Dedekind the above-mentioned demonstration concerning the points of the square and of its side).

The letter was sent on 20 June 1877 but Cantor was so impatient about it that he wrote to Dedekind again on 25 June 1877 urging him an answer:

> *«As long as you do not approve me I am bound to say: I see it but I don't believe it».*

Dedekind immediately answered back on 29 June 1877:

> *«Once again I examined your demonstration and I found no faults. I'm convinced that your interesting theorem is correct and I congratulate you».*

The route to infinity is definitely open (only the numerical infinity will be investigated in this chapter).

1.2.5 Cardinality

Let us consider the N set of natural numbers being **n** its cardinality or numerousity or power that we will call "of the numerable"; **n** is an infinite cardinality as it is larger than any given finite cardinality.

Be N_s the set (Galileo's) of perfect squares, N_e the set of even numbers, N_o of odd numbers, N_{Pr} of primes, … Each one of these sets can be put in biunivocal correspondence with N and therefore has the cardinality of the numerable **n**.

If A is a subset of N, infinite, then the cardinality of A is **n**.

In fact, as supposed, A = $\{a_1, a_2, \ldots, a_m, \ldots\}$ where a_i are elements of N. Let us consider the biunivocal correspondence $a_1 \leftrightarrow 0$, $a_2 \leftrightarrow 1$, …, $a_m \leftrightarrow m - 1$, …

Thus: **n** is the *smallest* infinite cardinal.

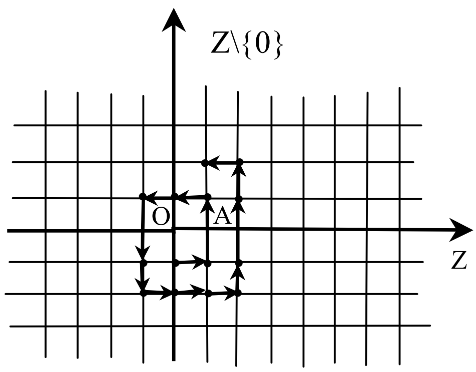
Let us consider the set Z of whole numbers; the biunivocal correspondence with N is created:

$0 \leftrightarrow 0$, $1 \leftrightarrow {}^{+}1$, $2 \leftrightarrow {}^{-}1$, $3 \leftrightarrow {}^{+}2$, $4 \leftrightarrow {}^{-}2$, …

Thus the cardinality of Z is still **n**.

In search of those values of infinity to which St.Thomas Aquinas referred to (see paragraph 1.1.1), let us have a try with Q, the set of rational numbers. Every rational number $\frac{^{+}a}{^{-}b}$ can be represented as a point P of $(^{+}a;\ ^{-}b)$ coordinates placed on a Cartesian coordinate system.

Then, all rational numbers can be positioned as shown in the figure below starting from origin in O.



Only some of these points represent rational numbers [the point A (1; 0) does not represent any rational, for instance]. The points representing rationals have been deeply marked in the figure. The outcoming rational sequence has as the first element $(0\ ;\ ^{-}1)$, the second $(1\ ;\ ^{-}1)$, the third (1 ; 1); …that represent 0, $^{-}1$, 1 ... So there is a biunivocal correspondence of Q with N since we “numbered” Q.

[As for the problem of “doubles” as in the cases of $(^{-}2\ ;\ 3)$ and $(2\ ;\ ^{-}3)$ that are different points representing the same rational; you may either count them twice or just skip them the second time you encounter them].

Thus the cardinality of Q is still **n**.

In addition, Cantor proved that the set of algebraic numbers (equation solutions) has against all expectations **n** cardinality.

It is precisely at the moment when **n** seems to be impossible to overcome that the key demonstration is achieved.

The set of real numbers included between 0 and 1 (end-points excluded) has a cardinality superior to **n**.

The demonstration is performed per absurdum. Assume per absurdum:

$0.a_{11}\ a_{12}\ \dots\ a_{1n}\ \dots$

$0.a_{21}\ a_{22}\ \dots\ a_{2n}\ \dots$

$\dots$

$0.a_{n1}\ a_{n2}\ \dots\ a_{nn}\ \dots$

$\dots$

all real numbers included between 0 and 1 (that is to say: let us suppose that they are a denumerable quantity). Let us consider the notation:

$0.b_1\ b_2\ \dots\ b_n\ \dots$

such that $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, …, $b_n \neq a_{nn}$, …;
then it is obvious that this notation:

- *is not* included in the preceding list of *all* real numbers between 0 and 1;
- is a real number included between 0 and1;

we found therefore a contradiction due to the assumption that real numbers between 0 and 1 would have **n** cardinality.
(Once again considerations on the double writing of rational numbers should be taken into account).

Thus: real numbers included between 0 and 1 are infinite although they do not form a denumerable infinity.
n is the smallest infinity and real numbers included between 0 and 1 constitute a *larger* infinity.
The exact date of Cantor's discovery is 7 December 1873. The date is known because on the following day he wrote a letter to the friend Dedekind to communicate his demonstration.
Observing that such a cardinality, that of reals between 0 and 1, is the same for all reals is banal. We would call it *cardinality of the continuum* and indicate it with **c**.
With a little abuse of symbolic language, we would write:

$$\mathbf{n} < \mathbf{c}$$

But **c** is also the cardinality of the points of a straight line, of those of a plane, of those of any continuous variety of *m* dimensions.

«It can be with no doubt affirmed that the theory of transfinite numbers works out or collapses together with irrational numbers; they share the same essence because these are anyway all examples or variants of actual infinity» (Cantor, 1932). [our translation]

Therefore Cantor was at that moment aware that there were at least two infinite numbers: **n** and **c**. His aim was to find a set S of **s** cardinality such that **n** < **s** < **c.**
He spent a long time working on that, but then a peculiar analogy raised his attention.

1.2.6 The Continuum Hypothesis

Let us consider the finite set I and its so-called power-set: P(I). From now on we will indicate the cardinality of a set with: | I |.
It can be demonstrated that:
$| P(I) | = 2^{| I |}$

Let us extend this concept to infinite sets.
According to Dedekind's method of cuts (or sections) used to introduce real numbers, R is nothing else but a class of classes of cuts in Q;
and therefore | R | = | P(Q) |
But then:

$$\mathbf{c} = 2^{\mathbf{n}}$$

this writing introduces an interesting demonstration:

$$\mathbf{c} \cdot \mathbf{c} = 2^{\mathbf{n}} \cdot 2^{\mathbf{n}} = 2^{\mathbf{n} + \mathbf{n}} = 2^{\mathbf{n}} = \mathbf{c}$$

(therefore the plane that is the set of all ordered pairs of real numbers and whose cardinality is **c** · **c** has **c** cardinality. This has been previously proved through the biunivocal correspondence between the points of a square and those of its side).

Once the meaning of the order of transfinite numbers has become clear, to continue with this procedure is an easier task. Let us consider the set F of functions from R in R.
We call **f** the cardinality of F:

$$\mathbf{f} = 2^{\mathbf{c}}$$

as well as the sequence of natural numbers 0, 1, 2, 3, … goes on adding 1 all the time, also the sequence **n**, **c**, **f**, **g**, … of transfinite numbers works in the following way:

$$\mathbf{n}, \qquad \mathbf{c} = 2^{\mathbf{n}}, \qquad \mathbf{f} = 2^{\mathbf{c}}, \qquad \mathbf{g} = 2^{\mathbf{f}}, \qquad \ldots$$

in a never ending process. If there were an end to it, we would in fact find out a paradox: an entity of the maximum possible cardinality, G for instance, which admits an increase going through its power -set P(G).

However, the aim was to find a set S such that: $\mathbf{n} < | S | < \mathbf{c}$:
Let us put it into more general terms:
we can try to find a set S_1 such that $\mathbf{c} < | S_1 | < \mathbf{f}$; and then another S_2 such that $\mathbf{f} < | S_2 | < \mathbf{g}$; and so on.
In 1883 Cantor wrote that he wished to be soon able to demonstrate that the continuum cardinality is the same of the second numerical class, that is to say that such a set S does not exist. His research produced no results: he could not prove it; nor could he prove the opposite (to demonstrate such S).
Then he developed a conjecture:
Cantor's hypothesis or *continuum hypothesis:*

c strictly follows **n** that is to say that there is no cardinal **s** such that $\mathbf{n} < \mathbf{s} < \mathbf{c}$.[19]

Now, if it is supposed that **c** strictly follows **n**, then why not generalise it?
Cantor's hypothesis or *generalised continuum hypothesis*:

c strictly follows **n**, **f** strictly follows **c**, **g** strictly follows **f**, and so on.

Therefore these are elements of a new sequence that can be re-named as follows:

$$\mathbf{n} = \aleph_0, \qquad \mathbf{c} = \aleph_1, \qquad \mathbf{f} = \aleph_2, \qquad \mathbf{g} = \aleph_3, \qquad \ldots$$

Thus $\aleph_{n+1} = 2^{\aleph_n}$

[Between 1938 and 1940 **Kurt Gödel** would demonstrate that, assuming the continuum hypothesis (CH) in the *set theory* (we will call it ZF by the name of the creators:

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[19] As Rucker reports (1991): *«Cantor firmly believed that* $\mathbf{c} = \aleph_1$ *was valid. Gödel, at a certain stage of his studies, believed that* ***c*** *was* $\aleph_2$ *and D.A. Martin wrote an article from which we could deduce that* ***c*** *is* $\aleph_3$*»*.

Zermelo (who developed the axioms in 1908) and Frankel (who further investigated the above-mentioned axioms in 1922 and then transcribed them in the language of the Calculus of Predicates), no contradictions are introduced (in other words, CH is compatible with ZF). Therefore: CH is either independent from ZF axioms or it can be demonstrated on their basis. To say it differently, this means that Cantor was not mistaken, i.e. from ZF is not possible to deduce that **c** is different from $\aleph_1$. At the same time to prove that Cantor was right is also not possible. In 1963 **Paul J. Cohen** showed (by means of a method called "forcing") that no contradictions to ZF are introduced if we assume the negation of CH. Thus, CH negation is compatible with ZF. Therefore it cannot be demonstrated if Cantor was right or wrong. In conclusion, CH has to be dealt with as a new axiom: if we add ZF it to we have the "*Cantorian* set theory"; whereas if we add its negation the "*non-Cantorian* set theory" (Gödel, 1940; Cohen, 1973)].

1.2.7 Giuseppe Peano (1858 - 1932)

Let us make a little digression with Peano. He also committed himself with questions related to infinity. His famous systematisation of natural numbers needed at some point an Axiom of Induction.[20] It can be even said that this is a basic and fundamental feature of the concept of natural numbers itself (Borga et al., 1985). Today the induction principle is a fundamental support for arithmetic and logical demonstrations and it recalls potential infinity.[21]

1.2.8 Cantor and the ordinals

Let us go back to Cantor (this part is an extract from Rucker, 1991; D'Amore, 1994).
Let us define ordinals by repeated steps:

0 is an ordinal

Principle 1:[22]: every ordinal number a has an immediate successor $a + 1$

[20] Be P a property that can apply to natural numbers. Let us assume that 0 possesses this property P, let us assume that for every natural number x, if x has the property P, then x + 1 will also have the property P, under these conditions we can state that every natural number has the property P.

[21] As far as didactics is concerned, Fischbein and Engel (1989) and Morschovitz Hadar (1991) worked on high school students' difficulty to accept the induction principle.

[22] The basic concept behind this Principle is that: *no ordinal number is minor than itself.*

Principle 2: given an increasing sequence of ordinals a_n, the minimum ordinal is defined [indicated as $\lim(a_n)$] that follows all the ordinals in the given sequence.

Starting from 0 and repeatedly applying the Principle 1, we obtain the ordinals 0, 1, 2, 3, …

Now if we want to overcome the infinite sequence of finite ordinals we need to use the Principle 2 to get $\lim(n)$ that we indicate with ω:

$$0 \quad 1 \quad 2 \quad \dots \quad n \quad n+1 \quad \dots \quad \omega$$

This is in its turn a new sequence of ordinals and consequently applying progressively the Principle 1 many more times, you obtain:

$$0 \quad 1 \quad 2 \quad \dots \quad n \quad n+1 \quad \dots \quad \omega \quad \omega+1 \quad \omega+2 \quad \omega+3 \quad \dots$$

This is the new increasing sequence of ordinals $(\omega + n)$ and therefore applying to it the Principle 2 $\lim(\omega+n)$ is created:

$$0 \quad 1 \quad 2 \quad \dots \quad \omega \quad \omega+1 \quad \omega+2 \quad \omega+3 \quad \dots \quad \omega+\omega$$

it can be also written down in this way: $\omega+\omega$ or $\omega \cdot 2$.

Adding and multiplying ordinals could be written down in this way:

$a+b$ ·= counting starting from $a + 1$ for b times

$a \cdot b$ ·= juxtaposing b copies of a

When we deal with finite ordinals, these operations will coincide with the usual sum or product and are commutative, but when it comes to their extension to transfinite ordinals the commutative property is not maintained.

Some examples:

$1 + \omega = 1\ 0\ 1\ 2 \dots =$ (counting again from the beginning) $0\ 1\ 2\ 3\dots = \omega$

$\omega+1 = 0\ 1\ 2 \dots 1 = \omega+1$

thus: $1 + \omega = \omega \neq \omega+1$

$2 \cdot \omega = \quad 2 \quad 2 \quad 2 \dots =$ (counting) $0\ 1\ 2 \dots = \omega$

$\omega \cdot 2 =$ (double juxtaposition of ω) =

$= 0 \quad 1 \quad 2 \quad \dots \quad 0 \quad 1 \quad 2 \quad \dots = 0 \quad 1 \quad 2 \quad \dots \quad \omega \quad \omega + 1 \quad \omega + 2 \quad \dots = \omega + \omega$

and thus: $2 \cdot \omega = \omega \neq \omega + \omega = \omega \cdot 2$.

We came to $\omega \cdot 2$; applying many times the Principle 1 we get:

$0 \quad 1 \quad 2 \quad \dots \quad \omega \quad \omega+1 \quad \omega+2 \quad \dots \quad \omega\cdot 2 \quad \omega\cdot 2+1 \quad \omega\cdot 2+2 \quad \dots$

and again the Principle 2, obtaining $\lim(\omega \cdot 2 + n) = \omega \cdot 2 + \omega$ that will be also called $\omega \cdot 3$.

Operating in the same way we get to $\omega \cdot n$, for every finite n and consequently we could use the Principle 2 to obtain $\lim(\omega \cdot n)$ i.e. ω copies of ω, that is to say $\omega\cdot\omega$ that we will also call ω^2. Continuing you easily get to a ω^3 and progressively to:

$$\omega^{\omega}$$

ω^2 can be conceived as the first ordinal a for which: $\omega + a = a$.

De facto: ω^2 is like $\omega + \omega + \omega + \omega + \omega + \dots$ and so it will make no difference if we put before another ω as addendum.

Analogously, the first ordinal a for which: $\omega \cdot a = a$ is ω^{ω}.

As a matter of fact, ω^{ω} can be thought as $\omega \cdot \omega \cdot \omega \cdot \omega\dots$, obtained by the juxtaposition of ω for ω times; therefore it will make no difference if we put before another ω as factor: $\omega \cdot \omega^{\omega} = \omega^1 \cdot \omega^{\omega} = \omega^{1+\omega} = \omega^{\omega}$, since that $1 + \omega = \omega$.

Let us consider the first ordinal a for which the equality: $\omega^a = a$ is valid

This number is:

$\omega^{\omega^{\omega^{\cdots}}}$ (in it the raising to power is performed ω times).

Nothing will change if we put at the base of this notation another ω: the exponents will be $1 + \omega$ that is always ω.

To this number it has been given the name ε_0 and for nearly 200 years it has been indicating real numbers small "as we please".

We introduce a new operation:

${}^{a}b\cdot = b^{b^{b^{\cdot^{\cdot^{b}}}}}$ (i.e. b elevated to itself, for a times counting the base).

Some examples will be provided only to give an idea of how numbers grow by means of this new operation.

${}^{3}3 = 3^{(3^3)} = 3^{27}$ nearly eight thousand billions.

According to the new operation, ω^{ω} is nothing else than ${}^{2}\omega$.

And therefore ${}^{3}\omega$ $\omega^{(\omega^{\omega})}$ is a number which is very difficult to imagine.

Let us go back to ε_0 that according to the new notation is ${}^{\omega}\omega$.

ε_0 is not the last ordinal, here we have an even larger one:

$$\omega^{\omega^{\omega^{\omega^{\cdots \omega}}}}$$

Every time that you come to larger ordinals, you need to stop for a while before envisaging the way of producing even greater ones, and this is only the beginning[23] (for a further investigation of this topic see Rucker, 1991).

1.2.9 Ordinals as cardinals

Let us go back to our subject matter using alephs.

ω is exactly $\aleph_0$, the first infinite cardinal.

But $\omega+1$, $\omega+2$, …, $2 \cdot \omega$ are also all $\aleph_0$.

$n \cdot \omega$, …, ω^{ω} are also $\aleph_0$

$\omega^{\omega}+1$, …, ${}^{\omega}\omega$ are also $\aleph_0$

Thus, ω is the smallest ordinal equal to $\aleph_0$; so far there was no growth however. Also ε_0 is nothing else than a "banal" $\aleph_0$. Every ordered set with cardinality ω, …, ${}^{\omega}\omega$ can be always put in biunivocal correspondence with N.

However, since it is possible to find ever-increasing ordinals, it is also possible to find ever-increasing cardinals:

$$\aleph_0,\ \aleph_1,\ \aleph_2,\ \ldots,\ \aleph_{\omega},\ \aleph_{\omega+1},\ \ldots, \aleph_{\aleph_1},\ \ldots,\ \aleph_{\aleph_{\omega}},\ \ldots$$

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[23] In Bachmann (1967) we are provided with probably the most exhaustive description of notation systems for denumerable ordinals. Whereas in Cantor (1955) we can find the clearest description of transfinite ordinals, he had ever produced.

We can also find a number θ such that $\theta = \aleph_\theta$

This θ is of this kind:

$$\theta = \aleph_{\aleph_{\aleph_{\aleph_{\aleph_{\ddots}}}}}$$

θ ends a cycle.

So far, as after $\aleph_\theta$ comes $\aleph_{\theta+1}$, …, $\aleph_{\theta+\omega}$, …

You never come to an end in the discovery of transfinite numbers.
Let us prove it:

- A set S is finite or denumerable if and only if $|S| \leq \aleph_0$
- The Principles 1 and 2 induce a stronger Principle i.e. the no. 3: for each set of *A* ordinals there exists the minimum ordinal that is major than every element of A and that we will call *supA*.

Let us consider the collection O*n* of all ordinals. If O*n* were a set, then according to the Principle 3 there would be an ordinal supO*n* (we call it Ω, the Absolute Infinity that is positioned at the end of the sequence of ordinals). But this is impossible because if Ω were an ordinal, then Ω would be an element of the collection O*n* of all ordinals and then it will be Ω < sup O*n* = Ω, a fact that contradicts a fundamental property of the ordinals according to which no ordinal can be minor than itself.[24]
The Principle 3 states that no set of ordinals can reach Ω.

We conclude with a passage from a Cantor's letter to Dedekind dated 28 August 1899:

> *«It may be legitimate to wonder if well-ordered sets or sequences corresponding to cardinal numbers* $\aleph_0$, $\aleph_1$, …, $\aleph_\omega$, … $\aleph_{\aleph_1}$, … *are real sets in the sense of being "consistent multiplicities". Is it not possible that these multiplicities are "inconsistent" and that the contradiction deriving from the assumption that these multiplicities exist in the form of unified sets has not been acknowledged yet? My answer is that the same question could be posed with regard to finite sets, and if you properly focus on it, it stands*

[24] Cesare Burali and Forti disclosed this situation in 1897, but Cantor had noticed it even before.

> *out clearly that not even for finite multiplicities a demonstration of consistency is possible. In other words: the consistency of finite multiplicities is a simple and improvable truth that we can call "axiom of arithmetic" (in the old meaning of the word). Analogously, the consistency of those multiplicities that have aleph cardinality constitutes "the axiom of arithmetic extended to transfinite"*» [our translation] (Meschkowski, 1967).

It seemed that here Cantor referred to simple and direct perception of the reality of cardinal numbers in the Realm of Thoughts. Moreover, in 1899 he proved his intellectual courage stating that: «*A number such as* $\aleph_2$ *is much easier to be perceived than a casual natural number of ten million digits*» (see Cantor, 1932). This Cantor's daring affirmation will prove quite difficult to be shared after the results regarding teachers' convictions will be illustrated in chapters 3 and 4.