# Chapter 3. Primary school teachers' convictions [35] on mathematical infinity[36]

The following reflections refer to a research study on mathematical infinity, which has been carried out over many years. As we have seen in chapter 1, this subject is still fascinating and provides the humankind with an opportunity for deep reflection.

One wonders why the specific and difficult subject of this research is addressed to primary schools. It is in primary school that pupils get in touch with infinite sets like the sequence of natural numbers 0, 1, 2, 3, … which is maybe the first and more spontaneous example of these kinds of sets.

From the early years of primary school onwards, teachers explain that this sequence does not finish, i.e. it has no "end". There will always be a "greater" number than the one taken into consideration: one has just to add a unit. This process can go on forever to "infinity". Teachers affirm that if we take into consideration any natural number $n$ we will always be able to find the next natural number $n + 1$; this process gives birth, step by step, to the sequence of natural numbers and represents the basis of one of the fundamental schemes of the mathematical reasoning: the principle of mathematical induction constituting a "delicate" axiom of Peano's axiomatic system (see paragraph 1.2.7) (Borga, Freguglia and Palladino, 1985).

Primary school children often talk of infinity with reference to numbers. For instance, during an experiment in the primary school of Mirano (Venice), Marco (an 8 year old pupil) wrote the following letter addressed to his classmates attending the first year, after the teacher had asked pupils to describe what most arose their curiosity:

> *Dear children of the first year, do you know what counting to infinity means? It means that if you count for 1000 years without a break, there will always be a*

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[35] We have chosen to talk about *convictions* instead of *conceptions* because we think that the interpretation of the first term that is generally provided is more consonant with our research. By *conviction (belief)* we mean: «*an opinion, a set of judgements/expectations, what is thought about something*» (D'Amore e Fandiño, 2004) whereas the interpretaion of conception that we make our own, and also more and more widespread and shared is the following: «*the set of convictions of somebody (A) on something (T) gives the conception (K) of A relatively to T; if A belongs to a social group (S) and shares with the others members of S that set of convictions relatively to T, then K is the conception of S relatively to T*» (D'Amore e Fandiño, 2004).

[36] This chapter has been published in Sbaragli (2003a).

*greater number than the number you have just counted up! There will always be a further number and it will go on like this forever. Close your eyes and count. When you grow as old as your grandfathers you will be still counting. And you will be old men with beards; you will be so old that your parents will not recognise you anymore!*

This word, infinity, is therefore fascinating even for primary school children. Already at this age children can perceive the mystery that goes along with this term.

Pupils, even in the early school years, often talk about this still unknown word, they feel its "power" and charm and this term will be present until secondary school or even at university. Still it often remains a concept that is not understood in a mathematical sense.

## 3.1 The mathematical infinity and the different nature of "obstacles"

At the root of the following considerations about infinity, there are studies in this field surveyed by many researchers in didactics of mathematics. They have analysed the problem of teaching and learning this subject, pointing out the mental processes of the students, their convictions and intuitions that are the results of widely spread misconceptions about different aspects of mathematical infinity [among many other examples, there are the classic works of Tall (1980), of Waldegg (1993) and the more recent ones of Fischbein, Jehiam and Cohen (1994, 1995), of Tsamir and Tirosh (1994, 1997), of D'Amore (1996, 1997), of Arrigo and D'Amore (1999, 2002), of Tsamir (2000)]. These researches involve different approaches to the theme of infinity and share the common aspect of "looking through the eyes of the students" in order to examine the reasons that render infinity such a complex subject to be learnt.

It is necessary to refer to the important field of didactics of mathematics concerning the study of the so-called *obstacles* hindering the construction of knowledge: ontogenetic, didactical and epistemological obstacles (Brousseau, 1983; Perrin Glorian, 1994; D'Amore, 1999), (see paragraph 2.5).

As to the treatment of mathematical infinity in primary schools, there are for sure *ontogenetic obstacles* bound to conceptual and critical immaturity. This is mostly due to

the age of the pupils (Spagnolo, 1998). This is not a good reason though to underestimate the first intuitions, the first images, the first models that take form in the mind of children since primary school as a consequence also of the spur of teachers.
Furthermore, international literature, starting from the historical development of this controversial subject (see ch. 1), managed to point out *epistemological obstacles* hindering the learning of mathematical infinity. This makes it possible to understand some of the difficulties encountered by students [see for instance Schneider (1991)].
In this work we aim to establish if it is possible to encounter *didactical obstacles*, perhaps even more influential than the epistemological, due to didactical choices of teachers which condition and strengthen pupils' first misconceptions (see paragraph 2.3). The presence of didactical obstacles in the learning process of mathematical infinity has already been noticed by Arrigo and D'Amore (1999 and, most of all, 2002).

In order to explain the aim of this work, we will make some considerations on epistemological obstacles. As to the historical development of a concept, we can assume that there has been a gradual shift in history from an intuitive "initial" phase to a final phase of the concept itself (maybe it would be better to call it "actual" or "advanced"), mature and structural (at the time of reference). It is of course only a scheme, since there are many other fundamental transformations, which allow us to reach the "actual" phase of the concept (Sfard, 1991) between the two phases considered as the starting and arriving point (when speaking about it).
What has happened in the history of mathematics can be also said for didactics. As a matter of fact, the first historical naive intuitions on infinity usually recur in the first considerations and convictions expressed by students in classroom.
From a didactical point of view, a similar situation is observed: during a first phase students approach intuitively a mathematical concept without possessing a complete and developed understanding of it.[37] Only successively, learning turns out to be fully-fledged and more mature (Sfard, 1991).

[37] This is also due to the necessity in mathematics of a coordination of semiotic registers, to be acquired only in the long term, which is a condition for the mastery of comprehension, being it the essential condition for a real differentiation between mathematical objects and their representations. (Duval, 1995).

Two "parallel" patterns can be envisaged: the first is related to the historical development of knowledge; the second concerns a pattern that is similar to what happens in didactics (Sfard, 1991; Bagni, 2001).
In didactics the transit from the "initial" phase to the "advanced" phase of knowledge can provoke doubts and reactions in students' minds. These can be also found in the corresponding transit of formation of knowledge.
It is important to underline that the "naive intuitive" phase seems to be in opposition to the "advanced" phase. Both in the history of mathematics and in the processes of learning and teaching, in primary school as well as in secondary school and in some cases even in further years of study, as models are still present in further education (Arrigo and D'Amore 1999, 2002).
These considerations could be useful when dealing with *didactical transposition* (see 2.4) that should begin with a first intuitive knowledge on the part of students, and successively address the students' initial convictions towards the "advanced" phase of the concept itself.

## 3.2 First research questions and related hypotheses

The first research questions and the corresponding formulation of hypotheses emerge from the above debated considerations concerning didactical transposition:
- Do primary school teachers know and are they aware of the "advanced" phase of the concept of mathematical infinity?
- In the didactical transposition, do teachers base teaching on real results reached in the "advanced" phase of the development of knowledge? Or do they strengthen the students' "naive intuitive" phase instead?
- And however, have teachers ever accessed the knowledge on infinity?

The hypothesis proposed here is that primary school teachers do not know the "advanced" phase of mathematical infinity concept. This is the reason why they are stuck to the "naive intuitive" phase of knowledge. In so doing, they strengthen the students' initial intuitive convictions without helping them in the transit to the "advanced" phase of the concept. Therefore, the didactical transposition, instead of moving from the students' "naive intuitive" phase to the concept "advanced" phase (meaning the "advanced" phase of knowledge), reinforces their naive convictions and keeps them to the intuitive phase. This attitude is considered in our opinion the source of didactical obstacles hindering the comprehension of the infinity concept.

The present research was firstly addressed to students attending the last year of primary school. Our intention was to retrieve the first images, the first intuitions and possibly the first difficulties encountered when students have to cope with the subject of mathematical infinity. These experiences, on which the following chapter is based, showed that already starting from the last years of primary school, the intuitive ideas possessed by students on this topic turn out to be most of the time false convictions. These beliefs are usually explained away with sentences like: *«The teacher told me that…», «In class we saw that…»* i.e. with attitudes similar to the famous case of Gaël[38] (Brousseau and Pères, 1981) that definitely confirmed the idea of didactical contract in the field of didactics of mathematics (see paragraph 2.1). Moreover, teachers were sometimes curious about the things we wanted to show children. They got informed about the object of our research and they frankly and honestly exposed their wrong beliefs on the matter. On the basis of such considerations, the core of our research shifted from students' convictions to teachers' convictions and consequently to possible didactical obstacles verifiable when introducing the concept of mathematical infinity.

## 3.3 Description of theoretical framework

Among the many publications in the international context, D'Amore (1996, 1997) has given an outstanding contribution, providing an accurate outlook on different research "categories" and a vast bibliography with more than 300 titles.

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[38] The case of Gaël was significant for the study of the causes of the elective failure in mathematics; researchers described Gaël's case as follows: instead of consciously expressing his knowledge he always just uttered it referring himself to terms that involved the teacher. The child experienced every didactical situation through the eyes of his teacher, till the researchers, thanks to adidactical situations, managed to get more personal interventions from him and, all in all, more effective from a cognitive point of view.

More precisely, the theoretical framework on which this work is based is mainly constituted by the fundamental considerations and results of Arrigo and D'Amore (1999, 2002) offering a crucial reference to this research study.
In particular, in the first work two phenomena have been described, based on the generalisation to infinite cases of what has been learnt on the biunivocal correspondence of finite cases and to which we will refer in the present work (Shama and Movshovitz Hadar, 1994; Arrigo and D'Amore, 2002):

• the first phenomenon is called by Arrigo and D'Amore *"flattening"* and has been already dealt with in other publications [Waldegg (1993), Tsamir and Tirosh (1994), Fischbein, Jehiam and Cohen (1994, 1995)]. This is about considering all infinite sets as having the same cardinality, that is to say that a biunivocal correspondence could be established between all infinite sets. In more detail, literature on this subject has showed that once the students have accepted that two sets such as N and Z for instance, have the same cardinality (thanks to the help of the researcher or teacher showing them the biunivocal correspondence between the two given sets), it is much more common that students tend to consider as true the generalisation that all infinite sets must have the same cardinality, which is not the case. The latter misconception is not only due to epistemological obstacles, of which we found evidence in the history of mathematics, but also to didactical obstacles as pointed out by Arrigo and D'Amore (1999 and in particular, 2002).

• the second phenomenon is that of *"dependence"*, as named by the two authors, according to which there are more points in a long segment than in a shorter one (Tall, 1980). This phenomenon can be observed not only in geometrical milieu, but it is also valid when referring to *dependence* of the cardinality on the "size" of numerical sets. For example, since the set of even numbers represents a sub-set of the natural numbers set, the former seems to be by implication formed of a smaller number of elements.

The above-mentioned attitudes have been surveyed and analysed in detail by Arrigo and D'Amore (2002). The two authors also pointed out that most difficulties encountered in the understanding of infinity are strictly related to students' intuitive models of geometrical entities (Fischbein, 1985) (see 2.2), in particular the point and the segment (see ch. 4). In our research work, we also based ourselves on the considerations reported in Fischbein (1993). He revealed, by means of some examples (some of these concerning the point), the complex nature of the relationships between figural and conceptual aspects, pertaining to the organisation of *figural concepts* and the fragility of such an organisation in the students' minds, has been underlined. On this latter aspect and from a didactical standpoint, Fischbein believes that teachers should systematically point out to their students the various contradictory situations in order to stress the predominance of definition on the figure. That is to say, students should be made aware of conflicts and of their origins, so that they can start being confident with the necessity for mathematical reasoning to depend on formal constraints. In addition, Fischbein (1993) claims that the integration of conceptual and figural properties into unitary mental structures, with the predominance of conceptual constraints on figural ones, is far from being a spontaneous process and in fact this could constitute a major continuous and systematic concern of teachers. To achieve this Arrigo and D'Amore (2002) suggest intervening in primary school teachers' preparation in this specific field. This latter aspect represents a crucial point in the present work, which is based on primary school teachers' beliefs on mathematical infinity; these convictions influence students' intuitive models resulting in situations of cognitive disadvantage. In order to modify and re-adjust these convictions, a new way of learning, only attainable thanks to suitable training courses for teachers enhancing a closer examination of the above – mentioned topics, is therefore required (see 4.1).

Another subject pertaining to this research is the classic philosophical debate about infinity in the actual and potential sense inspiring many authors: Moreno and Waldegg (1991), Tsamir and Tirosh (1992), Shama and Movshovitz Hadar (1994), Bagni (1998, 2001), Tsamir (2000). These authors point out that from both a historical point of view and that of the learning of infinity, the evolution of the actual conception of infinity is extremely slow and frequently contradictory and this is only possible thanks to a cognitive process involving cognitive maturation and systematisation of learning (for a historical excursus see in ch 1 the reflections of Aristotle, Euclid, Augustine of Tagaste, Thomas Aquinas, Galileo, Torricelli, Descartes, Gauss, Cantor). More specifically in

Tsamir (2000), difficulties encountered by teachers in training when faced with actual, rather than potential infinity have been highlighted. This is to be traced back to the previous considerations on the necessity of introducing these contents in the training of primary school teachers too.

## 3.4 Description of problems

This section provides a description of the problems inspiring the present research.

P.1 Are primary school teachers aware of the concept of mathematical infinity and of its epistemological and cognitive meaning?

P.2 Do teachers provide their students with some intuitive models on the topic since the first years of primary school? If they do, are they aware that these are misconceptions that will be awaiting a further systematisation, or do they believe these to be correct models that should accompany their students during their whole future educational career?

P.3 Could teachers' convictions be the cause of didactical obstacles responsible for the strengthening of the epistemological obstacles already pointed out in the research at international level?

## 3.5 Research Hypotheses

Here as follows we report the hypotheses related to problems described in 3.4:

H.1 We believe that mathematical infinity is a rather unfamiliar subject for most primary school teachers, both from an epistemological and from a cognitive point of view. We therefore thought that teachers would not be able to handle infinity and to conceive it as a mathematical object. Consequently, we assumed that teachers would stick to naive convictions as for example: infinity is nothing but indefinite, or infinity is synonymous with unlimited, or else infinity is a very large finite number [convictions that were present over the centuries throughout the history of this topic, see chapter 1, in particular they are to be traced back in the statements of Nicholas of Cusa (1400 or 1401-1464)].

H.2 We believe that primary schoolteachers normally provide pupils with intuitive models of mathematical infinity, starting from the early years of primary school.
Moreover, if teachers' naive convictions, assumed in H.1, were verified, they would condition (in our opinion) the models provided to pupils. We assumed that teachers provided intuitive models that they considered correct, but in fact they were based on misconceptions. In order to verify this hypothesis, we judged that it would be interesting to analyse accurately the teachers' statements and their way of expressing ideas.

H.3 We assumed that, if the two above-mentioned hypotheses had been proved true, beside epistemological obstacles that the study of mathematical history and the criticism of its fundamentals have highlighted, we would have been able to trace obstacles of didactical nature too. One of the obstacles we thought we would encounter is bound to a naive idea of infinity as a synonym for unlimited, a conviction which is in contrast with the concept of the infinity of points in a segment, a segment being limited though constituted of infinite points. One more obstacle we thought we would find is bound to the idea of infinity, considered as a large natural number [see ch. 1: Anaximander of Miletus (610 B.C. – 547 B.C.) and Nicholas of Cusa (1400 or 1401-1464)], it follows that the same procedures applied to finite sets are automatically transferred to infinite sets, which are seen as very large finite sets. Another didactical obstacle, often highlighted by Arrigo and D'Amore (1999, 2002), that we were confident we would come across, is the "model of the necklace" as the two authors call it. Students often point it out as a suitable model to visualise the points on a straight line, and they indicate their primary school teachers as the source of this model that withstands all subsequent attacks. (Arrigo and D'Amore, 1999; 2002) Our hypothesis was therefore that we would encounter didactical obstacles, deriving from typical models, usually introduced by primary school teachers.

If the above-mentioned hypotheses had been proved true, we would have gone ahead in our investigation and would have considered the possibility and the necessity of revising the didactical contents of primary school teachers' training courses. This is not meant to force teachers to change the contents of their didactical activity, but to prevent them from building intuitive models that could bring about situations of cognitive disadvantage for their students.

## 3.6 Research Methodology

### 3.6.1 Teachers participating in the research and methodology

As a consequence of the shift of present research focus on primary school teachers' convictions, our idea was to develop a questionnaire to use as a starting point for reflections and opinion exchange among teachers on matters related to mathematical infinity. The aim was to let their convictions, misconceptions and intuitive models with regard to this concept emerge.

For the outline of the questionnaire, several informal interviews with teachers were held to verify the text readability and understandability. The questions were all about those concepts that are usually dealt with in primary school and that create in students' minds, even without teachers' awareness, the first images to be transformed in intuitive models of geometrical entities or more in general of infinity.

The questionnaire and the following opinion exchange were administered to 16 Italian teachers of primary schools, different from those already involved in the initial phase [4 from Venice, 8 from Forlì (Emilia Romagna), 4 from Bologna].

The research was been carried out according to the following modalities: six meetings were organised. Two teachers attended each of the first four sessions, whereas four teachers at a time attended the two meetings left (16 teachers in total). Each meeting started with the questionnaire proposal: teachers were asked to read through the questionnaire and then to fill it in individually. After everybody had handed it in, open discussion among pairs or group of four people would start. During the debate teachers could express their convictions, doubts and perplexities in the presence of the researcher who intervened in the conversation only on certain occasions, in order to stimulate the discussion on some relevant aspects, but firmly trying not to modify teachers' ingenuous convictions. Discussion groups were organised as to allow confrontation between teachers that could already get on well together and were used to discuss and exchange opinions.

However, it was clearly stated right from the beginning that their names would not appear in the research work.

The teachers have judged the questionnaire easily "comprehensible". As a matter of fact, after a first reading of the questions, teachers unanimously affirmed that it was clear and of accessible interpretation, even though when it came to answer the very first question, 13 teachers out of 16, manifested great embarrassment: *«I don't know what to write, I never reasoned on this topic»*. Only after some self-assuring statements such as: *«I will write down what comes up to mind, even if it won't be well expressed»*, they started answering the first question.

Teachers had one hour for the questionnaire, so that they could read it through, reflect, think it over again and organise their answers with no pressure and taking their time. None of the teachers involved used all the time available.

As to the second phase, based on discussion and confrontation, there were no time restrictions; we adopted the technique of active open debate in groups of different size, using the tape recorder and leaving the researcher the task of highlighting contradictions and deeply rooted intuitive models.

This last discussion phase was the most fruitful and significant. As a matter of fact, since the very first interviews it was clear that a written text is not a suitable means to make real intuitive models emerge. A single answer, synthesised in most cases, is not enough to interpret teachers' real convictions. Such a complex and delicate topic needs a further and deeper investigation into teachers' individual and single convictions. To this aim, opinion exchange has proved to be a very useful means of revising and reworking the questionnaire's answers, to understand their intimate meaning, to verify their stability and to point out possible contradictions.

The decision of implementing confrontation between teachers, rather than between a single teacher and the researcher, is based on the necessity of collecting teachers' real convictions, otherwise difficult to be identified. When teachers are asked to express or defend their own opinions, in front of other colleagues with whom they feel confident and are used to arguing and sharing more or less the same knowledge, the expected outcome is that they would feel freer to manifest their ideas.

The applied strategy also served the aim of reducing some teachers' reactions such as: "trust in the researcher" or "trust in what mathematicians affirm" [often reported in literature; for example: Perret Clermont, Schubauer Leoni and Trognon (1992)], emerging not only when research is addressed to students but also when teachers are involved.

The complete documentation of these exchanges will not be provided in this thesis, only the most significant and recurrent sentences will be reported. Questionnaires and complete recordings will be at disposal of whoever is interested in further researching this topic.

### 3.6.2 Questionnaire content

The questionnaire contained 15 A4 sheets, one sheet for each question (with space for teachers to write their answers).

Here as follows the 15 questions will be transcribed together with some explanations on the methodology used for the compilation of the questionnaire:

*1) What do you think mathematical infinity means?*

*2) Has it ever happened to you to talk of infinity during the five years of primary school teaching? When? In what sense? How? Using what kind of support?*

*3) Does the term "infinity" in mathematics exist both as an adjective and as a noun?*[39]

*4) Are there more points in the AB segment or in the CD segment? (Write down on the sheet of paper everything that comes to mind).*

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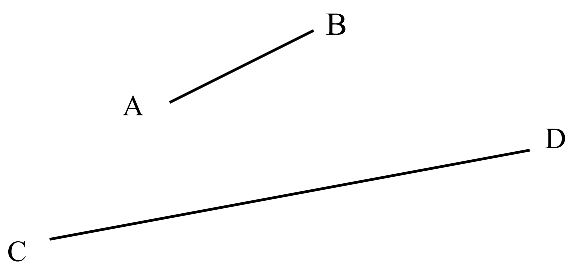
[39] Translator's note: the term *infinito* is used in Italian both as a noun and as an adjective, thus covering both meanings of the English words *infinity* and *infinite.*




*5) How many even numbers are there: 0, 2, 4, 6, 8, …?*

*6) How many odd numbers are there: 1, 3, 5, 7…?*

*7) How many natural numbers are there: 0, 1, 2, 3…?*

*8) How many multiples of 15 are there?*

*9) Are there more odd or even numbers?*

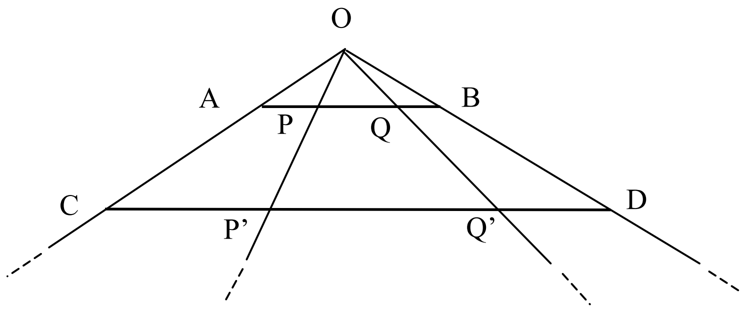
*10) Are there more even or natural numbers?*

*11) Are there more odd or natural numbers?*

*12) Are there more multiples of 15 or natural numbers?*

*13) Has it ever happened during your primary school teaching to compare the quantities of these numerical sets (even with odd, even with natural, odd with natural)? How? On which occasion?*

After teachers handed in the first 13 answers, they were showed Georg Cantor's demonstration (1845-1918) (see paragraph 1.2.3) related to question no. 4 that proves that there is the same number of points in two segments of different length. In order to illustrate it, teachers were showed the biunivocal correspondence on a sheet of paper containing two segments AB and CD (differently positioned on the plane from those concerning question number no. 4; they were shifted by hypsometry so that they appeared parallel and "centred" with respect to one another). At the beginning, with the help of a ruler, the point O of intersection between the straight lines AC and BD was drawn; successively from O the points of the segment AB were projected on the segment CD and vice-versa. In so doing, the biunivocal correspondence between the sets of points belonging respectively to segments AB and CD was demonstrated. It was therefore easy to observe that there is the same number of points in segments of different length.




Successively teachers received a sheet of paper with the following question:

*14) Try to be as honest as possible answering the following question: were you convinced by the demonstration that there are as many points in AB as in CD?*

After answer no. 14 had been handed in, teachers were shown the demonstration of question no. 10, that proves that the set of even numbers (E) is formed by the same number of elements than that of natural numbers (N), showing the related biunivocal correspondence (Tall, 2001a).

Let us illustrate the biunivocal correspondence showed to teachers:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| N | 0 | 1 | 2 | 3 | 4 | 5 | … | n | … |
| | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ |
| E | 0 | 2 | 4 | 6 | 8 | 10 | … | 2n | … |

This idea developed from the consideration made by Galileo Galilei (1564-1642) (even if Galileo talked about square numbers and not even numbers) (see paragraph 1.1.2): to each natural number corresponds a determined square and vice-versa. To each square number corresponds a determined natural number (its «arithmetic root»); thus there are as many natural numbers as square numbers.

As in the case of the previous question, teachers received a sheet of paper with the following question:

*15) Try to be as honest as possible answering the following question: were you convinced by the demonstration that there are as many numbers in the set of even as in that of natural numbers?*

Only after all sheets had been handed in, debate and opinion exchange between groups of two or four people started.

In consideration of the nature of this research and most of all the involved subjects' competences on this specific topic, the choice was not to establish a determined order of the questions to pose taking into account the preliminarity of concepts as the discrete that should precede the continuum. As a matter of fact, only question no. 4 seems to pertain explicitly and specifically to the field of "continuum".

## 3.7 Description of test results, opinion exchange and verification of hypotheses outlined in 3.5

From the answers given to the questionnaire, some rather generical affirmations emerged which have undergone further investigation thanks to the opinion exchange between teachers. Some of the answers given to each question have been selected and are provided below. Integrations to such answers obtained through verbal exchange during discussion are provided as well. The aim is to offer the widest and most representative view as possible of the respondents' convictions. Researcher's interventions and comments have been indicated in bold. It ought to be remembered that the researcher intervened only to stimulate conversation and to go more deeply into teachers' convictions.

### 3.7.1 Description of test results and related opinion exchange

1) As for the answers to the first question of the questionnaire, they can be all classified one way or another into convictions reported here as follows. It ought to be noted that none of the 16 interviewed teachers was aware of the “advanced” conception of mathematical infinity. It is important to remember that the operated classification should not be considered as definitive, since as we shall see later, some of the teachers’ affirmations, that were at the beginning dealt with as belonging to a specific category, have been also successively inserted in other ones as a direct consequence of the outcome of successive conversations.

• ***Infinity as indefinite.*** 7 teachers tend to consider infinity as indefinite, that is to say they do not know how much it is, what it is exactly, what it represents.

*R.: «To me it means without boundaries, with no limits like the space»*

***R.: «In the sense of indefinite?»***

*R.: «Yes without borders»*

*C.: «Something that you cannot say»*

***R.: «In what sense?»***

*C.: «You don’t know how much it is»*

*A.: «Something that cannot be written down»*

• ***Infinity as a finite large number.*** 3 teachers affirm that infinity is nothing but a very large finite number.

*A.: «To me it’s a large number, so large that you cannot say its exact value»*

*B.: «After a while, when you are tired of counting you say infinity meaning an ever-increasing number».*

• ***Infinity as unlimited.*** 5 teachers confuse infinity with unlimited, they think the term infinity can be attributed exclusively to the straight line, half-line and plane, i.e. everything unlimited. Therefore it is not possible to talk of infinity with regard to the points of a segment that is a limited entity. Curious enough is that if the researcher intervenes asking: *«How many points are there in a segment?»* teachers show to know the answer to the question that is: *«Infinite»*, but without understanding the real meaning of this statement. As a matter of fact, further investigating it, 3 of the 5 respondents affirm that in the case of the number of points of a segment, infinity is considered a large finite number whose exact value is unknown, whereas the remaining 2 see infinity as indefinite: you do not know exactly how much it is. So, all of the 5 answers are pertinent also to the other categories indicated for point no.1. To these teachers the following relationship seemed to be valid: in the cases of lines, planes and space, infinity and unlimited seem to be synonyms; in the cases of the quantity of numbers or points, they refer to infinity as a very large finite number or indefinite number.

*A.: «With no limits»*

*M.: «Something that I cannot quantitatively measure»*

(Also in this case, teacher M. associates the term infinity with unlimited without considering that a segment, for instance, though being limited and measurable in the sense understood by M., contains infinite points).

*N.: «Something unlimited»*

***R.: «So you will never use the word infinity referring to a segment?»***

*N.: «No, because it has a beginning and an end»*

***R.: «How many points do you think there are in a segment?»***

*N.: «Oh you're right, infinite. But it's just to say a large number, not as large as in the straight line. Even if you make very little points, you cannot fit in it more than that».*

[Teacher N. already reveals the conviction emerged also in answers to question no. 4, that is to say, that there are more points in a straight line than in a segment stressing the idea that to greater length corresponds a greater number of points. Points are therefore conceived not as abstract entities but as objects that should have a certain dimension in order to be represented (see ch. 4). These misconceptions are derived from teachers' models of fundamental geometrical entities such as point, straight line and segment].

*G.: «Unlimited»*

***R.: «How many points do you think there are in a segment?»***

*G.: «You say that in a segment there are infinite points because it is not known how many there are exactly».*

• ***Infinity as procedure.*** Only a single teacher talks of infinity in the first question referring to a never-ending process:

*B.: «I know infinity, it means to keep going on as with numbers… for ever».*

This conviction recalls the idea of potential infinity that we shall see illustrated in 3.7.3. Analysing the collected answers in more detail, it can be observed that also the answer given by B. belongs to the category of infinity as a large finite number. In B.'s answer (and also in answers of other teachers, which we will see later on) the conviction is to be traced back to potential infinity considered as an ever-lasting process.

2) Answers concerning the second question unanimously reveal, that the concept of infinity, under different forms, is dealt with by teachers from the early years of primary school, thus creating images of what is intended by this term. All of the 16 teachers affirmed that they mention and talk about infinity in primary school.

*A.: «Talking about the number, I show them the line with numbers and I say that they never end. And talking about infinity I show the difference among segment, half-line and straight- line».*

Once again the conviction of infinity as unlimited emerges. As a matter of fact, it was the same teacher affirming that infinity means lacking at least one limit. Therefore she explicitly says: *«You can talk of infinity only in the cases of the half-line and the straight line, but not in that of the segment»*, the segment being limited.

*G.: «I talk about it when we do numbers. I always say they are infinite»*

*A.: «In the third year I usually say that the straight line is infinite trying to evoke mental images rendering the idea of infinity like the laser beam»*

*M.: « I also use it for the parts I can make out of a quantity, I can keep on dividing always the same quantity».*

These are only some examples of the teachers' statements showing that they deal with the term infinity since the very early years of primary school, even without a complete awareness and correctness of its mathematical meaning.

3) The aim of the third question was to discover if awareness that infinity represents a mathematical object existed among teachers (Moreno and Waldegg, 1991).

For 13 teachers "infinity" in mathematics is only an adjective, the remaining 3 believe it is also a noun, but out of the latter 3, 2 of them conceive infinity as indefinite whereas the other one believes that you can use this word also as a noun but only meaning a very large finite number whose value is unknown.

*N.: «As an adjective»*

*M.: «In mathematics it exists only as an adjective, in the Italian language also as a noun»*

*A.: «In mathematics it is used as an adjective: infinite numbers, infinite space. As a noun in the Italian language: Leopardi's "Infinity"*[40]*; "I see infinity"; "I lose myself in infinity"»*

*B.: «Also as a noun to mean a large number»*

4) The fourth question was about the teachers' supposed conviction that two segments of different length should correspond to a different number of points [this idea already emerged from the answer to the first question provided by the teacher N. and illustrated in point 1) of this paragraph].

All of the 16 interviewed teachers affirmed that in two segments of different length there is a different number of points and more specifically, to a greater length corresponds a greater number of points (Fischbein, 2001). It stands out clearly that visually one segment appears to be included in the other and therefore in this case the figural model is predominant. This model negatively influences the answer and in fact the Euclidean notion: *«The whole is greater than its parts»* (see 1.1.1) cannot be applied to infinity.

Here as follows, we have reported some of the answers pertaining to the above-mentioned conviction:

*N.: «This makes me think that the different length of two segments should have some influence on the number of points»*

*B.: «In the segment CD; of course, it's longer»*

*G.: «In AB there should be a lot, in CD many more»*

*A.: «I'm not sure. Given that the segment can be considered as a series of points in line, I think that CD has more points than AB, even if I have learnt that the point is a*

[40] Translator's note: Giacomo Leopardi's *Infinity* is one of the most famous Italian poems of all times.

*geometrical entity, which, being abstract, is not possible to quantify it because it's not measurable. I would say CD, anyway»*

[The teacher A. showed inconsistency between what she affirms she had studied in order to take an Analysis exam at university and what she believes to be the most plausible answer according to common sense. Once again, the intuitive model persists and predominates. In this situation it is quite evident that there is no correspondence between the formal and the intuitive meaning (Fischbein, 1985, 1992; D'Amore, 1999)].

The above discussed intuitive conception represents a widespread misconception, it has already been mentioned in 3.3 and is called *dependence* of transfinite cardinals on factors related to magnitudes (the set of greater size has more elements). The teacher in question is therefore convinced that a greater length implies also a greater cardinality of the set of points. Accurate surveys have largely proved that mature students (those attending the last year of higher secondary school and the first years of university) do not succeed in mastering the concept of continuity because of the persisting intuitive model of a segment seen as a "necklace of beads" (Tall, 1980; Gimenez, 1990; Romero i Chesa and Azcárate Giménez, 1994; Arrigo and D'Amore, 1999, 2002).

This misconception will return also in the answers to questions no. 10-11-12, where *dependence* is to be intended as *dependence* of the cardinality on the "size" of numerical sets.

This conviction, as we have already observed in chapter 1 (that in a *longer* segment there are more points than in a *shorter* one) and notwithstanding several occasional episodes, has been definitely eradicated only in the XIX century therefore rather recently. Once again the history of mathematics has witnessed the presence of an epistemological obstacle highlighted in several research studies (Tall, 1980; Arrigo and D'Amore, 1999). The latter obstacle represents a misconception belonging to the common sense outside the mathematical world. Therefore, this phenomenon is to be traced back even within teachers' convictions, teachers who have not been given the opportunity to reflect on the "advanced" conception of this topic.

As a matter of fact, the epistemological obstacle, considered according to Brousseau (1983) (see 2.5) in its classic meaning, is a stable item of knowledge that has worked out correctly in previous contexts, but that is a source of problems and mistakes when trying to adjust it to new situations (dis-knowledge or *parasite model*). Furthermore, as stated by Arrigo and D'Amore (1999): «*… in order to overcome this kind of obstacle a new item of learning is needed*», in many cases this learning did not take place during the educational career and neither is favoured in further years of study.

Nevertheless, it seems quite difficult to figure out that teachers who have never reflected on such topics could possess an image of the topology of the set of the points of the straight line (and therefore at least their density) that enables them to understand the specific case of two segments of different length, for instance. To avoid that the above-mentioned convictions turn into incorrect models producing didactical obstacles (that in turn magnify the already highlighted epistemological obstacle), it is important to help the subject in question to detach her/himself from the model of the segment seen as a "necklace". In this way s/he comes to more appropriate images for the comprehension of the concept of non-dimensional points (see ch. 4). In order to do this, the subject should enlarge previous knowledge and build new items of knowledge but the only way to achieve this goal for her/him is to study theorems concerning the already mentioned topics.

5) – 6) – 7) – 8) As to the four following questions, 15 teachers answered in this way: «*Infinite*», with the exception of one of them who, after some hesitation, wrote: «*Quite a few!*», being afraid of saying something wrong. In mathematics it is a common and most widespread attitude to answer with "learnt by heart sentences" without proper awareness, or understanding of its real meaning according to the "advanced" conception of a concept (see 4.1). All of them remember that these sets are infinite, but they actually ignore the sense of such an affirmation. Almost everybody has memories of having studied that the point has zero dimension, but they do not know what this means, since the intuitive model of the point as the mark left by the pencil is still predominant in many cases.

9) This question and the following four envisaged the task of comparing some infinite set cardinalities that are often dealt with in primary schools. The collected answers have been classified in the following three categories:

• ***There are as many even numbers as odd numbers.*** 12 teachers out of 16 have this opinion.

*C.: «It's the same number to me»*

• ***It is impossible to make a comparison of the cardinalities of infinite sets.*** To 3 teachers a comparison of the cardinalities of infinite sets is not conceivable. As a matter of fact, in the logic of those who conceive infinity as indefinite or as something finite, very large but with an undetermined value, it is rather difficult if not impossible to make a comparison between cardinalities of infinite sets.

*R.: «You can not answer that, it is not possible to compare infinities»*

• ***The unsure.*** One teacher answered back with a question:

*A.: «I would say they are of equal number, the even and the odd numbers; but I have a major doubt: If they are infinite how can I quantify them?»*

(From this answer emerges the idea of infinity seen as indefinite).

10) – 11) – 12) The answers provided to these three questions belong to the following four categories. All of the 16 interviewed teachers are consistent always replying in the same way to all three questions:

• ***There are more natural numbers.*** 10 teachers answered that there are more natural numbers, supporting the common Euclidean notion: *«The whole is greater than its parts»*.

*C.: «The natural numbers»*

• ***You cannot compare infinite sets.*** The same 3 who in reply to question no. 9 could not conceive a comparison between cardinalities of infinite sets, remained firm in this opinion; this results in the idea that you can refer to cardinality only when dealing with finite:

*R.: «You cannot answer that, you can't make a comparison».*

• ***The unsure.*** The same teacher who answered to question no. 9 with another question, replied in the same way which shows consistency:

*A.: «I would say the natural numbers, but how can I quantify them? To say infinity means nothing».*

• ***They are all infinite sets.*** 2 teachers affirmed that all the sets in question are infinite and therefore they have all the same cardinality.

*B.: «They are both infinite. If two sets are infinite, they're just infinite and that's it».*

From the interview of these two teachers emerges the misconception of the *flattening* of transfinite cardinals illustrated in paragraph 3.3, resulting in the belief of considering all infinite sets of equal power. In other words, these teachers came spontaneously to the conclusion that being all the above-mentioned sets infinite, the attribute "major", in compliance with a passage of Galileo's, cannot be used when dealing with infinities (see 1.1.2). The direct consequence is that all of the sets of this type are nothing else than - banally - infinity.

***R.: «According to you, do all infinite sets have the same cardinality?»***

*B.: «What do you mean? The same number? Yes, if they are infinite!"*

13) This question has been posed in order to point out if some of the teachers interviewed had ever proposed the topic of the comparison of the cardinalities of infinite sets during the didactical activity in class. All of the 16 teachers answered that they had never proposed specific activities on that topic even if 3 of them admitted, during the open discussion, that they might ingenuously have said to their students that there are more natural numbers than even numbers. Such an affirmation is definitely a didactical obstacle to students' future learning.

14) In order to test to what extent teachers are convinced of their affirmations regarding the idea of point and segment (on which in particular question no. 4 is based) they were provided with the construction described in 3.6.2. This shows that there is the same number of points in two segments of different length and only afterwards question no. 14 had been distributed.

Answers have been classified according to the following categories:

• ***Not convinced by the demonstration.*** 5 out of 16 respondents were not convinced by the demonstration:

***R.: «Were you convinced by this demonstration?»***

*M.: «Well, not really; to me a point is a point, even if I make it smaller, it's still a point. Look!* (Drawing it on the sheet). *Then, if I make them all of the same size, how can they be of the same number?»*

***R.: «According to you, between two points is there always another one?»***

*M.: «No, no if draw two points one next to the other, very close, so close, practically stuck to one another there won't be any in between them»*

*B.: «Ummh! But in the segment AB you go over the same point when lines get thicker. I'm not convinced».*

As a matter of fact, to grasp the exact meaning of this construction has proved quite a difficult task for those teachers to whom the point is not conceived as an abstract entity with no dimension, but rather as the mark left by the pencil and therefore with its own dimension. More in general terms, teachers rejecting the above discussed construction are those who imagine the segment as the "model of the necklace of beads".

• ***Convinced by demonstration.*** 9 were convinced by the demonstration. The teachers A. and C., in particular, considered it crystal clear and extremely effective:

*A.: «That's nice! You convinced me»*

*C.: «You convinced me, it's exactly like that»*

*G.: «Yes, I'm convinced».*

Although these 9 teachers were promptly and immediately confident with the demonstration correctness some doubts and perplexities were provoked by questions such as: *«Are you really sure about it?»*. Our intention was to observe if teachers were inclined to change their mind showing by that not a profound and stable conviction. As a matter of fact, 3 admitted not being thoroughly convinced, returning to the initial affirmation that there are more points in CD. [On this aspect consult: Arrigo and D'Amore (1999, 2002)].

***R.: «Are you really sure about it?»***

*G.: «No, no! I'm still convinced that there are more of them in CD, you can see it»*

*R.: «I'm not so sure».*

• ***Trust in mathematicians.*** One teacher showed a sort of "trust in mathematicians", though not being totally convinced by the demonstration:

*A.: «If you mathematicians say that, we trust you. Me for sure, I won't get into these problems!»*

• ***The unsure.*** One teacher seemed to be in need of some kind of explanation, but after a little discussion claimed to be convinced:

*M.: «It's because you took that point over there, if you had taken another one it wouldn't have worked out… look!»*

(The teacher drew another point different from the projection point identified by the researcher and then drew lines intersecting the longer segment and not the shorter one. These considerations mirror the difficulty in understanding what it is and how a mathematical demonstration works).

***R.: «Yes, but if you want the projection point to be exactly the point you drew you have to perform a translation of the two segments and project right from that point*** (the translation on M.'s drawing was performed)***, however, the translation will not alter the number of points of the two segments»***

*M.: «Ok, you convinced me».*

15) The biunivocal correspondence was subsequently demonstrated to the 16 teachers. The biunivocal correspondence proved that the cardinality of even numbers is the same as that of natural numbers and then the question 15 was asked.

Teachers reacted in two different ways:

• ***The dubious.*** 6 expressed themselves as not being particularly convinced:

*M.: «Well, it's kind of a "strain"»*

*N.: «It's strange, in the set of even numbers all the odd numbers are missing to obtain the natural ones».*

• ***Those affirming to be convinced.*** 10 claimed to be convinced, but 2 in particular showed some trust in the researcher as the one who possesses Knowledge.

Furthermore, during interviews, it emerged that all the teachers who accepted the idea that some infinite sets are of equal power (as in the case of the even and natural numbers) are now convinced that this is bound to infinity and as a consequence they generalise that all infinite sets are also of equal number. This *flattening* misconception is seen as an "improvement" in comparison to the *dependence* misconception of the cardinality on the set "size". This change in attitude seems a slow and gradual approach towards "the correct and advanced model of infinity". The appearance of the *flattening* misconception of transfinite cardinals was not unexpected since primary school teachers ignore the set of real numbers, and therefore they opt for a generalisation of the notions related to the sets known to them.
Prove of that is given by the following conversation:

*A.: «Therefore all infinite sets are equal»*

***R.: «What do you mean? Do whole numbers have the same cardinality as natural numbers?»***

*A.: «Uhm, yes»*

***R.: «And the rationals? The fractions»***

*A.: «I think so»*

***R.: «And real numbers? The roots»***

*A.: «Yes all, all of them, they are either all equal, that is to say infinite or none of them is so».*

The aim of the proposed demonstrations was to show teachers that the primitive Euclidean property: «*The whole is greater than each of its parts*» cannot be applied to infinite sets: neither in the ambit of geometry [look at the proofs of: Roger Bacon (1214-1292), Galileo Galilei (1564-1642), Evangelista Torriccelli (1608-1647) and Georg Cantor (1845-1918)] nor to infinite numerical sets where one is a proper subset of the other.

Teachers' intuitive affirmations (misconceptions) seemed to be inconsistent, as the two contradictory misconceptions of *flattening* and *dependence* coexist in their mind. It has been observed a generalised difficulty of the teachers to realise when two affirmations are contradictory and we believe it to be the result of their lack of knowledge and of mastery of the concept of mathematical infinity.

In addition, it has also to be noted that the discussions among teachers brought no change of opinion when it had to do with the infinity issue. Some of the participants changed their mind only as a consequence of the two demonstrations showed by the researcher, whereas they showed somehow reluctant when stimuli to reflections came from the other colleagues.

### 3.7.2 The idea of point

Many of the teachers' affirmations, especially those related to the question no. 4 (reported in paragraph 3.6.2) based on the misconception that to a different segment length corresponds a different number of points, revealed how some of the convictions in question are related to the idea of point seen as a geometrical entity provided with a certain dimension, though small. This belief originates from the most commonly adopted representation of point conditioning the building of this mathematical object related image. As a matter of fact, this misconception seemed to be shared also by those not explicitly expressing it though stating with regard to question 4 that a longer segment has more points than a shorter one. In so doing they revealed "naive" interpretation of the idea of segment and point possessed.

Hereafter some of the affirmations related to question no. 4 are reported:

*B.: «In the segment CD, of course it's longer»*

***R.: «How many more?»***

*B.: «It depends on how big you make them»*

*M.: «It depends on how you draw them: distant or very close to one another; but if you make them as close as possible and all of the same size then there are more in CD»*

*G.: «In CD, it's longer»*

***R.: «But can you really see the points as graphically represented here?»***

*G.: «Yes, it's the kind of geometry we do that makes us see the points».*

Hereafter we report once again the statement mentioned in 3.7.1 concerning the question no. 1:

*N.: «Oh you're right, infinite. But it's just to say a large number, not as large as in the straight line. Even if you do very little points, you cannot fit in more than that».*

These affirmations are strongly influenced by the so-called "necklace model" to which we mainly referred to as source of obstacles in the understanding of the concept of mathematical infinity and of the straight line topology. De facto, a *parasite model* has been built in the students' minds (Fischbein, 1985) (see 2.2) as a consequence of the acceptance of the intuitive model seen as a thread of little beads. Most striking is that the "necklace model" represents not only a didactical device ingenuously invented by teachers in order to provide their students with just an idea of a segment, though being aware that the image in question is an imprecise, rough and quite distant representation of the real mathematical concept related to segment. On the contrary, this unfortunately represents the real model teachers possess of a segment and point. In addition, as emerged from discussion most of the teachers' deficiencies are particularly linked to the concepts of the straight line density and continuity.

### 3.7.3 Potential and actual infinity

The opinion exchange revealed how some of the teachers' convictions are definitely referable to the potential view of infinity. As a matter of fact, also in those cases when they adopted definitions ascribable to actual infinity such as: *«The straight line is formed of infinite points»*, they successively turned out to be inconsistent when declaring also that the term straight line is used only to indicate an ever longer segment, returning once again to the potential vision (in compliance with the Euclidean thought, see 1.1.1).

As pertaining to the potential use of the infinity concept the two following examples are reported:

*R.: «We use to say that natural numbers are infinite, but we know that this doesn't mean a thing as they can't be quantified! It's like saying a very large number that you cannot even say; that you can go on forever I mean. To say straight line is like saying nothing, it doesn't really exist, it's another way of saying an ever longer line».*

A further aspect originates from R.'s affirmation: the term infinity is mentioned but does not represent a quantity. Stating that natural numbers are infinite (a very commonly used expression pertaining, at a first glance, to infinity in its actual sense) is just another way to say a large finite number. Moreover, this affirmation seems to support the conviction that everything concerning the unlimited and infinity is perceived as non-existing since it is not to be traced in the sensible world. On the other hand, concepts such as segment, square, rectangle, for which it is possible to locate some approximated "models" surrounding us, are perceived as existing. The presence of such a conception implies that the real sense of mathematics and its related concepts too is mislaid. As a matter of fact, if you do not perceive mathematical entities as abstract but you remain stuck with the attitude of envisaging them as things existing in the sensible world, then to think of concepts such as mathematical infinity or the straight line topology happens to be cause of major disadvantage. The resulting problem is that some teachers think that most branches of mathematics are related to the concrete and sensible world and that there are some other concepts such as infinity or the straight line which are detached from the world of things and hence according to opinion not suitable to be dealt with in primary school. A teacher expressed this idea with the following words: *«If a thing doesn't exist as in the case of the straight line there is therefore no meaning to teach it?»*. The same considerations are also applicable to the following affirmation:

*N.: «I say that numbers are infinite, but I know it's only imagination, you'll never get to have them all, you use infinity to mean an ever-increasing number. No way you can reach infinity».*

The main consequence of such conceptions in the teaching activity is the risk of providing the students with images completely extraneous to mathematics and possibly turning into an obstacle to future learning both in analysis courses of higher school and even before in the lower secondary when concepts such as the density of Q, irrational numbers such as π, the ratio between the square side and its diagonal and many more, are introduced.

Interviewing teachers revealed the prevailing use of the potential infinity and with respect to this the outcomes of the discussions that have proved extremely interesting were aimed by the researcher at the comprehension of the double nature of infinity: actual and potential, in the same way as it appeared to Aristotle (see 1.1.1).

• 10 teachers are still stuck to potential infinity as shown in the following passages:

*M.: «To me there exists only the potential infinity, the other doesn't exist, it's pure fantasy, tell me, where is it?»*

***S.: «When talking of the straight line»***

*M.: «But where is the straight line? There is none. So actual infinity does not exist»*

***S.: «What do you think of the straight line?»***

*M.: «I think these kinds of things shouldn't be taught, at least not in primary school, poor children what can they do! Yes, of course you can also say that the straight line is formed of infinite points, but how are they supposed to understand that? (I don't believe it myself!), at their age they have to see things. They have to touch things with their own hands»*

*N.: «I think very large things though still finite exist, all the rest does not exist».*

• 6 teachers seemed to grasp the idea of actual infinity. In particular, three teachers showed a very enthusiastic reaction to their discovery of the distinction between the two conceptions of infinity: potential and actual.

*A.: «I never though about this distinction, but now I got it, I can imagine it»*

*B.: «I never even thought about it, nobody gave them the possibility of reflecting on this topic, but to be honest I always thought that it was meant only in the sense of a continuous and constant process. But now I've understood the difference».*

The latter statement shows the embarrassment felt by the teachers who were not given any possibility of reflecting on such fundamental topics they should be able to master in order to prevent the creation of students' misconceptions.

The crucial point is that "no sensible magnitude is infinite" and therefore the comprehension of such topics seems to go against intuition and everyday experience (Gilbert and Rouche, 2001). With reference to this, various research works [Moreno and Waldegg (1991), Tsamir and Tirosh (1992), Shama and Movshovitz Hadar (1994), D'Amore (1996, 1997), Bagni (1998, 2001)] pointed out that when acquiring the concept of *actual infinity* epistemological obstacles, deriving from an initial intuition, have to be encountered (and the history of mathematics itself confirms that). As a matter of fact, as the first chapter of the present work is meant to demonstrate, during the 2200 years from Aristotle till present time, the treatise of the concept of infinity underwent a very slow and not homogeneous evolution process.

Up until the XVIII century, infinity was considered only in its potential sense, and potential is still the approach of those who are led by intuition and lack an appropriate reflection on this topic. Yet the conception of actual infinity is fundamental for the study of Analysis, even if teachers tend to convey to their students only the potential use, as if it were the only way to conceive this concept. But problems come later, when students attending higher secondary school have to face the actual aspect of infinity, which may at that point turn out to be extremely difficult to accept. This as a result of the learning - in the previous years - of an intuitive model of infinity so deeply rooted and only representing its potential aspect. This model is only based on students' and their teachers' intuitions, but very distant from the world of mathematics.

Tsamir (2000) states: *«Cantor's set theory and the concept of actual infinity are considered as opposite to intuition and can raise perplexities. Therefore they are not easy to be acquired and some special didactical sensitivity is necessary to teach them»*.

Unfortunately, when the above-mentioned concepts have not been properly investigated in higher secondary school, the corresponding image, mainly based on initial intuition, remains linked to potential infinity.

In other words, if primary school teachers (and not only them) have never been taught the topic in question, they are obviously bound to refer when teaching these concepts to their intuitions. The history of mathematics has vastly proved these intuitions to be opposite to theory. Consequently, Tsamir's didactical sensitivity would hardly be developed, which causes didactical obstacles strictly related to the inevitable epistemological obstacles.

Didactical ones worsen epistemological obstacles. Intuition plays a dominant role, but it is also confirmed by the knowledge taught at school. This could be the case of a self-sustaining chain: teachers base their teaching actions on their intuitions. These were in turn strengthened by their teachers who in turn had previously based their teaching on their intuition and so on. Therefore there is an urgent need for breaking this chain. This goal can be achieved by highlighting teachers' deficiencies and introducing an infinity-targeted didactical activity addressed to both teachers with experience and those without, so that high school students' cognitive disadvantages and obstacles - pointed out by several research works - can be avoided.

### 3.7.4 The need for "concreteness"

During interviews with teachers a commonly shared opinion emerged. According to this view, primary school children need concrete models in order to understand mathematical concepts. That justifies the didactical choice of using a necklace of beads as a model of segment; or the mark left by the pencil or the grain of sand as a model of mathematical point. Unfortunately, not everything can undergo the process of modelling without consequences. It is not rare to verify that in the didactical transposition the teaching choices, based on major reference to everyday world, negatively condition students' future learning. Tests carried out with primary school children and with teachers willing to change their way of teaching, showed that children enjoy and find it easy to enter a dimension so far from the sensible world. It has also been observed working in this way, that teachers themselves find it easier to deal with mathematical concepts not making any more reference to the concrete world, as mathematical entities are abstract by nature. Such results raise the question whether children or rather teachers are those who feel this need for concreteness. And in fact, it stands out clearly that teachers find it difficult not to make reference to real things, whereas children at times are delighted in leaving the sensible world and feel completely at ease with that.

Two significant statements made by teachers during the interviews give evidence of this. These examples mirror two opposite points of view. The first one was already quoted and analysed from a different perspective in paragraph 3.7.3:

*M.: «I think these kinds of things shouldn't be taught, at least not in primary school, poor children what can they do! Yes, of course you can also say that the straight line is formed of infinite points, but how are they supposed to understand that? (I don't believe it myself!), at their age they have to see things. They have to touch things with their own hands»*

*A.: «You have to imagine these concepts rather than find them, I believe; the only way of doing that in Primary School is to make them use their imagination, which is so rich: "A straight line is a line that goes as far as the most remote infinite space", and they start imagining it … I tell them that you can't measure or weigh a point. It* *exists, but you can't see it, it's like magic. So it works, because they enter a world which is not any more that of concreteness. They need to enter the world of imagination, in order to make it».*

(The latter teacher took an Analysis exam at university).

## 3.8 Answers to questions formulated in 3.4

We are finally able to provide answers to the research questions formulated in 3.4.

P.1 The answer is with no doubt negative. There is a total absence of knowledge of what is intended by mathematical infinity, both in the epistemological and cognitive meaning. This deficiency surely derives from the problematic aspect of the subject matter thoroughly featured by epistemological obstacles and the lack of a targeted formation on this topic. To primary school teachers, infinity is an unknown concept, solely managed by intuition and for this reason considered as a banal extension of finite. That causes the creation of intuitive models that turn out to be thorough misconceptions. Teachers accept namely the Euclidean notion: *«the whole is greater than its parts»* for the finite and tend to consider it also valid for infinity, which is a *dependence* misconception. Expressions such as "to be a proper subset" and "to have less elements" should not be confused when dealing with infinite sets. Nevertheless primary school teachers, during their educational training, have only found evidence of what happens when dealing with finite and accepted it as an absolute intuitive model and consequently transferred to their pupils. In other words, if an A set is a proper subset of a B set, then the cardinality of B is automatically greater than the cardinality of A. In building such a misconception, teachers' intuitive model of the segment seen as a necklace of beads also plays a role, thus leading to the phenomenon of *dependence* on magnitudes. The *flattening* misconception also participates in this mechanism, but in the didactical repercussion it brings about less affecting consequences than *dependence* to primary school pupils. Also the straight line seen as an unlimited figure and the prolonged counting of natural numbers seem to make teachers consider infinity only in power and not in act, which results in major didactical obstacles (Tsamir and Tirosh, 1992; Shama and Movshovitz Hadar, 1994; Bagni, 1998, 2001; Tsamir, 2000).

P.2 The answer is affirmative. Students' intuitive images concerning infinity are continuously strengthened by teachers' stimuli, who tend to transmit to their students their own intuitive models, which are – without their being aware of that – thorough misconceptions (see P.1). Such convictions persist in students' minds and become so strong that they create an obstacle difficult to be overcome when facing the concept of actual infinity in higher secondary school. Intuitive models such as the segment seen as a lace for instance make the conception and understanding of the idea of density impossible. The latter is already introduced in lower secondary schools or even before, in primary schools. For example, when the so-called fractional numbers are positioned on the "rational straight line" $r_Q$, the necklace model resists and the density is limited to its potential aspect. To many students, density seems to be sufficient to fill the straight line and therefore the difference between $r_Q$ and r results incomprehensible, even when the set R and the definition of continuity are introduced a couple of years later: the intuitive necklace model still dominates.

P.3 The present research has clearly shown that, besides epistemological obstacles already pointed out in the international literature, there are serious didactical obstacles deriving from teachers' wrong intuitive models, that are in their turn transferred to students. In order to avoid such obstacles a better training is needed, so that a purely and exclusively intuitive approach to infinity can be averted. It is therefore necessary to reconsider the teaching contents for teachers in training (working for any educational level). In so doing, it could be avoided that students in higher secondary school would have to face the study of analysis with an improper background of misconceptions. The treatment of problems concerning actual infinity requires the development of different intuitive models, if not even opposite to those regarding finite. We believe that a suitable education on the subject of infinite sets should start at primary school, in order for students to start handling the basic differences existing between finite and infinite field, both in a geometrical and a numerical context.

## 3.9 Chapter conclusions

Many research works on the topic of mathematical infinity have revealed that the obstacles impeding the comprehension of this subject are mainly of epistemological nature.

The present research has identified primary school teachers' beliefs on infinity, which are supported by erroneous mental images, firstly influencing their convictions and subsequently their teaching activity, too. Many of the teachers involved in this research study, after some explanations were provided to them, have firmly admitted - and in that they showed a great professionalism - that their teaching was rich in wrong models. Such models were confirmed year after year, but they might have been - according to teachers themselves - the source of future didactical obstacles. We want to thank these teachers for their honesty and professionalism.

We believe that the difficulties encountered in the understanding of the concept of mathematical infinity are not exclusively due to epistemological obstacles, but didactical obstacles resulting from the teachers' intuitive ideas magnify them also. It is also very likely that surveyed deficiencies on this specific topic are not a problem exclusively affecting primary schools, but are instead rather widespread at every school level, among all those teachers who have never been given the opportunity to properly reflect on mathematical infinity.

So far it seems as if such a topic has been very much underestimated, above all as a subject for teachers' training. This deficiency is the main cause for the problems encountered by high secondary school students already possessing previous and strong convictions, which are unsuitable to face new cognitive situations. Models provoking obstacles in the teachers' as well as students' minds are necessarily to be inhibited and overcome. As we have seen many a time in this chapter, primary school teachers targeted training courses are required. These courses should take into account the several intuitive aspects and peculiarities of infinity as well as the outcomes collected by the researchers of didactic mathematics. They should be mainly based on open and free discussion; the historical aspects of the subject should be outlined too. They should start from initial intuitive ideas in order to transform them into new and fully-fledged convictions.

All of the teachers involved in this research have clearly voiced this necessity. In this respect, we report two teachers' opinions:

*M.: «Yes, it's the kind of geometry that we do that gets us to see the points. We need someone to help us reflect on such things and on the importance of transposing them in a correct way. In the mathematics we learned, they did not make us think about these things. We need some basic theory»*

*A.: «Our problem is that we try to simplify things, without some previous theory. We are sure we've got it, but in fact we don't have it. We are concerned with transferring it in a tangible way, without deeply investigating how it works».*

Such a specific training will enable primary school teachers to properly master concepts regarding the infinite sets, getting their students involved in meaningful experiences and activities implementing the building of intuitive images which are pertinent to infinite sets theory.