Chapter 4. Present and future research

4.1 The first training course on this topic

Conclusions to the preceding chapter underlined the fundamental importance of specific training courses addressed to teachers on mathematical infinity in order to achieve an "advanced" awareness of the topic. To reach this goal, in the last two years a tailored training "trajectory" has been implemented. The course has been addressed to 37 primary school teachers and 8 lower secondary school teachers of Milan. This has turned out to be an occasion for us to reflect on new aspects pertaining to the debated subject. The selection of participants, teachers, was the result of the attendance, within the framework of a series of conferences held in 2001 in Castel San Pietro Terme (Bologna): "Meetings with mathematics no. 15", to a seminar for primary and lower secondary school teachers called: *"Infinities and infinitesimals in primary and lower secondary school"*. At the end of the seminar a large group of teachers, curious about the topic dealt with during the meeting and whose convictions perfectly mirrored those misconceptions described in chapter 3, openly showed the need for a better understanding of the discussed subject matter which they were never given the opportunity to reflect upon. They have been chosen because they have turned out to be highly motivated and have a serious interest in that, up to that moment, unknown subject. From the researchers point of view, this situation represented such an ideal fertile ground to start up not only a training "trajectory" but also a real action research that day-by-day is proving fruitful and rich in stimuli. Before starting in 2001 the above-mentioned training course, teachers were asked to fill in the same questionnaire described and commented in paragraph 3.6.2 adopting the same methodology as in 3.6, in order to assess if these teachers' convictions were to be considered similar to those already collected and classified in chapter 3. With no hesitation, we can therefore assert, that the latter results go in the same direction as the former ones. Furthermore, we observed no relevant differences between primary school teachers' convictions and lower secondary ones. This latter aspect showed how deeply rooted and difficult naive intuitions are to be eradicated, because of the complex nature of concepts characterised

by epistemological obstacles. Only a proper and targeted training activity could help modify such convictions. Out of 8 people teaching in the lower secondary, 2 have a degree in mathematics and 6 in scientific subjects. None has ever attended a course on this topic at university or even successively. As a consequence, there was actually no difference whatsoever related to the different kinds of degrees of the participants. But the most striking aspect was that, teachers with a mathematics degree and people who had only attended a high school specific for teachers (until recent times in Italy such a high school diploma was a sufficient prerequisite to teach in primary schools) shared the same awareness, or better to say lack of it of with respect to these specific topics. It is really surprising to note how the epistemological complex nature of this subject really undoes all other knowledge items, creating a levelling of convictions.
The collected outcomes are made available by the author but the reader can also refer to the reflections reported in chapter 3.

The only real dissimilarity was mainly based on the different linguistic expressions, which emerged during discussions between groups of 4 secondary school teachers. They basically used "definitions" derived from the adopted textbooks and consequently passed on to their students. Definitions that turned out to be most of the times improper, badly expressed and managed.
We provide an example taken from the answer given by a lower secondary school teacher to question no. 4 of the questionnaire described in 3.6.2 and asking if there are more points in a longer segment rather than in a shorter one:

S.: «Well… provided that ***a point is a dimensionless fundamental geometrical entity****, I would say that it is not possible to establish if there are more points in AB rather than in CD. By the way, it is also true that* ***a straight line is formed of infinite points****, but we are talking about segments. Well, the fact of being of different length must mean something so I'd say there are a greater number of points in CD. Yes, yes, there should be more in CD».*

The bold type has been used to underline those expressions commonly considered by teachers themselves as "definitions" and not only in the lower secondary but also in the higher secondary schools as we shall see in paragraph 4.3.

This answer provides a good example that the management of such factual "knowings" at times improper, misunderstood in their real meaning and not internalised does not result in differences from those collected with primary school teachers. *«Knowledge is not in books, it is the understanding of books. If you consider scientific results, it has to be admitted that normally the one who is able to enunciate them without being aware of them, does not know them (…). Knowledge is neither a substance or an object, it is an activity of the human intellect performed by subjects that try to substantiate what they do and say (by means of demonstration and reasoning)»* (Cornu and Vergnioux, 1992) [our translation]. The secondary school teacher F. though not sharing the same view as the above-mentioned colleague's of primary school and contained in 3.7.2: *« Even if you do very little points, you cannot fit in more than that»*, she ends up saying all the same that there are more points in a longer segment than in a shorter, attributing to the point a nature that cannot be a-dimensional if it depends on size, as she affirmed at the beginning repeating a notion learnt by heart.

It clearly stands out that teachers are not aware, especially when dealing with delicate topics such as those concerning geometrical primitive entities, that in most cases they think they know some concepts but they actually do not. These considerations have inspired a new research work, still in progress at the moment, based on geometrical primitive entities and strictly linked to the concept of mathematical infinity treated in paragraph 4.3.

Cognitive deficiencies deeply influence the *didactical transposition* (see paragraph 2.4) whose choices may be resulting in misconceptions or even wrong models. These conceptions and models are at the basis of didactical experiences badly managed by teachers and presented year after year in the same way. As observed in paragraph 3.8, to many teachers and consequently also to many students, density seems to be sufficient to fill the straight line and therefore the difference between r_Q and r results incomprehensible even when the set R and the definition of continuity are introduced. The distinction between density and continuity is however not favoured by the a-critical use of the entity straight line that starts with the introduction of N since primary schools causing several didactical problems (Gagatsis and Panaoura, 2000) and continues in the following educational levels (Arrigo and D'Amore, 2002).

Another problem, common to all educational levels, is represented by the "natural" model of the order of Z, that due to its prompt understanding and extremely conceptual and above all graphic simplicity, at the end turns out to be univocal and impossible to overcome even when the biunivocal correspondence between the set Z and the set N, requiring a different order of the elements of Z in comparison with the "natural" order, is introduced (Arrigo and D'Amore, 2002) (see 1.2.5).

A further hint is provided by a lower secondary school teacher's answer to question no. 7 of the mentioned questionnaire: *How many natural numbers are there: 0, 1, 2, 3…?*:

F.: «Natural numbers are infinite because ***a set is infinite when is formed of infinite elements*** *and 0, 1, 2, 3, … are infinite».*

The same teacher confronted with question no. 10: *Are there more even numbers or natural numbers?,* affirms:

F.: «There are more natural numbers than the even, it's logic they are the double».

It is important to restate that most of the times "definitions" provided by textbooks are improper. A good example is contained in a lower secondary schoolbook in the section of arithmetic with the title*:* ***finite sets, infinite sets***:

"The sets we referred to are formed of a well determined ***finite*** *number of elements",* (this suggests that an infinite set like the natural numbers one is not formed of a specific number of elements as it is actually: a denumerable infinity. These considerations inevitably imply that infinity is associated with the indeterminate).

And furthermore:

"In mathematics there exist sets of an ***infinite*** *number of elements"* (in the text the term infinite was underlined). Therefore, infinite sets are introduced as sets of an infinite number of elements. The latter statement is to be found in many lower secondary school textbooks and is perceived by teachers as a "definition" (F.: «*It's in the textbook*») whereas other books used in higher secondary schools refer to the same affirmation classifying it as among the "primitive ideas".

The previously cited paragraph on finite and infinite sets continues and ends in the following way: *"A kind of infinite set is that of whole numbers for instance: no matter which you consider to be a finite set of whole numbers, it is always possible to find another whole number different from those already taken into account"* (this idea embraces an exclusively potential vision of infinity recalling the Euclidean approach of the "Elements"). The potential vision can be found in many books as well as in one in particular adopted in Italian lower secondary schools. In this book, just to give an example, the chapter on numbers begins in this way: *"The ultimate number would never be reached even if 1 keeps being added on and on and on…"* and follows with: *"the set of natural numbers is* ***infinite****"* (idea that could result in those misconceptions described in 3.7.1 such as: infinity as indefinite or infinity as a large finite number). As a matter of fact, the exclusively potential treatment of infinity will be passed on to teachers who in turn will pass it on to their students making them all think that infinity cannot be conceived as an object by itself, something definite and possible to grasp, "reach" and dominate.

Furthermore, some textbooks start with finite sets to successively define the infinite ones, others use the opposite method successively defining a *"finite set as a set non-infinite"*. Our objection is not addressed to the definition by negation, since in this case we subscribe to Bolzano's thought (1781-1848), reported in the bibliographical reference of 1985 and based on the consideration that if the so-called "positive concepts" exist, there should be no impediment for the existence of "negative" ones and for these latter concepts a definition in the negative is possible. As a matter of fact, the definition of an infinite set has in general a positive character, whereas the negative is attributed to finite sets, although philosophical texts usually attribute to the term "finite" the "positive" concept and to "infinity" (meaning non-finite) the "negative". [For a better understanding of the difficulties of defining the concept of "finite" see Marchini (1992)].

The crucial problem is to choose what definition of infinite set to adopt and to avoid the vicious circles that are triggered as a consequence of an initial definition for the concept not properly representing the concept itself. It has already been stated that false definitions magnify both teachers' and students' misconceptions.

To clarify the goals of a textbook may be of some help towards a better understanding of the issue at stake. As a matter of fact, a textbook is nothing but the result of a didactical transposition chosen by the authors and is therefore not to be interpreted by teachers as a book of mathematics where one can learn concepts. Knowledge should be already possessed and mastered by teachers when adopting textbooks. All the pieces of knowledge should only be refreshed and reinterpreted in the specific case of the didactical transposition decided by the text author and therefore and only successively personally adjusted to the specific case of class-context. With regards to mathematical infinity, very often teachers themselves do not seem to be confident at all with this knowledge and so they only attribute to the didactical transposition contained in the text the function of mathematical contents transmitters. Evidence of this is given by the frequent attempts to justify their answers concerning mathematical concepts with affirmations such as: *«It's written in the book we use»*. But when a concept or domain of knowledge is inserted into a textbook, it undergoes a massive transformation, i.e. its nature is changed in order to respond to another statute, another logic, and another rationality, influenced by school pedagogy requirements imposing a different form.

Returning to the "definition" of infinite set expressed by the concept that a set is infinite if it is formed of infinite elements, it clearly stands out that the latter cannot be considered a reliable definition, thus impeding the understanding that two infinite sets, as the natural and even numbers for instance are formed of the same number of elements. On the contrary, a definition that in the beginning may be somehow twisted and complex, but is in fact appropriate for defining infinite sets, is the definition called Galileo-Dedekind's (see paragraph 1.2.2): *"A set is infinite when it can be put in biunivocal correspondence with one of its proper parts"*. This implies, in fact, that the sets of natural numbers and even numbers may be formed of the same number of elements provided that the correct biunivocal correspondence is established between them (see 3.6.2).

The treatment of these subjects in textbooks is clearly problematic. Authors tend to diffuse rather delicate subject matters without the necessary critical caution that people who produce material destined for didactical use should have. Unfortunately readers, both students and teachers, do not display a sufficiently critical approach towards what is published. As we observed, they tend to accept everything they find in any textbook as trustworthy. Our future intention is also to analyse textbooks, in particular those generally chosen by teachers, to test them on the topic in question and to evaluate inaccuracy and defects, to find out, with suitable methodologies, the extent of teachers' reliance on this didactical instrument.

In 2001, after the questionnaire had been proposed, a training course was created on this topic. Initially 45 teachers, that have the same misconceptions, began to attend the course, recognising that, as was well expressed by the words of a lower secondary school teacher: *«As far as this is concerned, we are all in the same boat!»*. The course was organised in different meetings and is still running today involving a more limited number of participants. The course was initially conceived as based on the history of mathematics concerning this topic (see chapter 1).

This was due to the awareness that some convictions, influenced by strong epistemological obstacles, had to be eradicated. We were well aware of that, as we had encountered two of the features highlighted by the research body of Bordeaux and referred to by D'Amore (1999) as useful to spot epistemological obstacles:

• in the historical analysis of an idea, a fracture, a sharp gap, non-continuity in the historical-critical evolution of the idea must be traced; (the history of infinity is a good example of that).

• a mistake must recur over and over again, always in similar terms; (the same mistakes, coinciding also with the historical fractures, were traceable in the convictions of the involved teachers).

On the basis of this awareness, we thought it fundamental to build a strict connection between the history of mathematics and the didactical aspects during the course. We tried to join the two subjects through discussion and confrontation starting from teachers' primary intuitive ideas to develop them into new, more advanced convictions. This strategy proved fundamental to enable teachers to make a critical reading of their ideas, as they recognised them in some statements of the mathematicians of the past. This confrontation facilitated the eradication of the misconceptions that had emerged from the initial questionnaire. Moreover, the description of these historical fractures and discontinuity, highlighted some erroneous situations the mathematicians found themselves in and let some of the teachers understand the meaning of mistake in mathematics. (D'Amore and Speranza, 1989, 1992, 1995). The study of history turned out to be a sort of essential keystone for teachers' critical self-analysis. In particular, three primary school teachers, over recent past years, have spontaneously decided to keep track of their ongoing progression, writing down step by step the evolution of their convictions. It is our intention to publish the outcomes of the training course soon, seen through the eyes of one or more primary school teachers (in paragraph 4.4.3 you will read some extracts of self-evaluation made by two of the teachers who attended the course).

The effectiveness of the course can be seen in some of the following sentences by primary school teachers: «*I've learned more during these lessons, than I've ever learned in a whole life of mathematics teaching and refresher courses. I have the feeling I understand now for the first time what mathematics is, and think! I've been teaching mathematics for 27 years. This discovery has really upset my life*», «*I've understood what actual infinity is, and I've accepted it easily, since I had the courage to conceive it as a self-standing object, as a whole. Now I feel stronger*»; «*I've realised I'm much more attentive to my pupils and much more open to discussion. Above all, I try to work on their intuitions, as you did with us. They are happy with that*». Lower secondary school teachers said: «*You illuminated me! At last I've understood what I've kept on repeating and teaching without having seriously thought about it. I'm so thankful to you*»; «*I can't teach as I used to any more. I'm not satisfied with the way I taught before… I can't go back any more*»; «*What surprised me most, was to find out that I didn't even know what I taught. Do you know what in particular? I lost sleep over the discovery that* $3.\overline{9}$ *really equals 4 and there is absolutely nothing missing. It doesn't approximate it; it really equals it. I've always introduced the "rule of the recurring numbers". But I've never applied it to specific cases. To tell the truth I skipped them on purpose, in order to avoid confusion, I had therefore never noticed that myself*». Out of 8 lower secondary school teachers, just one, with a degree in mathematics knew that $3.\overline{9} = 4$, although at the beginning she honestly admitted that in her opinion this represented an exaggeration she accepted as a fact, whereas all the others initially argued something like: «*Over there* (indicating a point at "infinity"), *there must be something missing!*». Later on, when little by little they managed to accept the view of actual infinity, they succeeded also in conceiving and accepting this new discovery. Checking the results of the evolution in teachers' conceptions has been a slow, suffered but constant process. Signs of their progress have emerged and still emerge from the training course discussions. This experience has turned out to be really rich and significant from a scientific point of view and it has led to: the discovery that in training courses targeting schoolteachers there is a tendency on our part to take some fundamental pieces of knowledge for granted, that in fact are badly interpreted by our interlocutors; the discovery that teachers are sometimes totally unfamiliar with some subjects, and that this can cause fractions and incoherence in teaching and the basis for the creation of didactical obstacles; the discovery of the vital importance in this matter of addressing research to teachers convictions first, to focus subsequently on those of the pupils; the discovery that from a didactical point of view, there is a whole new world around infinity opening up that is still to discover.

We are still in touch with all 45 teachers, but we are cooperating in particular with a working-group of 5 primary school teachers with whom we have chosen to operate on a deeper level, from different points of view.

With these teachers we at last moved from teachers to students, that is to say, we went back to where we had started with our observations, back in 1996. As we showed in chapter 3, that year we investigated the convictions of primary school children,[41] who could not help reporting on this subject the knowledge they had learned from their teachers. [In this respect, on the connection between students' and teachers' convictions see the famous example of El Bouazzaoni (1988), dealing with the notion of continuity of one function]. Working with children we realised how big the potentialities of dealing with primary school children are, not only when referring to the "concrete" world, but also when having the courage of letting pupils explore the world of the "extra physical", as that of infinity for example, growing away from the physical world.

We will not go further into this matter in the present work, as we are focusing on teachers' convictions rather than on students'. However, we will just make some brief reference to the outcome of our previous research.

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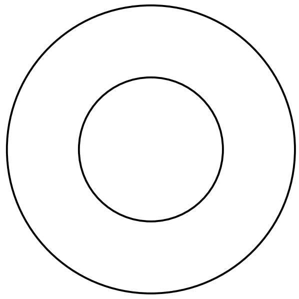
[41] As for specific research works on the subject of infinity for primary school children, please refer to following bibliography: Bartolini Bussi (1987, 1989), specific for primary and even nursery schools; Gimenez (1990) focussing on the difficulty of the density concept for primary school children; Tall (2001b) dealing with the evolution of the concept of infinity, from nursery school onwards, reporting the case study of a child called Nic.

4.2 Brief description of the research carried out with primary school children in 1996

This inquiry has been carried out with two classes of 10-year-old children in Forlì (Emilia Romagna). A total number of 38 children were asked to come out of the classroom in pairs and to work with the researcher. This the explicit agreement arranged with the children from the very beginning: all they would say outside the classroom would neither be evaluated nor told to their teachers. This in order to avoid that the stipulation of the didactical contract, depending on a classroom situation, could influence the experimental contract (Schubauer Leoni, 1988, 1989; Schubauer Leoni and Ntamakiliro, 1994) that students were establishing with the researcher. Our choice to work with pairs instead of with individuals was meant both to encourage children to undergo our inquiries and to trigger discussions, that could enable us to examine in depth the real convictions of interviewed children. The methodology we adopted consisted of letting two children enter a classroom and having them sit down at a table next to each other, in front of the researcher, who had a tape recorder without the children knowing. None of the classes that underwent our research had previously been proposed by their teacher specific activities on infinity.

The researcher started by handing out a sheet of paper where following two segments of different length were drawn and by asking: «*What do you think they represent?*»

After we got the answer and we clarified they were two segments, we carried on with the second question: *«Do you believe there are more points in this segment or in this other segment?* (pointing at the two segments)»; (this question is practically the same as no. 4 of the questionnaire in paragraph 3.6.2). After the children had given their answer and had discussed on it, they were presented Cantor's demonstration, showing that there is the same number of points in two segments of different length (see paragraph 3.6.2). Afterwards, the researcher handed out a second sheet where two concentric circumferences of different length were drawn and asked: *«Are there more points in this circumference or in this other one?* (pointing at the two circumferences)».



From time to time, during the discussions the researcher could make some more questions or remarks, with the aim of stimulating confrontation, but being careful not to influence the opinion of the interviewed children.

The children were subsequently asked: *«What is mathematical infinity in your opinion?»* and they were let free to discuss until the confrontation would stop.

Let us briefly report some of the results we gathered. In bold type there are the researcher's interventions during the discussion, meant to stimulate conversation and to inquire in depth into the children's convictions.

• Many children when asked to answer the first question: «*What do you think they represent?»* (showing them the two segments of different length) did not reply: *«Two segments»,* but often generically said: *«Lines»* or *«Straight lines»,* others noticed things based on knowings that were not included in our area of interest as:

S.: «They are bases, we revised them on Friday»

R.: «Bases of what?»

S.: «Of a rhombus, or better said, of a rhomboid»

R.: «Can you draw it?»

S. The child drew a trapezium.

Another child answered as follows:

F.: «They are parallel lines. Parallel lines aren't like concurrent lines that meet only in one point. They are infinite (in the sense that they do not meet)»; (here we can trace the use of "mathematese").[42]

[42] This word was minted by D'Amore (1993a) and refers to a sort of "mathematical dialect" used in classroom. A special language that the student considers correct, right, and appropriate to use in maths classes to fulfil "contractual" duties. Oppressed by the "burden" of this new language, the student often gives up the sense of the question or of her/his discourse.

• After having explained to the interviewees that they should concentrate their attention on the two segments, they were asked: «*Do you believe there are more points in this segment or in the other one?*».

Most of those interviewed answered something like: in the longer segment. Just one child claimed that there were more points in the shorter one G.: «*You just need to stretch it out and make it longer than the other one*». Whereas 16 children replied: «*Equal*»; 2 of them affirmed it without supporting their opinion with any reason and without saying that there are infinite points in both segments. Whereas as many as 14 children, all belonging to the same class, claimed that there is the same number of points in two segments of different length and more precisely, two points in both of them that mark the two end points of the segments. That highlights how teachers' didactical attention and consequently children's didactical attention often focuses on small details, conventions and non-significant formalisms, when dealing with the description of a concept. [On teachers' view on mathematics please refer to: D'Amore, 1987; Speranza, 1992; Furinghetti, 2002].

Let us now look at the extract of a conversation between two children, one of whom had expressed the above-mentioned interpretation.

R.: «Do you believe there are more points in this segment or in this other segment?»

M.: «In this one (pointing at the longer one)»

I.: «No, they are equal. There are two points»

R.: «What do you mean?»

I.: «In both of them there are the two end points that delimitate the segment. The teacher told us that»

R.: «So, how many points are there in a segment in your opinion?»

I.: «The same, they are always two».

Then there was the significant case of a child that, though he answered correctly to the first question: «*They are two segments. The teacher told us to write down that a segment is a set of infinite geometrical points*», he then claimed that there are more points in the longer segment, thus showing that he had failed to grasp the meaning of his previous statement. This reveals that one must be very careful when proposing definitions and above all when considering as satisfying the answers of a student, just because they coincide with the given or expected definition: repeating a definition does not necessarily mean understanding its meaning.

• Many misconceptions on geometric primitive entities emerge from children's conversations; these are false beliefs that negatively affect the subsequent learning process and that we are going to analyse in more detail in the next paragraph. We believe that this outcome underlines the importance of not leaving concepts to the sphere of mere intuition, and shows the importance to work on these pieces of knowledge, thinking of specific and structured activities, as those we suggested in paragraphs: 4.4.3 and 4.5.3.

• After having shown Cantor's biunivocal correspondence between the two segments of different length (see paragraph 3.6.2), most of the children immediately intuitively understood that both segments were formed of the same number of points, whereas in other cases, the discovery process was somehow slower, but it nonetheless occurred:

G.: «Here you are a triangle. Do you want to know the perimeter?» (G. had recognised a triangle in the figure after the researcher had drawn two semi-straight lines originating from the point of projection O and intersecting the two segments; the child was trying to use the knowledge acquired in class, and undervalued the question that was actually asked).

R.: «No, not the perimeter»

G.: «So the points are two in both of them»

R.: «Look!» (the researcher showed two more corresponding points of the biunivocal correspondence)

G.: «Then there are 3»

R.: «But there are also these two!» (the researcher showed two more points)

G.: «Then there are 4»

R.: «But there are also these ones» (the researcher showed two more points)

G.: «Ah, now I know: there are infinite»

R.: «Are there more points in this segment or in this one?»

G.: «The same»

R.: «Are you sure?»

Both of them: «Yes, yes»

Thanks to the demonstration some of the children understood that the two segments were formed of the same number of points, but they could not say the exact number, so they answered *F.: «They are the same. There are many points, but I don't know exactly how many».*

Except for 4 children, all pupils said they were persuaded by the truth of the new discovery, that is to say that two segments of different length are formed of the same number of points. Some pupils affirmed that in both segments there are infinite points, although when asked to explain what infinity is, they did not seem to be familiar with the advanced idea of the concept. It is interesting to notice that the explanation of a biunivocal correspondence did not surprise children, whereas it did surprise teachers, when some years later the same construction was proposed to them.
Only 4 children were perplexed and affirmed they were not convinced by the demonstration, this was mainly due to their strong misconceptions on the mathematical point:

A.: «I think it always depends. If you make smaller points here and larger points there, you never know how many they are» (from this reply we came to the conclusion that is important to focus on children's idea of mathematical point. Read more on this subject in paragraph 4.4).

• When the researcher showed the two concentric circumferences, almost every child immediately concluded that the number of points forming them was the same. Most of the children succeeded in building the biunivocal correspondence autonomously, starting from the central point, thus transferring a piece of knowledge they had learned before.

R.: «Now look at this (the researcher showed the sheet with two concentric circumferences of different length)»

M.: «This is a circular crown, we haven't done it» [manifesting a clause of the didactical contract (see paragraph 2.1) like: *«Only questions on subjects handled in class are allowed»*]

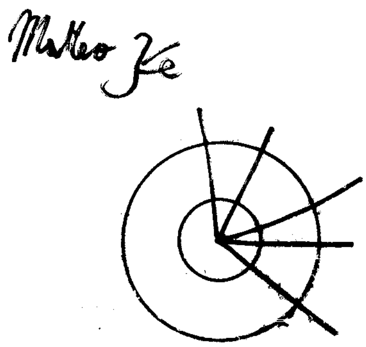
R.: « Are there more points in this circumference or in this other one?»

M.: «There is the same number of points, as before, it doesn't matter if one is small and the other is large»

S.: «I agree»

R.: «Why do you think it is so?»

M.: «It's like before, it goes like: you… you… you» (the child drew the biunivocal correspondence between the two concentric circumferences, starting from the centre)



R.: «What are you trying to show me?»

M.: «This little point corresponds to this little point, this… to this. Therefore they are perfectly the same. If one understands that, one understands this too»

S.: «You just need to understand one and you have understood them all»

R.: «Which one do you like most?»

M.: «The wheel, because I invented it» (this reply highlights how personally acquired pieces of knowledge are much more motivating and meaningful to a learner than any other proposed directly by the teacher).

In some cases the search for a demonstration has proved more difficult:

R.: « Are there more points in this circumference or in this other one?»

F.: «For me in this larger one»

M.: «No, in both of them (with a finger the child points at the biunivocal correspondence, but then covers it with the hand). *I just want to see one thing. I'll try»*

F.: «Yes, but if you make the little points smaller here and larger there»

M.: «It's different here, because it's a circle, it's closed»

F.: «You just need to make them more tightened here»

M.: «I just wanted to see how I had done the thing before. A thing like that (showing the biunivocal correspondence between the two segments). *If we do the same thing now.*

I wanted to see if we can do that, but here there must be something, because even if you try… I think they are the same and that's it»

R.: «You can draw, if you want»

M.: «This time it's different, because the circle is closed. I wanted to see how I had done the thing before like that, if we do the same thing now, I wanted to see if we can do that. But here there must be something, because even if you try. In that one there was a smaller one and a larger one while here there is a smaller one and a larger one, maybe it's more difficult to have the same number of points, in my opinion» (M. was considering an external point to both circumferences, not being able to find the biunivocal correspondence, he changed his mind and withdrew what he was saying at the beginning).

R.: «Why did you start from this point? Can't you think of another point from where it is more convenient to start?»

L.: «And if we draw a straight line?»

M.: «Ah, wait. We just need to do it in the middle, don't we?»

L.: «What are you doing, making a cross in the middle?»

M. built the biunivocal correspondence, discovering that there is the same number of points in two circumferences of different length.

Only 4 children, the same who claimed that there are more points in the longer segment, kept on arguing that the longer circumference is formed of a larger number of points. The reason for their choice derived from an erroneous idea of the mathematical point.

• 8 children spontaneously transferred the knowledge they had learnt in other contexts:

M.: «It's also like that and from the point you have always to go through both of them, so they have the same number of points (the child draws two concentric squares of different perimeters and builds the biunivocal correspondence)»

R.: «So in your opinion, there are as many points in a little square as in a big square»

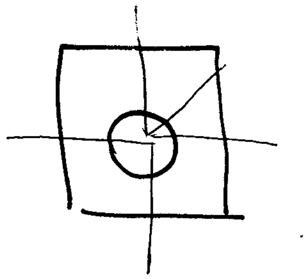
M.: «Yes, didn't you know that? No, because it seems like you don't know it»

R.: «It's that I didn't expect it, I couldn't believe you could have such an intuition»

M.: «I got so many intuitions, that's why I invented the wheel before» (meaning the demonstration related to the two concentric circumferences)

R.: «Now I ask you: are there more points in a " little circle" or in a "big square"?»

M. draws a "little circle" and a "big square" one inside the other, then he builds the biunivocal correspondence and answers:



M.: «These are the same too, because they have always to pass through here, here and here. It's easy though!»

• Only two children, particularly involved and open to discussion, have been further asked:

R.: «According to you, are there more points in this segment or in a straight line?» (on a sheet of paper a segment and a straight line parallel to one another have been drawn)

D.: «Now it's different! I don't know. Let's try»

F.: «You'd better use the ruler. But it's impossible to do it as before because there are some empty spaces here» (the child indicates that the segment is limited at both extremities)

D.: «Help, how can we do it?»

F.: «To me they are the same»

D.: «It's the same thing, you just have to join the points»

F.: «But the straight line never ends, there are more in the straight line»

R.: «How many points are there in a straight line?»

D.: «So many»

F.: «Infinite»

R.: «And in the segment?»

D.: «A lot but I think in the straight line there are always more, because the straight line continues to infinity whereas the segment stops» (here it is particularly evident the misconception of infinity as unlimited, see paragraph 3.7.1)

R.: «What if I told you that there are in both infinite?»

F.: «I wouldn't believe it. Here there are not infinite (indicating the segment)*, they will end sooner or later»*

R.: «But you told me before that in the segment there are infinite points»

F.: «Just to say so many»

R.: «So many? How many?»

F.: «Hey, do I have to count them?»

D.: «You think you can count them, but it's not because the point can be also extremely small, like that ·» (everything can be traced back to the misconception of point).

The researcher shows Cantor's biunivocal correspondence (see paragraph 1.2.3).

D.: «Then there are infinite, here and here»

F.: «So each straight line has the same points of the segment, because both of them have the same number of points»

D.: «Therefore for each line there is the same number of points because they are both infinite. So you cannot count them»

F.: «It's enough to say that the number of points is always the same even without looking at the length, one can be like that and the other like that (indicating different distances) *and you say infinite»*.

It ought to be observed that the word infinity is very frequently used, but from the investigation of children's convictions with respect to such topics emerged the same misconceptions showed by primary school teachers in paragraph 3.7.1.

Some of the answers collected are provided here as follows:

G.: «Something that never ends, the teacher told me. It's like a track with a beginning and an end but that you can go along it as many times as you want»

S.: «They are the lines, the straight lines, the curves, the polygonal lines»

F.: «So many as the points we talked about before»

M.: «To me they are the normal numbers 1, 2, 3, 4… that never end. Our teacher always tells us so»

I.: «It's a sphere getting bigger and bigger» (the potential infinity idea)

S.: «The darkness and the points of before»

A.: «Something that has a beginning, but the teacher says that it can't have an end»

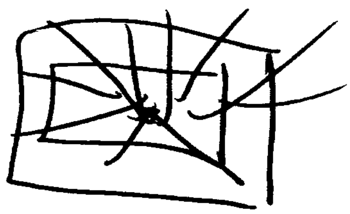
R.: «For example?»

A.: «Numbers, they don't have an end»

R.: «Is there not a last number?»

A.: «Infinity, it's the longest».

A significant case. During one of the first experiences carried out in order to test children's reactions before starting present research, we met Marco, a ten-year-old boy attending a different class from the others involved in the research. After he had been showed the biunivocal correspondence between two segments of different lengths, Marco, spontaneously, without having even seen the two concentric circumferences of different length, made the following drawing and affirmed by that:
«So, also this works»



Marco is of course an isolated case that therefore cannot be considered a prototype of what normally happens, but he deserves a mention as he inspired us researchers with trust and enthusiasm to embark on this project. We never came across any more "Marcos", especially since we shifted the focus of our attention from children's to teachers' convictions, in the latter case resulting much more difficult to try to break the erroneous models previously formed. Children showed to be extremely open-minded, flexible, willing to cooperate and to learn. Unfortunately all these attitudes were often negatively influenced by the kind of teaching received. It was not rarely observed that children, after having uttered sentences revealing misconceptions, also added: *«My teacher told me that»* and from the successive interviews conducted with primary school teachers we had evidence of this. That was the main reason why we have devoted our research to teachers whose convictions turned out to be more rigid and stereotyped.

4.3 Primitive entities of geometry

One of the research aspect we are currently investigating concerns students' and teachers' convictions not only regarding infinity but also geometrical primitive entities. This need originates from the frequent observation, when dealing with infinity, of the presence of misconceptions related to the point, the straight line, …

To achieve this goal, our decision was to use the methodology of TEPs: *«By TEPs we mean literally: students' autonomous text productions»*[43] (D'Amore and Maier, 2002). TEPs are therefore about written texts autonomously produced by students and regarding mathematical topics. TEPs are not to be confused with non-autonomous written productions (class tests, notes, procedure descriptions, …) since these productions are bound to certain constraints more or less explicitly given like such as direct or indirect assessments. In short, TEPs have to be considered as those productions that induce students to express themselves in a comprehensible way and use a personal language, accepting in this way to set themselves free from linguistic constraints and to employ spontaneous expressions instead.

In the article written by D'Amore and Maier (2002) some of the TEPs effects are listed and the following are the most interesting:

• TEPs production stimulates students to analyse and reflect on mathematical concepts, relationships, operations and procedures, researches and problem solving processes, which they get in contact with. In this way, every student can reach a better awareness as well as a deeper mathematical understanding of these concepts;

• TEPs give students the opportunity to constantly monitor their comprehension about mathematical topics by means of a fundamental and reasoned feedback with their teacher and classmates (self-evaluation);

• TEPs allow teachers to evaluate students' real personal knowledge and their understanding of mathematical ideas in a more detailed and accurate way than it would be possible with the analysis of common written texts, solely performed as not-commented problem solving activities.

TEPs production should provide the student's profound vision, detailed and explicit of her/his way of thinking and understanding mathematics, it is therefore necessary that the

[43] The German original term comes from Selter (1994).

student addresses her/his TEPs to whom needs all the pieces of information related to the subject of the writing. The addresser should be obviously someone but not her/his teacher.
The TEPs collected within the scope of this research are students' production starting from the nursery school (3 years old) and up to the higher secondary (19 years old). The idea was to start from nursery school in order to investigate if children of 3-5 years of age already possessed some primitive naive ideas related to these concepts. Additionally, our aim was to monitor the evolution of these ideas over the course of time, to this end nearly 350 TEPs have been produced and distributed to students of different educational levels. On the basis of the survey of these TEPs, our intention was to pursue the research object, more interesting in our opinion, of investigating teachers' real convictions concerning these mathematical objects, as direct consequence of the interpretation of the written performances of their students. In other words, after we handed to teachers the TEPs written by their students, transcribed on PC so that teachers could not be able to recognise students' handwriting, teachers were asked to read them, provide an interpretation of them and analyse them in detail on their own. Starting from the analysis carried out by teachers, the researchers' investigation could begin with the aim to assess teachers' convictions on the mathematical objects we proposed. The research had been performed by means of interviews, since we were afraid of exclusively taking into consideration the written answers that might be subjected to those factors already pointed out in international literature and namely: time pressure in finishing the assigned task, superficial answers, fear of being judged, etc. The joint use of the TEP and the interview, instead, especially if performed with all the necessary calmness and with no time pressure, has the advantage of making the subject feel at ease and therefore favour the investigation of the real, deep and hidden competences of the subject in question.
At the same time, our aim was to evaluate, in general, how teachers analyse and interpret a TEP, what their point of view is and finally their skills and ability in interpreting them.
Before handing out the TEPs, students were explained that no evaluation on the part of teachers had been envisaged for that work, and only after that clarification pupils,

starting from those attending primary schools, were asked to provide written answers to the following questions:

• *Imagine you have to explain to one of your classmates what mathematical infinity is. What would you say?*

• *How would you explain it to a classmate of… years old?* (primary school pupils were asked to consider children two years older than them, whereas secondary school students had to think of younger students)

• *Imagine you have to explain to one of your classmates what a point in mathematics is. What would you say?*

• *Now explain it to a child of ... years old*

• *Explain to one of your classmates what a straight line in mathematics is*

• *Now explain it to a child of ... years old*

• *Finally, imagine you have to explain to one of your classmates what a line in mathematics is. What would you say?*

• *Now explain it to a child of ... years old*

As for nursery school the decision was to pose the following explicit questions: What is mathematical infinity for you? What is a point in mathematics for you? What is a straight line in mathematics for you? What is a line in mathematics for you? In addition, for each question, children were asked to draw if they felt like it.

Here as follows the outcomes of this research will be provided followed by only some general remarks, as the above-mentioned results have been not yet analysed in detail. It is however important to underline that the original texts handed in by children contain some grammatical mistakes that might not appear in the translation. These works are made available by the author to whom is interested in consulting them.

♦ The results collected in nursery school showed that 4-5 year old children already possess some first intuitive ideas related to these concepts, these are convictions that can serve as the basis for future misconceptions. For instance, the majority of children tend to associate the mathematical point with the graphic sign of a pen and answer with sentences such as:

«They're little spots» (Loris, 4 years old)

«They're some small and big dots» (Andrea, 5 years old)

Here are some of the answers concerning mathematical infinity:

«It's infinite line. That never ends. Universe is infinite. Numbers go to infinity» (Federico, 5 years old)

«It's when one never stops doing maths» (Riccardo, 5 years old)

Answers concerning the straight line:

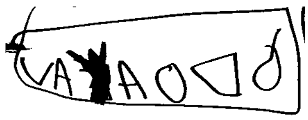
«It's a line that is straight» (Marco, 5 years old)

«When I'm hungry and I ask my grandpas and they don't give me anything, I have to wait till the cooking is ready»[44] (Riccardo, 5 years old)

Answers concerning the line:

«It's a line dividing the numbers» (Anna, 4 years old)

«The mathematical line is a meter» (Riccardo, 5 years old)



♦ It has been noted that starting from the lower secondary school, the texts produced by the students did not really assume the form of real TEPs, even if the motivation was to explain some concepts to one of their classmates. As a matter of fact, students tend to answer in a direct and concise way, adopting most of the time some supposed definitions, even when they have to address their explanations to pupils younger than them. Only in some specific cases, students decided to use the drawing for younger pupils, as a privileged form for making themselves understood and in so doing revealing severe misconceptions.

This phenomenon may be depending on the kind of topics dealt with, so specific and targeted or on the motivation chosen. In order to discover this, our future aim is to try to change the students' motivational aspect using a different strategy that has often proved itself extremely involving: "Pretend to be a teacher, a mother, a child of … years old

[44] Translator's note: in Italian the term straight line is "retta". This word is also used in an idiomatic expression and namely "dare retta" which means to obey to someone.

…" (D'Amore and Sandri, 1996; D'Amore and Giovannoni, 1997) to verify if the students' approach and consequently also their related written productions change.

♦ It has been revealed strong misconceptions belonging both to students and teachers concerning these mathematical objects and deriving from the visual images and the use of these terms in other contexts different from the mathematical one (for a better treatment of this aspect see paragraph 4.4). Researchers were considerably surprised by a particular aspect and namely the fact that formulating the questions as belonging to the mathematical field rather than the more specific one of geometry, has turned out to be misleading for students as well as for some teachers. Let us try to shed light on this latter aspect. In Italy, in primary school there is the most widespread attitude of creating at least two different subjects of study within the field of mathematics: geometry and the so-called "mathematics", meaning arithmetic. When activities are introduced in primary school one of the children's most common attitudes is to ask: *«Are we going to do mathematics or geometry?»*, *«Do we have to take the exercise book for mathematics or for geometry?»*. There exist, consequently for both children and teachers two separate worlds and according to the world chosen, there are different behaviours and attitudes to adopt: you are ready to do calculation if the field is that of "mathematics" or you expect to make a drawing if the subject is that of geometry.
Therefore to the question:
Imagine you have to explain to one of your classmates what a point in mathematics is. What would you say?
Children provided answers such as:
«To me the point in mathematics is an important thing. But it can mean three things to me:
a) The point in a large number like 143.965.270.890 in such a number the points are useful to be able to read the number;[45]
b) Someone, instead of × uses the point for example 144 · 5 = 620 in this multiplication the point is used as abbreviation;
c) Somebody else uses instead the point as a comma, for example194,6 or 194.6

[45] Translator's note: in the Italian language the comma is used to separate decimals instead of the decimal point, whereas the dot is used to separate large numbers with more than three figures.

To me the most useful of all is the first case» (10 years old)

The majority of children attending the last years of primary school make no reference to the point in its geometrical sense, as they believe it as not a part of mathematics, but they rather look for the use of point in the field of arithmetic. In the nursery school instead, as well as in the first years of primary school, no distinction has yet taken place between "mathematics" and geometry as a consequence of the teaching received and the choice is mainly for the geometrical field.

Teacher's comment: *«This* (referring to the child who wrote the above quoted TEP) *has correctly identified the point in mathematics. If he were asked in geometry then it would have been another matter, but as for mathematics he is right: the point is this one».*

Here as follows there is an attempt made by a child to join the two fields: geometrical and "mathematical":

«The point in mathematics is a very, very little spot that can become a very high number» (10 years old).

Among the few children of the last years of primary school that opt for the geometrical field, the largely discussed and pointed out misconceptions are to be traced:

«I would say that the point is a small element, round, the beginning and the end of a straight line» (10 years old).

Teacher' comment: *«If he is referring to the point in geometry, then what he says is ok, he explained it in detail, but the question was about the point in mathematics».* The teacher demonstrates misconceptions related to the point and distorted ideas of mathematics.

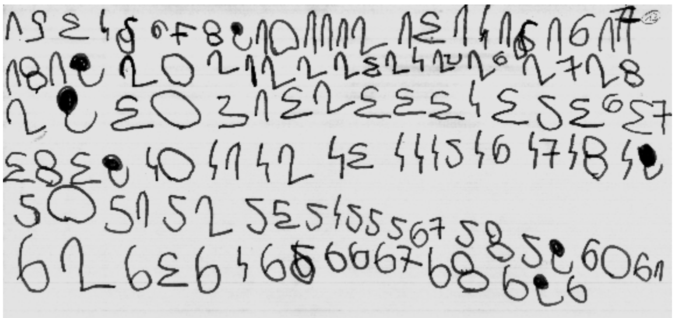
So being the straight line included in the arithmetic field, it becomes: *«The line of numbers»* and the line becomes: *«A symbol used in operations or in fractions».*

We believe these considerations are crucial, from a didactical point of view, to highlight the importance of the context that will be dealt with later in 4.4.

♦ Student produced TEPs do not show any evolution in the course of years concerning what is intended by mathematical infinity: the misconceptions underlined are always the same as the teachers' ones on which paragraph 3.7.1 is about: accepted, shared and confirmed by teachers themselves. Here some of most significant examples are provided:

«They are the angels that live for ever» (6 years old)

«I thought about numbers» (6 years old)



«Difficult works like doing 60 sheets of exercises in one day» (7 years old)

«To me mathematical infinity is an infinity of numbers and problems to solve. I'm not very good at it and so to me it never ends» (8 years old)

«To me there is no infinity in mathematics, because numbers in mathematics do not start and they never end» (9 years old)

«In my opinion mathematical infinity is like space, it never ends, numbers cannot end, combinations of numbers cannot end. But I think that the characteristics of mathematical infinity are not only numbers, they can be also shapes, and we know some of the many geometrical shapes. Infinity in mathematics is difficult to explain because mathematics is everywhere, even only to calculate the depth of a picture you need mathematics, to see how large a classroom is, you need to calculate the perimeter or the area.

There is one thing I've always asked myself: who's got evidence that mathematics is infinite? I know well that it is infinite but is there any evidence?» (10 years old)
«It's a thing that goes on forever, it gets so far» (11 years old)

«I'm sorry but nobody has ever taught me what infinity is, I think it's something whose well defined quantity is not known» (12 years old)

«Infinity is something which has no end, e.g. numbers, after the last number you think there is always another one and you can get to count with no end (that is to say to infinity)» (13 years old)

«I would say it's nothing but it's everything at the same time. That is why is not possible to imagine it» (first year of gymnasium)

«Infinity in mathematics in an undetermined set, like that of natural numbers or of the points of a letter of the alphabet» (second year of gymnasium)

«It's a set whose elements are uncountable» (third year of gymnasium)

«Infinity, vast concept pertaining to the mathematical field and constituting a conceptual limit» (fourth year of gymnasium)

«Think about the greatest number you can ever conceive. Imagine to surpass it and to make it grow as much as you can: that number tends to infinity» (last year of gymnasium)

♦ The TEPs obtained at the higher secondary school concerning primitive entities are mainly based on the use of supposed "definitions" proposed or accepted by the teacher, that have however in most cases not a proper and real meaning in the mathematical sense, or even if they have, are not thoroughly internalised and accepted by students.
Here are some examples:

«The point is a geometrical entity belonging to a set defined as space. It is indicated with capital letters» (second year of scientific high school).

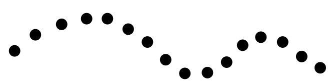
There is no actual explanation of the specific characteristics of mathematical point; in the first part of the above quoted affirmation the straight line, the plane, … may be included; nevertheless the teacher commented in the following way: *«I believe this is an acceptable definition of point, to me it's clear that the student understood what it is»*.

Another example:

«The line is an infinite set of points» (second year of scientific high school).

Teacher's comment: *«This is not good, I would not accept it because it doesn't say how the points are located»*, therefore the teacher assumes this statement as incomplete.

On this reply: *«I would say that it is an infinite set of points not necessarily in line»* (second year of scientific high school) the teacher commented in this way: *«This is ok, it's the one written in the book and that I asked them to write in the exercise book. I accept this one, because it makes clear that the points can be not in line»*. And yet this way of conceiving the line, could make one even think of a plane or points arranged in the following way:



A further example: *«The straight line is the set of points joined to one another so as to stay aligned»* (second year of scientific high school)

Teacher's comment: *«This is ok for me, I would accept it because it is clear that she understood what is meant by straight line, even if she uses some improper terms»*. The largely mentioned "model of the necklace" is clearly revealed by this way of conceiving the straight line.

Let us put an end here to such considerations that are still largely up for discussion. The aim of present work was to simply underline how TEPs are a useful device for researchers in order to obtain more detailed information concerning students' as well as teachers' knowledge and comprehension of mathematical concepts. We intend to publish the results of this research as soon as possible.

4.4. The discovery of the relevance of context: the point in different contexts

4.4.1. Where the idea of point in different contexts originates from

In consequence of the training course involving the teachers of Milan and the research in progress on the primitive entities of geometry, significant points for reflection have been emerged leading the analysis towards several directions: among these the investigation of the point used in different contexts. Starting with the *dependence* misconception (see 3.3), which was initially manifested by the teachers involved and according to which there are more points in a longer segment than in a shorter one, it has been highlighted how influential the figural model is, negatively conditioning in this case answers such as: to a greater length correspond a major number of points. This phenomenon is connected to the idea of straight line seen as "the necklace model" (Arrigo and D'Amore, 1999; 2002), being the latter proposed by the teachers as the suitable model to mentally represent the points on the straight line.

It is exactly in this model that the different convictions pertaining both to students and teachers can be traced back. Such convictions are related to the idea of point as having a certain dimension, though very small; such beliefs derive from the commonly used representation of point (for a better treatment of this specific topic see 4.5) that influences the image for this mathematical object.

The TEPs, collected during the research work concerning the primitive entities of geometry and reported in the preceding paragraph, have shown that the ideas of point manifested by students of whatever class-age are usually linked to the graphic mark left by the pencil or to their personal idea of point that can be traced in different contexts. These contexts are at times very far from the world of mathematics and students tend to directly transfer them in the mathematical field:

«A point in mathematics is a point with some numbers inside» (6 years old)

«I think that the mathematical point is a point that makes a mathematical sentence end and also makes numbers finish» (8 years old)

«It's not yet exactly known what a point is but to me it's just a point on a sheet that can have different dimensions» (9 years old)

«The point in mathematics is a little sign like that · or the question to solve.
The point in mathematics is also the one that is put on certain numbers for ex. 1˙000.
The point in calculators is considered a comma.
The point is also that of the equation for ex. $100 \times \ldots = 200$*»* (10 years old)

«One point in mathematics is important to get a good mark and be happy» (11 years old)

«The point in geometry is the reference point of a figure» (12 years old)

«It's a round point that forms the lines» (13 years old)

«A point is a part of an undetermined plane because it can have different dimensions, it constitutes the beginning, the end or both of a segment, a straight line, etc» (13 years old)

«A point is a small sign and is a fundamental geometrical entity» (first year of scientific high school)

«A point is the smallest element taken into consideration» (second year of scientific high school)

«A point is a geometrical entity, the smallest conceivable one tending to 0. Between two points there is always a third one» (third year of scientific high school)

«*A minimum point* · » (fourth year of scientific high school)

« • ⟵ *this is a point*» (last year of scientific high school)

«*A geometrical entity infinitely small, that located on a Cartesian plane has 2 coordinates (x, y)*» (last year of scientific high school)

As we have underlined in 4.3, these ideas are in some cases accepted and even shared by teachers of different educational levels (as to the idea of point showed by primary school teachers see 3.7.2).

In order to avoid that these convictions become the basis of incorrect models possessed by both teachers and students, it is therefore necessary to help subjects take the distance from the model of the segment as a "necklace" and from distorted visions concerning geometrical primitive entities, creating more suitable images allowing them, for instance, to conceive points without thickness. To this end, subjects should be supported in overcoming their previous knowing in order to build a new knowing. The questions arisen by this consideration were the following: When should this knowing be introduced and how? Which is the right direction to follow when introducing it? Where are the learning difficulties for these "delicate" mathematical objects mostly hidden?

4.4.2 Reference theoretical framework

Both teachers' and students' affirmations made us focus on the importance of context, following a situational and socio-cultural approach of social constructivism. According to this view knowledge, in particular the mathematical one, should:

- be the product of the student's active construction (Brousseau, 1986);
- have the characteristics of referring to a specific social and cultural context, though remaining in constant relation to other contexts;
- be the result of special models of cooperation and social negotiation (Brousseau, 1986);
- be used and further readjusted to other social and cultural contexts (Jonnasen, 1994).

On the basis of the above-mentioned considerations, we embraced an "anthropological" vision thoroughly oriented on the learning subject (D'Amore, 2001a; D'Amore and Fandiño Pinilla, 2001; D'Amore, 2003), rather than on the discipline, favouring "the relation and use of knowledge", rather than the "knowledge". This kind of approach is a philosophical choice of pragmatic nature (D'Amore, 2003). It is de facto the "use" that conditions the meaning and therefore the value of a given content and in this specific case, we would deal with the points used in different contexts. However this idea could be enlarged, in general, also to all geometry primitive entities and not only to them. In this perspective, we perceived the necessity and importance for the teaching activity to focus on the different contexts and forms of "uses" of an item of knowledge that determine the meaning of objects.

As a matter of fact, within the pragmatic theory we opted for as possible reference for the analysis, linguistic expressions, single terms, concepts and the different strategies to solve a problem, etc. assume different meanings according to the context in which they are used. And that is why they should be properly decoded, interpreted, selected and managed by the student. As stated by D'Amore (2003), according to this theory no scientific observation is possible since the only kind of possible analysis is "personal" or subjective and in any case circumstantial and not to be generalised. The only way is to examine the different "uses": the set of the "uses" determines in fact the objects meaning. This however should not mean, according to our view, which the teacher has to address the learning activity towards a mere act of intuition or a student's mere personal interpretation. Especially when dealing with mathematical concepts which entail the risk that the student's intuitive image turns into a parasite model (D'Amore, 1999), as it has been largely proved in this research work. As stated by D'Amore (2003): *«One of the main difficulties is that in the idea of "concept" participate several factors and causes; to express it briefly and therefore also incompletely, it seems not correct to affirm for instance that "the concept of straight line" (assuming that it exists)* (the example could be obviously also generalised to the point) *is that inhabiting the scholars' minds who have dedicated to this topic their life made of study and reflection; it seems rather more correct to affirm instead that there is a predominant so-called "anthropological" component that stresses the importance of the relations between* $R_I(X,O)$ *[institutional relation with that object of knowledge] and* $R(X,O)$ *[personal* *relation with that object of knowledge]* (in this case D'Amore explicitly refers to those symbols and terms dealt with by Chevallard, 1992) (…) *Therefore, according to the direction I chose, in the "building" of a "concept" would participate the institutional part (the Knowledge) as well as the personal part (belonging to anyone who accesses the Knowledge and thus not only to the scholar but also the student)»* [our translation].

But what does traditionally happen for the geometrical primitive entities? In particular, in the cases of the point and the straight line? The feeling is that in the case of these mathematical objects, the subject in question is simply left to the "personal aspect" and its comprehension is simply due to an act of intuition.[46]

Unfortunately, this approach bears the risk of severely reinforcing in the students' minds some parasite models such as the so-called "necklace model" which turns out to be binding for future mathematical learning, with a predominance of the figural aspect on the conceptual one (Fischbein, 1993) and being source of intuitive distorted ideas that will constantly and continuously clash throughout the student's education pathway and even get in conflict with the other branches of knowledge. We believe, from a didactical point of view, that it is important to follow a pragmatic approach with a constant mediation activity on the part of the teacher, in order for mathematical objects and their related meanings to overcome the "personal" phase and become "institutional" (Chevallard, 1992; D'Amore 2001a, 2003; Godino and Batanero, 1994). In order to obtain this result, teachers should be aware of the "institutional" aspect of knowledge, but this phenomenon as we have observed in the preceding paragraph does no take place in the case of the geometrical primitive entities. Once again it has to be noted that the choice of leaving primitive entities to the mere "personal" aspect is not a conscious didactical choice, aimed at sidestepping very delicate questions related to the attempt to "define" such objects. This choice actually derives from the passive acceptance of well-

[46] In Borga et al. (1985) it is highlighted how Pasch, already in 1882, clearly called for the opportunity to avoid any recourse to the meaning of geometrical concepts and to refer only to mutual relations among them, explicitly formulated in axioms. Peano's contributions on the fundaments of elementary geometry are bound, not only ideally, to the works of Pasch. This leads to the creation of a hypothetic-deductive system, where primitive concepts, generally without any meaning, are considered as implicitly defined by axioms. Bertrand Russel had exactly this in mind, when he expressed following paradoxical sentence: *«Mathematics is the science, where one does not know what s/he is talking about and does not know if what is said is true»* (Enriques, 1971).

established misconceptions, which have turned into wrong models held by teachers themselves. G.: *«For thirty years I've been telling my children that the point is what you draw with a pencil, I cannot change it now. And after all, I'm convinced that this is the real meaning of a point. Why, isn't it like this anymore?»* (primary school teacher). Rather curious is the question posed by the teacher G.: «*Why, isn't it like this anymore?»* that highlights not only the false convictions related to the idea of point, but also the personal beliefs concerning the idea of mathematics [on teachers' personal view and their "implicit philosophies" see Speranza (1992), on the ideological beliefs see Porlán et al., 1996)]. It seems as if the idea possessed by the teacher exactly coincided with the *didactical transposition* of mathematical knowledge that is usually proposed by the *noosphere* (see 2.4). For the teacher G. there is therefore no distinction between a mathematical concept and consequently its transposition deriving from a particular didactical choice: these two aspects are one single thing to her/him.

The direction we wish teachers would adopt, for themselves as well as for their pupils, follows a "pragmatic" approach according to which the notion of the object meaning (knowledge in general, mathematical knowledge in the specific case) is not more interesting than that of relation, "relation to the object". The latter should however be consistent with the basics of the reference discipline. For more than 2000 years, mathematicians have been trying to introduce the linguistic device of simply using words such as "point", "line", "straight line", "surface", "plane", "space" without providing an explicit definition, basing themselves on the hypothesis according to which more or less all the people that use them (children included) have an idea of their meaning: as a matter of fact, they will learn what they are just by using them. But is it sensible to consider this strategy a winning one, after having analysed children's and most of all teachers' affirmations? To avoid severe misunderstandings such as those revealed in these chapters, teachers should be firstly aware of the "institutional" meaning for a particular mathematical object that they intend to implicitly define, secondly they should convey the use of such objects into a critical and confident way so as to remain consistent with respect to the related discipline.

Also Fischbein's considerations embrace this view (1993): *«High school students should be made aware of the conflict and of its origin, so as to keep the emphasis in their minds on the necessity to base themselves, when dealing with mathematical*

reasoning, mostly on formal constraints. All this leads to the conclusion that the process of building in students' minds figural concepts should not be considered as a spontaneous effect of plain and common geometry courses. The integration of the conceptual and figural properties into some unitary mental structures, with the predominance of the conceptual constraints on the figural ones, is not a natural process. This should be a major systematic and continuous concern of the teacher» [our translation]. The considerations of Fischbein referred to high school students, should be in our opinion transferred also to all the other educational levels, or better still, we do believe it is essential that teachers pay this didactical attention already since primary school.
In a fourth grade of a primary school of Rescaldina (Milan) after having asked children: *«How big is a mathematical point?»*, we obtained the following answers: *«A point could be big, it depends on the felt-tip because it has different sizes»; «To me the point could be a very, very big thing or microscopic because it is like a circle of different sizes»; «It depends on how you make it»; «To me the point is big according to what you compare to it. If you compare it to an atom, it's very big. If you compare it to a wardrobe, very little».*
Moreover, to the question: *«How many points are there in a plane?»* it has been answered in the following way: *«It depends, if the little points are very close to one another there could be 100, even more»; «It depends on how many of them we want to make, we can make them very close and they become quite a lot. If we want to make them distant there are a few»; «It depends on the plane to me, the bigger it is the more there can be»; «In my opinion they can be infinite, in this plane they can be infinite, because a little point always finds some room»; «No plane is made of points, this sheet has been printed as a whole, it is not made by little points»; «According to me there can be more little points if the plane is large, it depends on how we draw them. There can be a big one and many little ones».*

What has emerged from all these reflections is the awareness of the necessity of not taking for granted the ideas pertaining to both teachers and students about infinity and the other primitive entities of geometry. Another fundamental aspect that has emerged from this research concerns the fact that it is essential, from a didactical point of view,

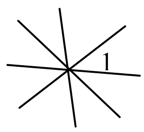
that these ideas are conveyed towards the "institutional" facet, showing the featured properties and the relationships that connect them.
In addition, we do consider as crucial that the teaching activity starting from primary school should focuses on several aspects such as: the importance played by the different contexts which is bound to the different "uses" of the knowledge on the part of students. It has therefore been identified as necessary for teachers who participated in the training course we organised, to put together with the researchers some activities to be practiced with their students. Some interesting proposals have been developed all regarding this topic and addressed to both primary and lower secondary school pupils. The choice was to start with these educational levels since as clearly revealed by the outcomes collected, misconceptions concerning the several geometrical entities are already to be spotted in primary school. It is about naive ideas most of the time linked to different contexts but which are nevertheless nonchalantly transferred to the mathematical world especially for the linguistic analogy.
Our educational and didactical goals regarding the structuring of activities with teachers are not limited to the students' acquisition of knowings, skills and competences but are targeted to develop one's own individual "use" of knowledge. The activities are meant not only to teach something but also to teach students how to manage one's knowledge and consequently to allow them to be able to make the right choices when confronted with a complex amount of information or a problematic event. All this in accordance with Gardner's words (1993): *«One of the basic targets when educating to understand or teaching to understand is: to train the child's skill to transfer and apply the acquired item of knowledge to various situations and contexts»* [our translation].
With regard to the different contexts, already pointed out by students' TEPs, and in particular as to the item of point, it suffices to look up in any Italian dictionary as for instance "Il Grande Dizionario della Lingua Italiana" (The Great Dictionary of the Italian Language) published by UTET, to find nearly 40 different meanings for the word "point". Additionally, if you look up for idioms and common expressions, at least 200 different contexts for the use of this term are to be found. Among all these, there is obviously also the definition of mathematical point but from a didactical point of view the latter is usually left to intuition, dealt with only successively and almost neglected in

some respects when compared to the other uses. The main effect is an exclusive sedimentation of all the other meanings for the term in question.

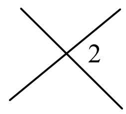
As a matter of fact in primary school, the difference between the mathematical point and the point used in other contexts (e.g. the figural one) is seldom highlighted. As a main consequence, when the point is finally dealt with in a more sophisticated way during high school, it is too late for students: all other meanings prevail and as a result the idea of a new meaning is unacceptable contradicting those already designated up to that moment (we recall the distinction between image and model reported in 2.2).

4.4.3 A provocation

In an article due to be published: *"The discovery of the relevance of context: the point in different contexts"* (Sbaragli, 2003b), we started off the treatment in question with a provocation we held as particularly significant. The reader is firstly asked to observe the following two figures and subsequently to answer the following questions borrowed from the fundamental work on figural concepts of Fischbein (1993):



3a



3b

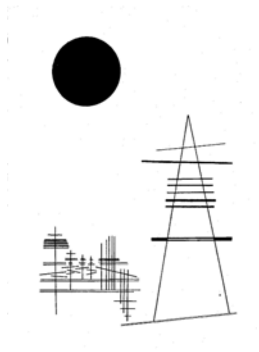
«In 3a there are four intersecting lines (point 1). In 3b, there are two lines (point 2). Compare the two points 1 and 2. Are these two points different? Is it one of them larger? If so, which one? Is it one of them heavier? If so, which one? Have the two points got the same shape?».

The experiment goes on with a series of reflections and provocations such as:

- by answering the preceding questions, what context would the reader have been thinking of?;

- how would Seurat, the "pointillist" painter have reacted to these stimuli? To Seurat, would the point have been conceived as an abstract concept or would its dimension have assumed a great importance? Can we affirm that Seurat did not succeed in “conceptualising”[47] as intended by Fischbein? Let us reflect on the effect that the illustrated below Seurat’s painting would have had if it were “represented” with a conception of the point only as a position but dimensionless;

- what would Kandinsky (1989) have thought of the questions formulated by Fischbein, since he called one of his paintings: “*The subtle lines stand up to the heaviness of the point*?”;



- what would the Australian Aboriginals think of the point since they use it as the basis to represent every single image?;

- in addition it is reported the answer given by a land surveyor of “the old guard” with more than 40 years of experience to the questions posed by Fischbein: *«It’s obvious that a point is… a point, but in drawing it changes according to which kind of pen-nib you*

[47] We are well-aware that we are using the rather complex and delicate term of *conceptualisation* in a rather simplicistic way: *«Entering this adventure leads one to become at least aware of something that is to say that the question: What is it or how does conceptualisation occur? remains basically a mystery»* (D’Amore, 2003). We owe our choice to the use that Fischbein (1993) made of the term in question in the example of a point individuated by the intersection of different segments, which we have transferred in our considerations.

use and so a point can get larger or smaller. If you use different pen-nibs or if you go over it with an ever-increasing number of lines, the point becomes visually bigger». And it is legitimate to ask oneself if the surveyor has still to conceptualise or if the conceptualisation depends on the context.

The above-mentioned provocations offer a possibility to reflect on the context importance which seems to be even more evident thanks to the accurate analysis of Fischbein's (1993) above-mentioned article, where the experimental situation of the two points is introduced: one identified by the intersection of four segments and the other by two segments. This research was addressed to subjects whose ages are included between 6 and 11 years old and who had been asked the above-mentioned questions with intentional ambiguity. Fischbein himself affirms that these questions could have been considered either from a geometrical point of view or from a material (graphic) one. The research aim was to reveal the evolution linked to the age in the subject's interpretation and the presumable appearance of the figural concepts (point, line).
As Fischbein states, the results showed a relatively systematic evolution of the answers from a concrete representation to a conceptual-abstract one. But are we really sure that the conceptual interpretation is exclusively the abstract one, or does this depend on the context? It has been with certainty acknowledged that in the mathematical field the conceptualisation of point takes place when the subject is able to make abstractions and conceive the point as a dimensionless entity. However, in the question the mathematical point was not mentioned therefore the attention could be focused on any kind of point: as meant by the land surveyor, the painter, the Aboriginal, the designer, the musician, the geographer, ... In our point of view a surveyor of "the old guard" who is used to draw with pen-nibs but is not able to distinguish the different sizes of a point, for instance a point obtained using a pen-nib 0.2 or 0.8, has not succeeded in conceptualising in her/his own field. Hence, conceptualisation depends on context, and therefore it seems essential to us that Fischbein's question should evidently clarify the reference context.
It is legitimate to wonder: is it true all the time that the graphic perception is less conceptual than the abstract or does this depend on the context of reference? In some specific contexts, the graphic for example, could the figural aspect be considered more

conceptual than the abstract one? In our opinion, to note the different dimensions of two points requires a particular sensitivity, some keenness and a certain degree of "conceptualisation" which proves fundamental in certain fields. It clearly emerges therefore the necessity for teachers to be aware of the reference field when posing questions to students and even introducing a particular context to them, in order to make sure that the students' unexpected and unhoped-for answers, are not a consequence of the interviewee referring to a different context from that envisaged by the interviewer. In some respects it would be as if you expected to obtain solutions for an equation of a certain set without explicating to which set should the solutions belong to.
In this perspective it could be risky if what Fischbein hopes for (1993) is generalised to every field. Namely that the point is to become disconnected from its context so as to prepare the concept of geometrical point. As matter of fact, we believe as fundamental that students are aware of the context in which they move and that they have a conception that is consistent with the related context. At the same time, we do hope that students are also able to distinguish and therefore employ the different "uses" within the same context and even within all the different contexts.

Returning to infinity. When the researcher, even if considered an expert of mathematics, posed teachers the following question not providing the context:

R.: «What is infinity for you?» s/he collected the following answers, all pertaining to other fields different from the mathematical one:

A.: « Leopardi's Infinity» (primary school teacher)

F.: «The space or better still, the universe surrounding us» (primary school teacher)

C.: «Your e-mail address»[48] (primary school teacher)

G.: «The infinite love I feel for my daughter» (lower secondary school teacher)

A.: «The trust in God I feel inside» (lower secondary school teacher).

It is definitely true that, when dealing with infinity even when the context is explicitly provided to teachers: i.e. the mathematical one, the collected answers are not consistent with the context taken into account (see 3.7.1). Nevertheless in most cases, some generically and completely unexpected answers like those mentioned above are averted.

[48] Translator's note: www.infinito.it is an Italian webmail provider.

The considerations regarding the point as seen in different contexts can be also valid when introducing to students the concept of infinity during their educational career, making clear reference to the use of this term in different contexts: philosophical, religious, mystic, linguistic, mathematical, …

Back to the point. As a consequence of these considerations concerning the point used in different contexts several activities have been planned. At the moment the teachers from Milan are experimenting these activities with primary school students. The experiment in progress would successively turn into a proper research activity as the future intention is to collect the results gathered during these years of study in this specific field, observing the didactical repercussion originating from this "new" didactical transposition. Our attempt would therefore be to survey the transformation teachers' and students' misconceptions will undergo, what kinds of images of the different mathematical concepts they will possess (in particular of the mathematical infinity and of the fundamental entities of geometry) and finally at that "point" what their idea of mathematics will be like. The specific treatment of these activities is outside the scope of the present work, our focus is on the "use" of the word point in different contexts: in music, language, geography, arts, drawing, mathematics, …, analysing in depth the characteristics for each context. For example when talking about painting, we highlight the aspect that we are dealing with a point with some peculiar characteristics such as size, shape, weight, colour, …, all depending on the drawing tools; the point in question has a different meaning according to what the artist intends to express. In the world of mathematics instead, the focus is on Euclid's choice of considering the point as dimensionless. Euclid assumed this principle as "starting point", the "primary rule" of the great game of mathematics which children are invited to play. But every game in order to be called a real game and to allow active participation needs the acceptance and attendance of some of its "rules", which in this specific case will lead participants "to see with the mind's eyes". The ability of accepting, respecting and sharing the others' choices and to make explicit the characteristics pertaining to different contexts, are in our opinion fundamental elements. These elements allow children to detach themselves from the physicality of the points that they normally draw, in order to accept a different world, that of mathematics with its own “rules” different from everyday rules.

To enter the world of mathematics and to accept the “rules” of the game represent one of the main goals we try to achieve during our mathematics training courses addressed to teachers belonging to any educational level. The aim is to make the topic of infinity, so distant from the finite one, more accessible. What has resulted from the research study conducted in these years is a somewhat wider scope that focuses more generally on the basic idea that teachers have of mathematics before starting to work on their misconceptions related to infinity.

G.:«Now I understood what mathematics is, nothing can surprise me anymore» (primary school teacher).

Since this work is addressed to teachers, our intention is to point out how fundamental it has turned out to be for teachers themselves to cooperate with us in the planning of students' activities, because this served as an opportunity for them to reflect on their teaching method. Here we report two extracts from a “logbook” some of the teachers wrote during this experience that well express the feelings of two primary school teachers:

L.: «We primary school teachers are skilled designers of techniques and didactical material that are even regarded with admiration for their ingenuity. But sometimes we fall too deeply in love with our “expedients” and we use them with too much confidence. It's true, to talk about mathematics we need some models, we have very young students, and therefore we always think it necessary to materialise concepts for them. In this way we do not realise the unintentional trickery in which we get our students involved in: through our material didactics they get convinced that mathematical objects are real objects and that it is the way they should be treated. Me too in my primary school teaching history I happened to turn to material didactics and I spent time and energy to make it even more effective. But at a certain point of my teaching career, I realised that manipulating, doing, building, do not necessarily lead to mathematical knowledge. (…) I finally understood that the concept does not exist in the mind, unless you are not able to imagine things; the manipulating hands are of no help in building mathematical concepts. (…) I was *satisfied and it seemed to me that everything was working well. But it was just an impression due to my mathematical ignorance, I don't blame myself, I'm just stating something.*

Two years ago at a meeting in Castel San Pietro I heard Silvia Sbaragli talking about mathematical infinity: potential and actual infinity: I entered a new world. Silvia Sbaragli was talking and I was reevaluating myself as a maths teacher: too much confusion and many things taken for granted.

I ploughed my didactical soil well and I was able to immediately grasp the message contained in the speaker's words: a fruitful and fertile thought at the right moment.

That moment was crucial for my teaching experience: that was the start of a cooperation that enlightened and is still enlightening my profession (…) helping me a great deal to organise my still fragmented, confused and incomplete ideas.

I finally understood for good and all what is the right approach to do mathematics. I tried to make my children see the light in the same way as I did: I think I made it. I'm happy with the transformation that allowed me to peep into the world of mathematics with the correct outlook that washes away my ignorance in one go. This approach won't let me make big mistakes and omissions with my children and it will allow me and my students to enjoy such a rich knowledge in which and with which the human mind can play and have fun».

C.: «I have to admit that, many times, with the intention of facilitating the learning and of fulfilling the students' need for clarity and concreteness, we would rather favour our need for confidence trying to find contacts with or evidence in the real world. All this reassures us, we have a major control over it, it's there, its' visible, you can't make mistakes. To face the world of the non-sensible is scary and in every way we try to transfer the objects and rules of Mathematics to the real world, for even mathematical concepts need to find a real justification to exist. It was very challenging to work on representations, to understand that they are useful, to give shape to something which is not concrete. However, these representations can also be weak because to present concrete models in mathematics does not guarantee correct learning all the time, but this method rather conditions and even hinders it. I *remember Elena who after having reasoned, talked and reflected once again on the mathematical point told me: "Yes I don't see it, but I do understand it"».*

The only way to comment on these two so meaningful reflections is through a short, though effective Japanese proverb: *"To teach is to learn"*. This proverb is also valid for us researchers every time we come into contact with the fruitful world of didactics.
Our aim seems to be at least partly achieved: to substantially affect teachers' convictions in order to successively and indirectly affect the students' convictions.

At this point, it should be legitimate to ask oneself the reason why a thesis centred on primary school teachers' convictions on mathematical infinity has focused this chapter almost entirely on the point and its didactical transposition. As a matter of fact, though bearing always in mind that the main subject matter is that of infinity, these were the aspects we encountered along our pathway over the years. It has been proved impossible to deal with the mathematical infinity in the geometrical field without making any reference to the primitive geometrical entities. The majority of the wrong beliefs concerning infinity originate from some misconceptions related to these mathematical objects. Furthermore, we are convinced that these proposals constitute a new way of working in class, more flexible, closer to "our" idea of mathematics, capable of getting both teachers and students closer to the concept of infinity.

4.5 A further fundamental aspect: different representations of the point in mathematics

Another fundamental and delicate aspect connected to the preceding treatment concerns the different representations of the point in mathematics. The choice was once again to investigate the point but the following reflections can obviously be applied also to all of the other primitive entities of geometry, and not only to them. Where do the following considerations come from? In these chapters we have repeatedly shown that the majority of difficulties and misconceptions especially those regarding mathematical infinity mostly depend on the visual representation provided for geometry primitive

entities. But what kind of representation is this provided by teachers and accepted by the *noosphere*? As for the primitive entities of geometry, is there a tendency to provide one single representation or rather several, even adopting different semiotic[49] registers? How does the representation provided by teachers influence students' convictions?
Before answering these questions, let us first analyse in depth the reference theoretical framework.

4.5.1 Reference theoretical framework

As for this treatment we referred to Duval who highlighted the fact that in Mathematics the conceptual acquisition of a piece of knowledge should necessary firstly go through the acquisition of one or more semiotic representations. The issue of registers was introduced in the famous articles of 1988 (a, b, c); and in the following work of 1993. Therefore borrowing one of Duval's affirmations: there is no ***noethics*** (conceptual acquisition of an object) without ***semiotics*** (representation by means of signs), we made the following considerations.

As D'Amore stated in his book of 2003:

- every mathematical concept has references to "non-objects"; therefore conceptualisation is not and cannot be based on concrete reality-based meanings; in other words in Mathematics broad references are not possible;
- every mathematical concept is forced to make use of representations, as there are no "objects" to put in their place or to recall them; therefore conceptualisation should go through representative registers that according to various reasons and especially if having a linguistic nature, cannot be univocal;
- in Mathematics we usually talk more of "mathematical objects" rather than "mathematical concepts" as Mathematics would study the objects rather than concepts; *«The notion of object is a notion that we are forced to use right from the moment in which you question yourself on nature, on conditions of validity or on the value of knowledge»* [our translation] (Duval, 1998).

[49] Taking as a starting point the framework of Duval that we are going to describe when talking about a "register of semiotic representation", we refer to a system of signs enabling us to fulfil the functions of communication, treatment, conversion and objectivation.

As revealed by this latter point, to Duval the notion of concept assumes a secondary importance, whereas priority is attributed by the Author to the couple (sign, object). Vygotskij quotes the importance of the sign also in a passage from 1962, mentioned by Duval (1996) where it is stated that there is no concept without sign. If we assume this as true, the didactical consequence is to pay special attention to the sign choice, or better still to the sign system representing the mathematical object selected to be taught to students. The above-mentioned attention is often underestimated or taken for granted. D'Amore (2003) reported Duval's thought stating that there is a group of didacticians that tend to reduce the *sign* to the *conventional symbols* that identify directly and singularly some objects but that can turn into misconceptions since they become the unique representatives of a given register. We believe this is exactly what happens with geometric primitive entities. The point is perceived as and referred as the unique representation that is commonly provided by the *noosphere*: a dot on the blackboard; the straight line as a continuous line, of variable thickness, straight and formed of three initial dots and three final. No one dares to take the distance from these representations. Teachers and indirectly also students perceive them as the only plausible and possible representations. As a consequence, the point is associated with the unique image provided for it: a "round" sign left on a sheet of paper, of variable diameter and with a certain dimension.

A.: *«I don't think there are other ways of representing the point other than that of gently touching a sheet with a pen»* (primary school teacher)

R.: «Can't you think of anything else? What do you do with your students?»

A.: *«If you ask me how I represent it, in order to produce it I make a little sign on the blackboard but if you mean what I say when describe it, I usually tell them to think of a grain of sand or of salt».*

Among the models selected by teachers to represent the point they are all the time "round like" images because misconceptions concern also the idea that the shape of a point is "spherical":

R.: «According to you, is it legitimate to represent a point as a star?»

A.: *«As a star? Of course not, what kind of question is this? A point is not in the least a star!»*

R.: «Why, is the point this: •?»

A.: «Yes, the point is spherical, it's not definitely star-shaped».

4.5.2 A particular case of Duval's paradox: the primitive entities

Let us analyse the famous Duval's paradox (1993) (quoted in D'Amore 1999 and 2003): *«(…) On the one hand, the learning of mathematical concepts cannot be other than a kind of conceptual learning and, on the other hand, it is only by means of semiotic representations that an activity on mathematical objects can be carried out. This paradox could represent a real vicious circle to learning. How is it possible then that learners should not confuse mathematical objects with the related semiotic representations if the only representations they come in contact with are the semiotic ones? The impossibility of a direct access to mathematical objects, beyond every semiotic representation, makes confusion almost inevitable»* [our translation]. This confusion is magnified in the case of primitive entities, as these are most of the time simply left to an act of intuition. Furthermore, the learning of these mathematical objects is made more complicated by the decision of providing the students only with some vain and univocal conventional representations, which are therefore blindly accepted because of the *didactical contract* constituted in class (see 2.1) and of the phenomenon of *scholarisation* (see 2.4).

The paradox continues as follows: *«And, on the contrary, how could they* (learners) *acquire the mastery of mathematical treatments, necessarily bound to semiotic representations, if they do not already possess a conceptual learning related to the represented objects? This paradox is even stronger if both the mathematical and conceptual activity are considered as one single thing and if semiotic representations are considered to be of minor importance or extrinsic»* (Duval, 1993). [our translation]

We take into account the latter paradox with reference to the mathematical point: we wish as teachers that students would conceive the mathematical point conceptually, considering it as a dimensionless object, although it is only by means of semiotic representations that an activity on mathematical objects can be carried out. The learners will surely tend to confuse mathematical objects with their semiotic representations, but this phenomenon may take place especially when the provided representations are almost exclusively univocal and conventional. For instance in the cases of the point and the straight line and when teachers do not perform a mediation activity between the “personal object” and the “institutional object” (Godino and Batanero, 1994, Chevallard, 1991). So when dealing with primary school children, what is the right strategy to talk about the point without drawing it in only one way on the blackboard? How is it possible to be free from this representation that is fixed and stable transforming itself into an erroneous model for both teachers and students? How can students possibly acquire some mastery of mathematical *treatments*[50] and *conversions*[51] linked to semiotic representations when what is provided for geometrical entities is basically one and only one conventional representation?

Difficulties are not only due to the impossibility for students of having from the beginning a conceptual knowledge of mathematical objects but are nevertheless magnified by the revelation that most of the times even teachers do not possess this conceptual knowledge. Therefore they tend to confuse the mathematical object they intend to explain to their students with its representation (see chapter 3).

The constant and continuous cooperation over the years with teachers has quite often revealed that some of them tend to attribute the existence of a mathematical object to its possibility of being represented by means of images or concrete objects:

S.:«To me infinity doesn’t exist, you cannot in the least see it» (primary school teacher)

R.: «Why can you see the number 3?»

S.: «Of course, you just need to show 3 fingers, write 3, show 3 objects. But how can you do it with infinity?»

R.: «So according to your way of thinking, you just need to write this: ∞»[52]

S.: «No, that’s different you can’t even touch it with your fingers. The 3 exists to me and infinity not».

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[50] By the word *treatment* we refer to a cognitive activity, typical of semiotics, which consists in the passing from a representation to another within the same semiotic register.

[51] By the term *conversion* we mean a cognitive activity, typical of semiotics, which consists in the passing from a representation to another, in different semiotic registers.

[52] Rucker (1991) presents a curious observation: the symbol ∞ first appeared in 1656 in a treatise by John Wallis on conical sections, *Arithmetica Infinitorum* (see: Scott, 1938). It was soon spread everywhere as the symbol for infinity or eternity in the most diverse contexts. In the 18th century, for example, the symbol for infinity appeared on the tarot card of the Fool. It is interesting to note that the cabalist symbol associated to this card is the Jewish alphabet letter aleph $\aleph$.

These affirmations underline once again the false convictions teachers have of mathematical objects and more in general of mathematics itself.

As Fischbein (1993) affirmed it is important to underline that: «*In empirical sciences the concept tends to approximate the corresponding existing reality, whereas as far as mathematics is concerned it is the concept that, through its definition, dictates the properties of the corresponding figures. This will lead to a fundamental consequence. The entire investigating process of the mathematician can be mentally carried out, in compliance with a specific axiomatic system, whereas the empirical scientist has to, sooner or later, return to the empirical sources. To a mathematician, reality can be source of inspiration but in no case a research object leading to mathematical truths and by no chance a final example to prove a mathematical truth. The mathematician, like the physician or the biologist, makes use of observations, experiments, inductions, comparisons, generalisations, though her/his research objects are purely mental. Her/his laboratory is, on the whole, confined to her/his mind. Her/his pieces of evidence are never empirical by nature, but exclusively logical*» [our translation].

Duval's paradox is even more evident if teachers let the concept coincide with its related representation and if they have never reflected on the topic and structure the didactical transposition taking into account the meaning and importance of semiotic representations.

The considerations collected so far are once again strictly connected with the issue of *figural concepts* as illustrated by Fischbein (1993): «*A square is not an image drawn on a sheet of paper; it is a shape controlled by its definitions (even if it can be inspired by a real object). (…). A geometrical figure can be thus described as bearing some intrinsic conceptual features. Nevertheless a geometrical figure is not a pure concept. It is an image, a visual image. It possesses some characteristics not belonging to usual concepts, that is to say it includes the mental representation of spatial properties. (…). All geometrical figures represent mental constructions that simultaneously possess both conceptual and figural properties. (…). In geometrical reasoning the objects of study and manipulation are therefore mental entities which we call figural concepts and that mirror spatial properties (shape, position, size) but that also possess, at the same time, some conceptual properties such as: ideality, abstractness, generality, perfection. (…). We need some intellectual effort in order to understand that the logic-mathematical* *operations manipulate only a purified version of the image, the spatial-figural content of the image. (…). Ideally, it is the conceptual system that should completely control the figures' meanings, relationships and properties. (…). But in general the evolution of a figural concept is not a natural process. As a consequence, one of the main tasks of the didactics of mathematics (in the field of geometry) is to create some didactical situations that would systematically require a close cooperation between the two aspects, up until their fusion into unitary mental objects*» [our translation]. It is right on the basis of these considerations that we are working together with three teachers from Milan on the creation of suitable activities. These experiences are aimed at valorising and highlighting, as far as the primitive geometrical entities are concerned, several semiotic representations of different registers, letting students' imagination free and helping them to detach themselves from some false stereotypes. Such stereotypes are already accepted as conventional and in so doing they reach the *knowledge institutionalisation* that leads to the institutionalised knowledge of the various mathematical objects. It is exactly letting both teachers and students get rid of given and stable representations that constitute "wrong models" (see chapter 3) that it is possible to build an idea closer to the mathematical object to learn. Our aim is to try to inculcate first in teachers and then in students, the idea that the nature of a concept is independent of the kind of representation selected in order to represent it. The issue of infinity would be consequently easier to accept.

4.5.3 Some activity proposals

The activities structured together with the primary school teachers from Milan are intended to make students perceive the "weakness" characterising the mathematical representations, so as to make students grasp what lies beyond a specific concrete model (not only figural) and attribute a conceptual meaning, from a mathematical perspective, to the different images. In this way students will be able to see with "the mind's eyes" and find out the right connection between all different aspects by means of the use of various language codes: verbal, sign, figural, mental, ... In particular, we maintained it necessary to structure activities targeted at the formation of figural concepts as intended by Fischbein (1993).

At the same time, our aim is to enable students to "dare" to invent different representations for the same concept. This will allow students to perform *treatments* that is to say to pass from one representation to another within the same semiotic register for the same concept, as well as to perform *conversions* between one representation and another using different semiotic registers. *«Further more: knowledge "is" the intervention and use of signs. So, the mechanism of production and use, subjective and inter-subjective, of these signs and of the representation of the "objects" of the conceptual acquisition is crucial to knowledge»* [our translation] (D'Amore, 2003).

In order to do this, students should be able to validate[53] and socialise their choices defending their opinions with the appropriate argumentation but they should be even able to accept the other's motivations, so as to create some shared and conventional representations within the class. These representations will be at a later stage compared to those selected by the noosphere. As D'Amore affirms (2003): *«During the learning process of Mathematics, students are faced with a new conceptual and symbolic world (representative in particular). This world is not the result of a solitary construction but the outcome of a real and complex interaction with the members of the micro society which the learning subjects belong to: their classmates and teachers (and the noosphere, at times in the background, at times in the foreground)* (Chevallard, 1992). *It is thanks to a constant social debate that the learner becomes aware of the conflict existing between "spontaneous concepts" and "scientific concepts". Teaching is not a mere attempt to generalise, magnify, and develop in a more critical way the students' "common sense", teaching is about a much more complex action, ... Therefore learning seems to be a kind of construction subordinate to the need for "socialising". The socialising activity takes place thanks to a means of communication (language for instance) and that in Mathematics will be influenced by the symbolic mediator's choice, i.e. the representation chosen register (or imposed, by several factors or even simply by the circumstances)»* [our translation].

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[53] *Validation* is the process adopted and followed to reach the conviction that a specific obtained result (or the construction of an idea) responds exactly to the requisites explicitly brought into play. This can happen when a student proposes her/his conceptual construction to the others, explicitly in a communicative situation, focusing her/his attention on the transformation of a piece of personal and private knowledge into a communicative product and defending her/his opinion (or solution) from sceptics (that is to say validating her/his reasoning).

Why is the point in mathematics represented only as a "round" sign? Does its "round shape" constitute one of its specific mathematical properties?
A point in mathematics should be an a-dimensional entity, therefore its representation, necessary to refer to this concept, can be of any kind since it should not stick to any specific characteristic but the one of not being represented. In our opinion, the varieties of representations allow students to purify the object from those features that are not proper to it: size, weight, colour, dimension of its diameter, … From a didactical point of view, it is sufficient to establish a position in the space to identify a point whereas as for the representation of this position, it will be the children's task to use their imagination and according to their wish and taste, they might represent the mathematical point as the end point of an arrow, the intersection of a cross, the centre of a little star, …
The creation of the wrong model deriving from the point's univocal given representation constitutes an analogous situation to what happens in nursery school when the teacher tries to make the pupil learn to recognise the square shape always providing the same model for it: red, made of wood, with a specific extension and thickness, … The child would believe that the square's characteristics are exactly those of being: red, made of wood, with that specific dimension. In order to purify the concept provided for the square from features that do not characterise it, students should be given the opportunity to "see" different images acting in different contexts which will allow them to attain those characteristics of ideality, abstractness, generality, perfection.[54]
«The point exists only in my mind, it's like a little ghost. A little ghost can pass through infinite little ghosts» (Luca, third year of primary school).

One day entering a third year class where teachers were adopting the above-mentioned approach, children asked the researcher the following question:
«Try to understand what it is». And they drew on the blackboard the following image:

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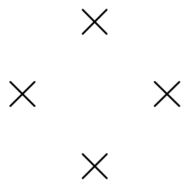
[54] In this respect Locke (1690) asserted: *«As for the general terms (common nouns), … the general and the universal clearly do not belong to the sphere of real things, but they are inventions and creatures of the intellect made up for its own purposes, and they just pertain to the signs, be they words or ideas»* [our translation].



The answer was: *«Square»*.[55]

Successively they posed the following question:
«What is this?»

And the researcher's answer: ***«Two points»,***
children reacted in this way: *«No, try again»*
R.: «Is it the segment that has those two points as end-points?»
B.: «No, try once again. C'mon you can get it!»
R.: «The straight line passing by those two points»
B.: «Well done Silvia, now we can draw it»

Children demonstrated to have chosen an alternative way also to represent the straight line. These proposals imply students' "personal risk", their commitment and their direct *involvement* in learning manifested with the breaking of the didactical contract (see 2.1):

[55] A propos of this, Speranza (1996) wrote: *«Let us go back to the Ancient Greece. In The Republic Plato wrote: «Those who deal with geometry… make use of… visible figures and they reflect on them but in fact they think of what they represent, reasoning on the square itself and on the diagonal rather than on the drawings…». (…). Plato spoke about "square itself", of which the drawings of the square are "images". This recalls the "myth of the cave": the true reality is that of general ideas, that exist by themselves: sensible things are just "images" we can see, they are like shadows of the real entities which are outside projected on the back of a cave»* [our translation].

«The need for such a break can be summed up by the following aphorism: believe me, says the teacher to his pupil, dare to use your knowledge and you will learn» (Sarrazy, 1995).

If it is true what Duval claims (1993) that the creation and development of new semiotic systems is the symbol (historical) for the progress of knowledge, we intend, by means of these activities, to activate such a progress within the classroom adequately, considering all three cognitive activities "proper to semiotics": *representation, treatment, conversion*. In particular, to conversion we attribute a major position, according to the grounds provided by D'Amore (2003) and even before by Duval (1993). Among these reasons we consider it fundamental that such a specific cognitive activity enables to define some independent variables concerning both the observation and the teaching activity. This will favour the "conceptualisation" which is actually activated, or even simply sketched, through the coordination of two distinct representation registers.

Other activities structured by teachers are focused on the main differences between the finite and the infinite field as to avert that the infinity concept is banally reduced to an extension of the finite. The treatment of the issues concerning infinity requires the development of different intuitive models at times even opposed to those used when dealing with the finite. According to our point of view, in order to avoid the creation of misconceptions regarding this topic, a proper education centred on the handling of infinity sets should be started already in primary schools. This approach would allow students to begin observing the principal difference between the two fields. The goals we have set when structuring these activities are mainly intended to enable students to: grasp the real essence and charm of mathematics; distance themselves from the everyday routine regarding the finite; transfer biunivocal correspondence from finite to infinity; sense the meaning of infinity both in the numerical and geometrical field.
In this respect, we reported some of the statements given by a lower secondary school teacher as the result of having introduced to her/his students some of the biunivocal correspondences between infinite sets (in particular, between the set of natural numbers

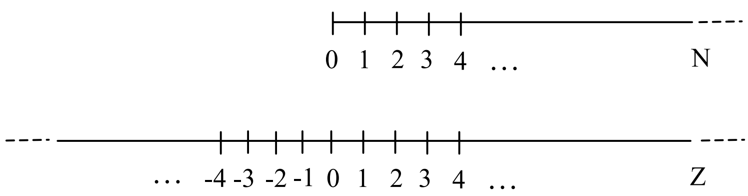
and that of even numbers, natural and odd, natural and whole) learned during the training course:

C.: «I feel a kind of relief since I discovered I can position numbers in different ways. The fact of knowing that it is possible to play with the order of numbers is amusing for me as well as for my students. School tends to be very rigid, you never depart from the norms».

Here the teacher in question was referring to the possibility of finding a different order from the "natural" one for infinite sets, such as the set of whole numbers. This allowed the teacher to show children the biunivocal correspondences between infinite sets.

C.: «After having shown this order: 0, + 1, -1, +2, -2, +3, -3… related to the set of whole numbers, a child spontaneously modified it in this way: 0, -1, +1, -2, +2, -3, +3… Children had found of some help the example of the game of bingo, where numbers are contained in a bag in a scattered order. This activity did not surprise them as it surprised me when attending the course I saw the biunivocal correspondence between N and Z: it seemed students had accepted it with no problem at all. I also tried not to make them concentrate on the common and so misleading graphic representation».

The teacher was referring here to the following graphic representation:



This latter representation makes teachers as well as students believe that the number of whole numbers is twice as that of natural numbers (someone also makes a clarification: with the exception of the zero which is the neutral element).

In this school year, the mentioned activities and others more, constituted the core issue of many articles centred on didactical workshops and published in a widespread Italian journal called: *La Vita scolastica* (The school life) addressed to primary school teachers. This represents, in our opinion, a great result as the issues of geometrical primitive entities and mathematical infinity will have a much more influential didactical repercussion and at the same time will push teachers to approach these issues and all the related topics too. Further, it is meaningful that the articles in question are structured as workshops,[56] where students shave an active role *building,* even literally, objects that try to eradicate several misconceptions. Special relational mechanisms are therefore enhanced (teachers-students) as well as cognitive relationships (student-mathematics) of major theoretical interest (Caldelli and D'Amore, 1986; D'Amore, 1988, 1990-91, 2001b).

4.6 The "sense of infinity"

The last aspect to be pointed out in this work regards a research study still going on today and that involves 9 researchers working for the following organisations: NRD (Mathematical Didactics Research Group, Mathematics Department, University of Bologna, Italy), DSE (Education Sciences Department, Ministry for Education, Bellinzona, Switzerland), ASP (Pedagogical Specialised School of Canton Ticino, Locarno, Switzerland), Mescud (School Mathematics University of Distrital, Univ. Distrital "Francisco J. de Caldas", Bogotà, Colombia).

The idea developed by D'Amore proposes to investigate, in different contexts and involving a broad sample of participants, whether or not a "sense of infinity" exists. In order to understand what is meant by this expression a clarification of the concept of "estimate", as intended by Pellegrino (1999) is needed: *«The result of a process (conscious or unconscious) that aims at identifying the unknown value of a quantity or magnitude»*. It is therefore about to sense the essence of the cardinal of a collection. The necessary skills to be a "good connoisseur of estimates" as highlighted by Pellegrino (1999), regard different factors: psychological, metacognitive, emotional and mathematical. These are some of the most important questions we asked ourselves:

[56] As D'Amore (2001b) claims: *«The "Workshop" is an environment where objects are produced, where people concretely work, where they build something; the most peculiar feature of a workshop must be some sort of creative practice; in a Workshop there must be a tendency towards ideation, planning, creation of something which is not repetitive or banal, otherwise a factory… would be enough»*.

what happens if this *unknown value* is infinite? Does a “sense of infinity” exist, as does a “sense of the finite number”? If it exists, how does it manifest? If not, why? Is it possible to convey an intuitive meaning to the difference between the denumerable infinity and the continuous infinity?

During the research carried out in 1996 with primary school children, we came across some statements, spontaneously reported, that showed certain confusion between finite and infinite numbers. A good example is provided by a conversation that took place with the researcher and two children after they had been shown two segments of different length and were asked the following question: ***«According to you are there more points in this segment or in this other one?»***

M.: «We studied that a line is a set of points»

I.: «The line, not the segment»

R.: «Do you know what a segment is?»

I.: «Yes, it's a line which starts and ends with two points and the points have letters» (I.'s answers are inconsistent: the line is formed of points, the segment, though still being a line, is not formed of points)

R.: «And in a line?»

I.: «There are many points»

R.: «How many?»

M.: «Infinite points»

R.: «So also here in the segment there are infinite points»

I.: «No, it's limited» (also in children emerges the idea of points as unlimited as already spotted in teachers' convictions in 3.7.1)

M.: «There won't be so many as in this one (indicating the longer segment)»

R.: «So, you think there are more here (indicating the longer segment) ***than here*** (pointing at the shorter segment)»

M.: «It depends on how large they are, if one is one km large and the other one mm, there can be two or one million» (this affirmation recalls those of the teachers reported in 3.7.2 and underlines also a lack of “sense of measure”)

R.: «And in the line?»

M.: «Billions».

M., although claiming that the number of points of a line are infinite, subsequently affirms that in a line there would be billions of points. Where does this inconsistency come from? Does it depend only on the misconceptions concerning the geometrical primitive entities pointed out in the preceding paragraphs or also, by any chance, on a total inability of creating an image for infinity in its actual meaning? Wouldn't it be, by any chance, also the complete impossibility of *estimating infinity*?

The research studies carried out by Arrigo and D'Amore (1999, 2002) with higher secondary school students mirrored the same kinds of misconceptions.

Let us once again turn to the aspect that we held as most important: teachers. The present work has already clearly shown (paragraph 3.7.1) that teachers provided some curious estimates about infinity. Some examples are reported here as follows.

To the question: *«What do you think mathematical infinity means?»*

A primary school teacher answered:

C.: «Something that you cannot say»

S.: «In what sense?»

C.: «You don't know how much it is».

Some other primary school teachers affirmed:

A.: «To me it's a large number, so large that you cannot say its exact value»

B.: «After a while, when you are tired of counting, you say infinity meaning an ever-increasing number»

M.: «Something that I cannot quantitatively measure»

D.: «It's something so big that, no matter how much you try, it's impossible to classify it thoroughly. Mathematics, with its discipline, attempts to study a part of it».

Whereas two lower secondary school teachers wrote:

L.: «Mathematical infinity is when it never ends, it's a convention. When I cannot indicate the "beyond" I use this term: infinity and I indicate it with this sign: ∞»

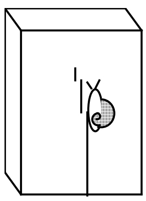
F: «Mathematical infinity is a world constituted by elements that are impossible to think in their totality».

Could the fact of conceiving infinity as a large finite number, or as the unlimited, or also as a kind of process as reported in 3.7.1, be caused by an inability of *estimating infinity*?

The goal we are aiming at is to try to provide plausible answers to a number of questions that, as suggested by D'Amore, we chose to classify into two main groups: one of intuitive and linguistic nature and the other basically of more refined and technical nature. The first category concerns students not particularly skilled in mathematics or people with not a good knowledge of mathematics (students with not a solid educational background, primary school teachers, people with no connections with the world of school or the academic world, people that have a medium-high cultural level); the second one is mainly centred on evolved students or people with a good mathematical background [as for example, secondary school teachers, undergraduate students of mathematics (III and IV university years) and postgraduate students attending specialisation courses]. The TEPs (D'Amore and Maier, 2002) methodology together with the interview technique was once again preferred on the basis of what has been already described in 4.3.

We will not provide a detailed report on the research questions as well as the TEPs contents and the interview topics: they all had a shared and common beginning but then they developed differently, according to the interviewee's skills and educational level. Consequently, the results will not be reported in this work as they will soon be published but since we are dealing with teachers' convictions on mathematical infinity, we will conclude this thesis with two interviews.

After having showed the following TEP…



A snail wants to climb a wall.
In the first hour it gets up to the half of it.
In the second hour, the snail being tired, gets only up to the half of the space covered before.
In the third hour, even more tired, it performs the half of the distance covered in the preceding hour.
And so on …

I don't think it will ever get to the top

Of course it'll get there: if you consider that after two hours the snail has already walked along three quarters of the pathway …

What do you think?

Primary school teacher:

K.: «Is there a time limit?»

R.: «No, there are no limits»

K.: «So why shouldn't it get there, it will make it»

K.: «½ + ½ of ½ + How much is this sum? I can't say it really. It should result the length of the wall, but the time employed does not matter»

K.: «Let me think… the wall has an end, the height does not depend on the fact that it succeeds in getting there or not, it influences the number of hours. Yes, I think it'll make it»

R.: «In your opinion how much is the sum you told me: «½ + ½ of ½ + …?»

K.: «It can be the wall's height»

R.: «In what sense?»

K.: «The snail will get there»

R.: «And how much is this sum exactly? Could you tell me?»

K.: «Infinity? I don't' exclude it can be infinity. Yes, I think so… but also I don't think so, I don't know. I'm not really good at these kinds of things».

A lower secondary school teacher:

L.: «According to me the snail will never get there. I do not know anything about the series, but let's try it.

For each hour you have to put the covered pathway plus the half of it and so on the thing goes on to infinity, so it will not get there».

In the meantime the teacher was writing on a sheet of paper:

x =

1° hour= ½ x

2° hour= ½ x + ¼ x

3° hour=½ + ¼ x + ½(½x + ¼ x)

4° hour = ½ x + ¼ x + ½(½x + ¼ x) + ½…

L.: «The result is always a fraction of the whole pathway which is x and the more the time goes by the more it still remains a fraction. According to calculations it seems as if the snail can't make it: you have to add ever-smaller fractions of the pathway *but it will never get there. The stretches become smaller and smaller but you add them up. It will never get there».*

Does therefore a *sense of infinity* exist? Our surveys are still focused along this direction and the results of this curious research work will be soon published.

We have introduced in this chapter several research lines that we are following at present and that we intend to investigate in the near future. This complex outline demonstrated that as far as the infinity issue is concerned, there is always a new world to discover, study, analyse and investigate in depth. The feeling we receive is that year after year we are just at the beginning of such an "infinite" pathway: *«Infinity! No other problem has ever so deeply shaken the spirit of the humankind; no other idea has ever so profoundly stimulated their intellect; and nevertheless no other concept is so in need of* ***clarification*** *as infinity»* (Hilbert) [our translation].

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