Chapter 2. International research context

The research work on mathematical infinity, which will be described in chapters 3 and 4, is to be considered within today's scenario of mathematical didactics. This seems to be aiming at focusing on the phenomenon of learning, the latter seen from the point of view of *fundamental didactics* (Henry, 1991; D'Amore, 1999). This notion includes all the basic elements related to the research in mathematical didactics, deriving from the numerous and complex analyses of so-called "triangle of didactics" (see paragraph 2.4). A brief outline of the major topics concerning this research field will be provided in the next chapters and will recall mainly the writings of D'Amore (1999, 2002, 2003) where the author builds up a personal trajectory in the field of didactics of mathematics. We fully subscribe to the ideas developed by D'Amore.

2.1 The didactical contract

The first attempt "to define" the didactical contract is the following: *«During a teaching class prepared and held by a teacher, the student is generally given the task to solve a problem (a mathematical one), but access to the assigned task is made possible through interpretation of the questions posed, the pieces of information provided and through the fixed steps imposed by the teacher's method. The (specific) teacher's behaviours expected by the student and the student's behaviours expected by the teacher constitute together the didactical contract»* (Brousseau, 1980a; our translation). The latter idea has been shared by various scholars from all over the world and has become part of the language spoken by the whole international community since the late Eighties (Brousseau, 1980b, 1986; Brousseau and Pères, 1981; Chevallard, 1988; Sarrazy, 1995; Schubauer-Leoni, 1996). The original idea of didactical contract has been often reinterpreted and modified by various authors over the years, even with very different modalities and approaches as stated by Sarrazy (1995). Going back to the original principle, the "expectations" to which Brousseau refers to, are in most cases not due to explicit agreements imposed by school or teachers and negotiated with the students but they are strictly related to the way school, mathematics and the repetition of modalities are conceived (D'Amore, 1999, 2002, 2003). Over the last decades, the analysis of phenomena related to such students' behaviours has yielded significant results favouring the interpretation and explanation of various behaviours that were still considered inexplicable or due to lack of interest, ignorance or students' immaturity until recent times (Baruk, 1985; Spagnolo, 1998; Polo, 1999; D'Amore, 1999). The above-mentioned research study revealed that children and young people have specific expectations, general schemes and behaviours having no relation with mathematics though depending on much more complex and interesting motivations emerging from the didactical contract set up in the classroom (D'Amore, 1993b; D'Amore and Martini, 1997; D'Amore and Sandri, 1998). In order to modify these behaviours, students should be able *to break the didactical contract* (Brousseau, 1988; Chevallard, 1988), being personally responsible for their choices. As a matter of fact, through the breaking of the didactical contract students create a new situation that contrasts their expectations, habits and all the clauses that have been set so far in didactical situations. To achieve this goal students should be determined enough to try themselves out and be in the front line, going against the given contract clauses. This phenomenon can happen only if the teacher favours such a breaking.

2.2 Images and models

With respect to *"image"* and *"model",* we will use the following terminology and treatment adopted by D'Amore (1999, 2002, 2003):

The *mental image* is the figural or propositional result produced by an (external or internal) impulse; cultural influence, personal style[25] and feature condition it. In short, it

[25] By cognitive style we intend all those personal features that an individual, more or less consciously, has got and implements, when involved in a learning process; these characteristics seem not to depend just on "natural" proclivities, but also on mood and temporary situations, disposition, interest, motivation, … For example, one can gradually get to know how to learn acoustically or visually and get familiar with learning by manipulating images or symbols, … (De La Garanderie, 1980; Gardner, 1993; Sternberg, 1996).

is the typical product of the individual but it still presents common and constant connotations shared with other individuals. The mental image undergoes different levels of conscious elaboration (this skill is also related with the individual).

All elaborated mental images (more or less consciously) connected to the same concept form the (internal) *mental model* of the concept itself. As a matter of fact, students build for themselves the image of a concept. They believe it to be stable and definitive but at a certain point in their cognitive history they receive information on the concept that is not included in the image they have constructed. Therefore students have to adjust the "old" image to a new wider image that contains both the previous one and new pieces of information. This fact is caused by a *cognitive conflict* triggered by the teacher (see paragraph 2.3). The process can take place many times during the student's "educational history". Most concepts in mathematics are formed only through the constant transit, over years, from an image to the other, the latter being more powerful than the former. One can visualise these subsequent conceptual constructions as a sequence of images, which get "closer and closer" to the concept.

During the sequence of images you reach a certain point when the image you have come to after several passages "resists" different stimuli, and turns out to be "strong" enough to contain the new argumentation and pieces of information gradually encountered. These are related to the concept, which is represented by the image itself. Such a stable and no longer changing image can be addressed as *model* of the concept. Therefore, "to construct a model out of a concept," means to successively revise several (weak and unstable) images to come to an ultimate strong and stable image.

It can be verified that:

- *the model is created at the right time*, i.e. it is just the correct model aimed at for that specific concept of mathematical knowledge. The didactical action has worked out: the student has built a correct model of the concept;

or:

- *the model is created too early*, i.e. the image is still weak and needs to be widened. In this case reaching the concept turns out to be difficult because the stability of the model is an obstacle to future learning.

The name *intuitive model* is given to those models that fully respond to intuitive stimuli and are immediately and strongly accepted. That is to say that there is direct correspondence between the suggested situation and the mathematical concept used. When a teacher suggests a strong and convincing image that becomes persisting and is continuously confirmed by numerous examples and experiences of a concept, the image develops into an *intuitive model* (Fischbein, 1985, 1992). Still this model could not correspond to the model of concept expected. There is also the category of *parasite models*, created through repetition, but not at all desired (Fischbein, 1985). Examples of this kind can be found in D'Amore (1999).

From a didactical point of view, it is advisable that the misconception-image does not become a model (see paragraph 2.3), for, due to its own nature, it is awaiting a definitive collocation. In this case, assimilating the new situation to adjust the former model (strong and stable) to the new one proves quite a difficult task. It is advisable to let students keep unstable images until proper and meaningful models, which are suitable to the expected level of mathematical competence, are created. Thus, it is important that the teacher avoids providing explicitly unreliable and wrong information as well as autonomous building of information helping create parasite models in students' minds. In order to succeed in reaching this difficult goal, the teacher should be confident and skilled not only in the field of mathematics but also in didactics of mathematics.

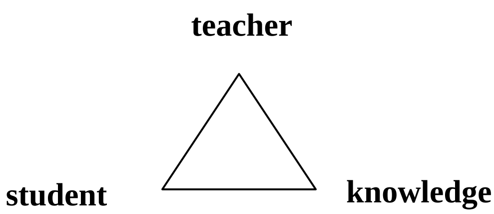
2.3 Conflicts and misconceptions

Another subject dealt with in didactics of mathematics and pertaining to this work is that of *cognitive conflicts* (Spagnolo, 1998; D'Amore, 1999, 2003). Over time the student constructs a concept and then builds an image of that concept; during the school years this image can become stronger and be validated through tests, repeated experiences, figures and exercises, especially those assessed and marked as correct by the teacher. It can also happen that such an image turns out to be inadequate sooner or later when compared to another one relating to the same concept. This latter image can be suggested by the teacher or anybody else, it is unexpected and in contrast with the previous one. A *conflict* is born between the previous image, which the student thought to be definitive, and the new one; this generally happens when the new image expands the scope of applicability of the concept or provides a comprehensive version of it. Therefore, the *cognitive conflict* is an "internal" conflict between two concepts, two images or a concept and an image.

Misconceptions are at the basis of conflicts. These are temporary, incorrect conceptions, which await a more elaborated and critical cognitive collocation (D'Amore, 1999). A misconception is a wrong concept and therefore it normally represents an event that must be avoided; this situation should not be considered completely negative: a temporary misconception, which is undergoing the process of finding its cognitive collocation, can be necessary to reach the construction of a concept. In some cases images become real misconceptions, i.e. wrong interpretations of the information received. To call them *mistakes* is somehow to make things too easy and banal, as even very young children have naive but deep mathematical conceptions (Aglì and D'Amore, 1995) that are obtained empirically or through social exchanges. As to mathematical knowledge, it is reasonable to think the whole school career of an individual as a continuous transit from misconceptions to correct conceptions. The passage from a first elementary conception (naive, spontaneous, primitive, etc.) to a more elaborated and correct one is a delicate and necessary phase.
Examples of conflicts and misconceptions can be found in D'Amore (1999).

2.4 The triangle: teacher, student, knowledge

Over the last twenty years, research in the field of didactics of mathematics has deeply and profoundly investigated the aspect of what is hidden behind the "triangle" whose "vertices" are: student, teacher and knowledge (Chevallard and Joshua, 1982; Chevallard, 1985; D'Amore, 1999; D'Amore and Fandiño, 2002).



According to *fundamental didactics*, this is a *systemic model* used to locate and analyse the many relationships established among the three “subjects” that represent the “vertices” of the triangle. The complex nature of the systemic model is due to the necessity of simultaneously considering all the mutual relationships among the “vertices” including all the implications of different natures.

“Vertices”

In this paragraph we shall refer to D’Amore and Fandiño’s synthesis (2002) where every “vertex” of the triangle acts as a pole:

- *knowledge,* be it academic, official or university, represents the ontogenetic or epistemological pole. It is around this vertex that the *epistemological obstacles* theory (see paragraph 2.5.) is situated. Those obstacles are the ones related to the concept’s intrinsic nature, to its evolution and to formal complexity of its structures.

- *student* represents the genetic or psychological pole. This vertex is about personal, cultural or cognitive projects filtered by the *scholarisation*[26] relationship that makes learning subject’s personal experiences not free from constraints. It is around this pole that *ontogenetic obstacles* theory is situated (see paragraph 2.5).

- the *teacher* is the functional or pedagogical pole. This vertex is about those cognitive and cultural projects which are highly influenced by all pedagogical expectations (not always explicit), beliefs linked to knowledge, professional convictions and “implicit philosophies” (Speranza, 1992).[27] It is around this pole that the *didactical obstacles*

[26] Referring back to D’Amore’s idea (1999): *«By “knowledge scholarisation” I refer to the mostly unaware act in the social and school life of a student (occurring almost always during primary school), through which s/he delegates to school (as an institution) and to the school teacher (as a representative of this institution) the task of selecting relevant knowledge issues (relevant from a social point of view, for acknowledged status and approved by the noosphere), thus giving up direct responsibility to choose learning contents according to personal criteria (taste, interest, motivation,…)»* [our translation].

[27] We are referring to the “philosophies” that Speranza describes as “implicit”, in other words to those philosophies that exist and are influential, although they are not implemented in didactical praxis.

theory (see paragraph 2.5) is situated, as the teacher is responsible for didactical projects' choices.

"Sides"

D'Amore and Fandiño (2002) provide an explanation of the "sides" that highlights the relationships between pairs of poles:

• *teacher-student* could be summarised with the verb *"animate"* (a term linked to *motivation*, interest, *volition,*[28] ...). This verb recalls the following concepts:

- *devolution* is the action of the teacher on the student. The teacher tries to involve students in the didactical project proposed. Therefore this is the process or the responsibilisation activity that the teacher uses in order to get students personally involved in a cognitive activity that consequently becomes one of the cognitive activities of students themselves;

- *involvement* is the students' action exerted on themselves: students accept devolution, i.e. they become personally responsible for the construction of their own knowledge;

"To animate" can be therefore interpreted as a thrust towards personal involvement favouring devolution.
Midway between devolution and involvement, *adidactical situations*[29] (Brousseau, 1986) are to be found. These are situations favouring the "passage" from devolution to

[28] It is important to draw a distinction between *motivation* and *volition* as in Pellerey (1993). The former refers to: *«The formation of intentions, that is to say the elaboration of reasons inveigling someone into doing something»,* whereas the latter refers to *«The concrete will to achieve the aim expressed in the intentions»*. Being motivated to do something, like learning for example, does not necessarily mean being ready to do that or able to persevere when facing up to the first difficulties or failures. This distinction was introduced in the didactical context by the work of D'Amore and Fandiño (2002).

[29] In an environment which has been organised for the purpose of learning a special subject, we can talk about an *adidactical situation,* when the didactical intention is no longer explicit. The teacher suggests an activity without declaring the purpose of it; the student is well-aware that all activities in the classroom are meant to build up new knowledge, but in this case s/he does not know exactly what s/he is going to

involvement. When students are faced with a *didactical situation*[30] structured according to specific “rules of the game”, the knowledge acquisition is not guaranteed unless a confrontation of students with an *adidactical* situation is not foreseen. It is as if the teacher-student relationship were interrupted in favour of the student-situation relationship: students produce their knowledge as a personal response to *milieu*[31] requirements rather than to teacher’s expectations. The milieu is not “constructed” by the teacher. It preexists to the didactical situation and in general terms is referred to the collection of objects (mental and concrete) known to system-subjects independently of the fact that those objects are at that moment part of the knowledge acquisition process in act.

Elements characterising this side are the following:
-. didactical contract (see paragraph 2.1);
- didactical obstacles (see paragraph 2.5);
- pedagogical relationships;
- valuation (Fandiño Pinilla, 2002);
- scholarisation;
- devolution or lack of it;
…

learn. If s/he decides to participate, accepting to get involved, then s/he frees her/himself from “contract” constraints (see paragraph 2.1) and participates in an adidactical activity. In this case, the teacher is just a spectator, that is to say, s/he is not explicitly involved in the knowledge management. The teacher dissimulates her/his didactical purpose and her/his will to teach, in order to make the student accept the cognitive situation as her/his own responsibility.

[30] We talk about *didactical situation* when we analyse an explicit education context, for example when a teacher playing in the role of a teacher openly informs her/his students about the knowledge content that is at issue in that moment.

[31] In the Theory of Didactical Situations, Brousseau (1989) introduces the notion of *milieu*, in order to stress the systemic nature of his approach: *«For the researcher a modelling of the environment and of its pertinent responses as far as a specific learning process is concerned, is just one of the components of a [didactical] situation. (…). It plays a fundamental role in the learning, as it is the cause of adaptation (for the student), and in the teaching, as reference and epistemolocical object »*. (our translation)

• *student-knowledge* is characterised by the verb *"to learn"*. The prevailing activity is *involvement*. It favours the access to "personal knowledge" which will be *institutionalised* (see teacher-knowledge side) by the teacher through the implementation of knowledge construction. On this side are positioned the images students possess of school, culture, ...; the specific personal relationship with mathematics and overall with knowledge institutionalisation (mainly depending on age), previous experiences; family and society, …

Elements characterising this side are:
- various learning theories;
- role and nature of conceptions;
- epistemological obstacles theory;
…

• *teacher-knowledge*. The main verb is *"to teach"* and the featuring activities are: *knowledge institutionalisation* (Chevallard, 1992) and *didactical transposition* (Chevallard, 1985, 1994; Cornu and Vergnioux, 1992).

Knowledge institutionalisation [32] is a process complementary to devolution and involvement that takes place when the teacher recognises that the student's personal acquisition of knowledge is legitimate and usable in the school context, once devolution and student's involvement have been verified.

The more general activity characterising this side is the *didactical transposition* (Chevallard, 1985) that is intended as the adaptation activity, transformation of knowledge into a teaching object according to place, audience and the didactical goals expected. The latter aspect will be fundamental for the treatment of this thesis (see ch. 4). The teacher should therefore operate a transposition from *knowledge* (originating

[32] According to Brousseau (1994): *«Knowledge institutionalisation is the social act through which teacher and student recognise devolution»*.

from research) to *taught knowledge* (knowledge taking place in the classroom)[33]. As a matter of fact, the passage is much more complex because it goes from *knowledge* (that of the discipline experts that structure and organise such knowledge) to *knowledge to be taught* (that decided by institutions) to *taught knowledge* (chosen by teachers as specific object of their didactical intervention).

The passage from *knowledge* to *knowledge to be taught* is filtered by teachers' epistemological choices which depend on their convictions, on their "implicit philosophies", on their idea of didactical transposition, on the influence of the *noosphere*[34]*, ...*

Therefore elements characterising this side are teachers' beliefs about knowledge, pupils, learning, educational goals, school, …

In this analysis the function of the "triangle" is not explicative or descriptive of educational experience, but mainly methodological: each "vertex" of the system is the observer that looks at the relation between the other two. Though, none of the elements involved can be completely separated from the others. Furthermore, its implicit effort is to fill this scheme with as many elements (or variables) concerning the educational experience as possible. This experience has to be understood as problematic.

In this systemic model act at least three categories of entities:

- *elements* (which are "vertices" or "poles");
- *relationships* among elements (which are the "sides");
- *processes* which are the modalities for the system to function (e.g.: devolution, didactical transposition, didactical engineering, ...).

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[33] The teacher is never an isolated individual, when extracting a knowledge item from her/his social or university context to adapt it to the always unique context of her/his classroom. In fact it is the collective community, the institution that provides an objective definition of school knowledge in its specificity, its methods and rationality. The didactical transposition produces a certain number of effects: simplification, de-dogmatisation, creation of fake objects or production of totally new ones.

[34] The *noosphere* is a sort of intermediate zone between the school system (and the teacher's choices) and the wider social system (outside the school). In this zone, relationships as well as their conflicts between these two systems operate. The noosphere could be described as «*The external sphere containing all the people who think about the teaching contents and methods*» (Godino, 1993).

Over the whole triangle gravitates the noosphere with its burden of expectations, pressures and choices.

2.5 Obstacles

Building models, especially models concerning mathematical infinity, is not an easy task, as we shall see in the later chapters. This depends on the fact that every concept, even if it seems an easy one at a first glance, is wrapped in fluctuating and complex surroundings of associated representations, creating multiple levels of formulations and integration of the concept (Gordon and De Vecchi, 1987). Therefore the first step is to "clean up" the concept from this halo that seems to conceal its intimate meaning. And this is what we tried to do with teachers when dealing with mathematical infinity (paragraph 4.1).

Moreover, the *obstacles* to learning that should be taken into account, as firstly described by Guy Brousseau (1983, 1986), are of primal interest for this research (Ferreri and Spagnolo, 1994; Spagnolo, 1998).

«Obstacle is an idea that, at the moment of formation of a concept, has been able to cope with the previous problem (even if this has a cognitive nature), but has failed to cope with a new problem. Given the success obtained at this stage (in fact, because of this), there is a tendency to keep the idea already acquired and tested and save it, despite its failure. This ends up by being a barrier to following learning processes». (D'Amore, 1999; our translation).

Brousseau makes a distinction among three types of *obstacles*:

- *obstacles of ontogenetic nature*;
- *obstacles of didactical nature*;
- *obstacles of epistemological nature.*

• *Ontogenetic obstacles* are linked to pupils and their maturity. During the learning process every individual develops skills and competences suitable to their mental age (which is different from the chronological age). As for the acquisition of some concepts, these skills and competences can not be sufficient and create obstacles of ontogenetic nature. For example the student can have neurophysiological limitations, which may even depend only on their chronological age (Spagnolo, 1998).

- *Didactical obstacles* depend on the teacher's strategical choices. Every teacher chooses a project, a curriculum, a method, personally interpreting the didactical transposition, according to personal, scientific and didactical beliefs. The teacher believes in the choice made, considers it to be effective and thus proposes it to the class; but what has proved effective for some students, may not be effective for all the others. For some others the choice of that particular project may turn out to be a didactical obstacle. This kind of obstacles would be the core of our research (see chs. 3 and 4).

- *Epistemological obstacles* depend on the nature of the subject itself. For instance, when in the evolution history of a mathematical concept a non-continuity, a fracture, or some radical changes of the concept are singled out, then that concept presumably bears internal obstacles of epistemological nature, as far as understanding, acceptance and finally learning by the mathematicians' community are concerned (Spagnolo and Margolinas, 1993; Spagnolo, 1998; D'Amore, 1999). Mathematical infinity provides an emblematic example (see ch. 1). This last point is manifested, for instance, in typical and recurrent mistakes made by different students in different classes over the years (see ch. 4). Discontinuity is revealed not only in the concept of mathematical infinity but also in teachers' convictions (see chs. 3 and 4) or in the beliefs of anybody else that has dealt with this subject (Spagnolo, 1995).