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FIBERWISE BIRATIONAL REGULAR MAPS OF FAMILIES OF ALGEBRAIC VARIETIES

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Abstract. Given two regular maps of algebraic varieties $g : X \to S$, $h : Y \to S$ with irreducible fibers and a surjective regular map $f : X \to Y$ such that $g = h \circ f$ and the restriction of f on each fiber is birational, we give sufficient conditions for f to be an isomorphism. Similar problem is studied for schemes.

Introduction

A well-known fact is that every regular map from an open subset of a complete nonsingular algebraic curve to a complete algebraic variety may be extended to a regular map of the curve. This is proved by considering the closure of the graph of the map in the product of the curve and the variety and proving that the projection of the closure to the curve is an isomorphism by means of Zariski's main theorem. If one tries to apply the same argument to families of curves one encounters the following problem. Given two proper families of curves $g : X \to S$, $h : Y \to S$, where h is smooth, and a commutative diagram of regular maps as in (1) below, such that f is surjective, with finite fibers, and $f_s : g^{-1}(s) \to h^{-1}(s)$ is birational for every $s \in S$, is it true that f is an isomorphism? If the scheme-theoretic fibers of g were reduced this would follow from Proposition 4.6.7 (i) of [7]. This condition, however, might be difficult to verify. In fact a weeker condition on g suffices to conclude that f is an isomorphism. One of our results is the following theorem.

Theorem 0.1. *Let k be an algebraically closed field and let X, Y, S be algebraic varieties over k. Let*

(1)
$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & S & \end{array}$$

be a commutative diagram of morphisms such that:

(a) *h is flat and $\dim_k \Omega_{Y/S}(y) \leq \dim_y Y_{h(y)}$ for $\forall y \in Y$;*
(b) *g is proper, f is surjective and $|f^{-1}(y)| < \infty$ for $\forall y \in Y$;*
(c) *for every $s \in h(S)$ the fiber $g^{-1}(s)$ is irreducible and there is a point $x \in g^{-1}(s)$ such that g is flat at x and $\dim_k \Omega_{X/S}(x) \leq \dim_x X_{g(x)}$;*
(d) *for every $s \in g(X) = h(Y)$ the map $f_s : g^{-1}(s) \to h^{-1}(s)$ is birational.*

Then $f : X \to Y$ is an isomorphism.

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We prove in Theorem 2.2 and Theorem 2.4 analogous statements for schemes over a field of characteristic 0 and for schemes of finite type over a perfect field respectively.

When the relative dimension of $g : X \to S$ is one, Theorem 9 of [12] applied to $Spec\,\mathcal{O}_{X,x} \to Spec\,\mathcal{O}_{S,s}$, where $x \in X$ is arbitrary, $s = g(x)$, implies that $g : X \to S$ is smooth and therefore $f : X \to Y$ is an isomorphism by Proposition 4.6.7 of [7]. There are however two questionable points in the proof of Theorem 9 of [12], one of which is a gap. We discuss them in Section 3.

The paper is organized as follows. In Section 1 we prove the main result, Theorem 1.1, from which all other theorems in the paper are deduced. The arguments of part of the proof are similar to those of Kollár (see [12] p.719). Our contribution is in avoiding the use of the (S_2)-property by means of Proposition 1.3. Theorem 1.1 is a statement concerning affine schemes. In Section 2 we apply it to obtain the results of Theorem 2.2, Theorem 2.4 and Theorem 0.1 that we discussed above. In Section 3 we discuss Theorem 9 of [12].

Notation. If $\varphi : A \to B$ is a homomorphism of rings and $I \subset A$, $J \subset B$ are ideals we denote, following [14], $\varphi(I)B$ by IB and $\varphi^{-1}J$ by $A \cap J$. If $\mathfrak{p} \subset A$ is a prime ideal, then $k(\mathfrak{p})$ is the quotient field of $A/\mathfrak{p}$. The term variety over an algebraically closed field is used in the sense of [16]. We do not assume that the varieties are irreducible.

1. The main theorem

Theorem 1.1. *Let $A \to R$ be a flat local homomorphism of Noetherian local rings. Set $\mathfrak{m} = rad(A)$, $\mathfrak{n} = rad(R)$. Suppose that A is reduced. Suppose that:*

(a) *$R \otimes_A k(\mathfrak{m})$ is a regular ring and $k(\mathfrak{n})$ is separable over $k(\mathfrak{m})$;*
(b) *$R \otimes_A k(\mathfrak{p})$ is a regular ring for every minimal prime ideal $\mathfrak{p}$ of A.*

Set $S = Spec\,A$, $s_0 = \mathfrak{m}$, $Y = Spec\,R$, $y_0 = \mathfrak{n}$, and let $h : Y \to S$ be the associated morphism of affine schemes. Suppose there is a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & S & \end{array} \tag{2}$$

such that:

(i) *f is finite and surjective;*
(ii) *X_{s_0} is irreducible and generically reduced;*
(iii) *g is flat at the generic point of X_{s_0};*
(iv) *if $\eta \in X_{s_0}$ and $\zeta = f(\eta) \in Y_{s_0}$ are the generic points of X_{s_0} and Y_{s_0} then $f^{\sharp}(\zeta) : k(\zeta) \to k(\eta)$ is an isomorphism;*
(v) *every irreducible component of X contains $g^{-1}(s_0)$.*

Then Y is reduced and $f \circ i : X_{red} \to Y$ is an isomorphism. Moreover the open set of reduced points of X contains the generic point of $g^{-1}(s_0)$. Assumption (a) implies Assumption (b) if R is a localization of a finitely generated A-algebra, or if A is a G-ring (cf. [13] *§ 34, or* [14] *§ 32).*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of A. One has $A_{\mathfrak{p}_i} = k(\mathfrak{p}_i)$ for every i since A is reduced. Tensoring $0 \to A \to \oplus_i A_{\mathfrak{p}_i}$ by R one obtains by Assumption (b) that R is reduced. Assumption (i) implies that $X \cong Spec\, B$, where $B = \mathcal{O}_X(X)$ is a finite R-module and moreover $f^\sharp(Y) : R \to B$ is injective, since f is surjective and R is reduced. The points of X where f is not flat form a closed subset $Z \subset X$. The image $f(Z)$ is closed in Y since f is finite. Let $Y' = Y \setminus f(Z)$, $X' = f^{-1}(Y')$. Then $f|_{X'} : X' \to Y'$ is finite and flat.

We claim that $\zeta \in Y'$. One applies [13, 20.G] to $\mathcal{O}_{S,s_0} \to \mathcal{O}_{Y,\zeta} \to \mathcal{O}_{X,\eta}$ and $M = \mathcal{O}_{X,\eta}$. By hypothesis $\mathcal{O}_{Y,\zeta}$ and $\mathcal{O}_{X,\eta}$ are flat $\mathcal{O}_{S,s_0}$-modules. Furtermore Y_{s_0} is integral by Assumption (a), so $\mathcal{O}_{Y,\zeta} \otimes_{\mathcal{O}_{S,s_0}} k(s_0) \cong \mathcal{O}_{Y_{s_0},\zeta} \cong k(\zeta)$ is a field. Hence $\mathcal{O}_{X,\eta} \otimes_{\mathcal{O}_{S,s_0}} k(s_0)$ is a flat $\mathcal{O}_{Y,\zeta} \otimes_{\mathcal{O}_{S,s_0}} k(s_0)$-module. Therefore $\mathcal{O}_{X,\eta}$ is a flat $\mathcal{O}_{Y,\zeta}$-module. The hypothesis that $g^{-1}(s_0)$ is irreducible implies that $f^{-1}(\zeta) = \eta$, therefore $\zeta \in Y', \eta \in X'$.

Assumption (i) and Assumption (v) imply that every irreducible component of Y, being an image of some irreducible component of X, contains $h^{-1}(s_0)$ and in particular ζ. Therefore the open set Y' is connected and dense in Y. We claim that $f|_{X'} : X' \to Y'$ is an isomorphism. First, $f_*\mathcal{O}_{X'}$ is a locally free sheaf of a certain rank $d \geq 1$, since $f|_{X'}$ is finite, surjective and flat, and Y' is connected. One has $d = \dim_{k(y)} \Gamma(X_y, \mathcal{O}_{X_y})$ for every $y \in Y'$. Let $y = \zeta$. Then $f^{-1}(y) = \eta$ and furthermore $f : X \to Y$ is unramified at η. Indeed, it suffices to verify that $f|_{X_{s_0}} : X_{s_0} \to Y_{s_0}$ is unramified at η. This holds since $\mathcal{O}_{Y_{s_0},\zeta} = k(\zeta)$, $\mathcal{O}_{X_{s_0},\eta} = k(\eta)$, for by hypothesis X_{s_0} is irreducible and generically reduced, and furthermore $k(\zeta) \to k(\eta)$ is an isomorphism by Assumption (iv). We obtain that for $y = \zeta$, $X_y = Spec\, k(\eta)$ and $d = \dim_{k(y)} \Gamma(X_y, \mathcal{O}_{X_y}) = 1$. This shows that the morphism of sheaves of rings $f^\sharp : \mathcal{O}_{Y'} \to f_*\mathcal{O}_{X'}$ makes $f_*\mathcal{O}_{X'}$ a locally free $\mathcal{O}_{Y'}$-module of rank 1. This implies that $f^\sharp : \mathcal{O}_{Y'} \to f_*\mathcal{O}_{X'}$ is an isomorphism, hence $f|_{X'} : X' \to Y'$ is an isomorphism. Since Y is reduced this implies that the open set of reduced points of X contains X', in particular the generic point η of $g^{-1}(s_0)$ is a reduced point of X.

In order to prove the isomorphism $f \circ i : X_{red} \xrightarrow{\sim} Y$ we replace X by X_{red} and observe that all the assumptions of the theorem hold for $f \circ i = f_{red} : X_{red} \to Y$ and $g_{red} : X_{red} \to S$. We may thus assume that $X = Spec\, B$ is reduced, so Assumption (v) holds for every $V(\mathfrak{p})$, where $\mathfrak{p}$ is an associated prime of B.

Let I be the radical ideal $I = I(Y \setminus Y') \subset R$. We will prove below (see § 1.4) that if $I \neq R$, then $depth(I, R) \geq 2$. Assuming this statement one proves that $f : X \to Y$ is an isomorphism as follows. Consider the exact sequence of finite R-modules

$$0 \to R \xrightarrow{f^\sharp(Y)} B \to Q \to 0. \tag{3}$$

Let R' be the image of R. By way of contradiction let us assume $Q \neq 0$. Let $J = rad(Ann(Q))$. One has $V(J) = Supp\, Q$, so $Y \setminus V(J) \subset Y'$. The isomorphism $f^{-1}(Y') \xrightarrow{\sim} Y'$, proved above, shows that $Y' \subset Y \setminus V(J)$. Therefore $I = J \neq R$. The stated inequality $depth(I, R) \geq 2$ implies by [14, Theorem 16.6] that $Ext^1_R(Q, R) = 0$. Hence (3) splits, $B \cong R' \oplus Q'$. Let $\mathfrak{p} = Ann_R(x)$ be an associated prime ideal of Q. Since $Ass(Q) \subset Supp(Q) = V(J)$ (cf. [14, Theorem 6.5]), one has $\mathfrak{p} \supset I$. Let $\mathfrak{p}' \subset R'$ be the image of $\mathfrak{p}$ and let $x' \in Q'$ be the preimage of x. Since $\mathfrak{p}' \cdot x' = 0$ the subset $\mathfrak{p}' \subset B$ consists of zero divisors, so $\mathfrak{p}' \subset P_1 \cup \dots \cup P_m$, where $P_i, i = 1, \dots, m$, are the associated prime ideals of B. Let $\mathfrak{p}_i = R \cap P_i$. Then

$\mathfrak{p} \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m$, so $\mathfrak{p} \subset \mathfrak{p}_j$ for some j. Let $P = rad(\mathfrak{m}B)$. Assumption (v) means that $P_i \subset P$ for every i. Hence $R \cap P \supset \mathfrak{p}_j \supset \mathfrak{p} \supset I$. Since $P \in Spec\, B$ and $R \cap P \in Spec\, R$ are the same as $\eta \in X$ and $\zeta \in Y$ respectively, one obtains that ζ belongs to $V(I) = Y \setminus Y'$. This contradiction shows that $Q = 0$ and therefore $f^\sharp(Y) : R \to B$ is an isomorphism. The isomorphism $f \circ i : X_{red} \to Y$ is proved.

It remains to prove the last statement of the theorem. Let C be a finitely generated A-algebra. Let $T = Spec\, C$, $u : T \to S = Spec\, A$ be the morphism corresponding to $A \to C$. Suppose there is an A-isomorphism of R with $\mathcal{O}_{T,z}$ for some $z \in T$. Let $j : Y \to T$ be the composition $Y \xrightarrow{\sim} Spec\, \mathcal{O}_{T,z} \longrightarrow T$. One has $u(z) = s_0$ and $\mathcal{O}_{T,z} \otimes_{\mathcal{O}_{S,s_0}} k(s_0) \cong \mathcal{O}_{T_{s_0},z}$. Assumption (a) implies, according to [11, Corollaire II.5.10], that the fiber T_{s_0} is smooth at z, so u is smooth at z by [1, Theorem VII.1.8]. Smoothness is an open condition so we may, replacing T by an affine neighborhood of z, assume that $u : T \to S$ is smooth. It is moreover surjective since the flat morphism u is an open map. Every fiber $T_s, s \in S$ is geometrically regular. Let $s \in S, y \in h^{-1}(s), t = j(y)$. One has $\mathcal{O}_{S,s}$-isomorphisms

$$\mathcal{O}_{Y_s,y} \cong \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{S,s}} k(s) \cong \mathcal{O}_{T_s,t}.$$

Therefore $\mathcal{O}_{Y_s,y}$ is geometrically regular. This means that $R \otimes_A k(\mathfrak{p})$ is geometrically regular ring for every $\mathfrak{p} \in Spec\, A$, in particular Assumption (b) holds.

Suppose now that A is a G-ring. Then A is quasi-excellent (cf. [13, § 34]. Let $k = k(\mathfrak{m})$, $\mathfrak{n}_0 = \mathfrak{n}/\mathfrak{m}R \subset R \otimes_A k$. Assumption (a) implies that $R \otimes_A k$ is formally smooth with respect to the $\mathfrak{n}_0$-adic topology (cf. [13, § 28.M] Proposition). Hence by Théorèm 19.7.1 of [8, Ch.0] the homomorphism $A \to R$ is formally smooth with respect to the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies of A and R. A theorem of André [2] yields that $A \to R$ is a regular homomorphism, so $R \otimes_A k(\mathfrak{p})$ is geometrically regular for every $\mathfrak{p} \in Spec\, A$. This implies Assumption (b). □

Our next goal is to prove that $depth(I, R) \geq 2$ provided $I \neq R$, a statement used in the proof of Theorem 1.1. It is proved in [7, Ch.0] Proposition 10.3.1 that, given a Noetherian local ring $(A, \mathfrak{m})$ and a homomorphism of fields $k(\mathfrak{m}) \to K$, there exists a Noetherian local ring (B, J) and a flat local homomorphism $(A, \mathfrak{m}) \to (B, J)$ such that $\mathfrak{m}B = J$ and $k(J)$ is isomorphic to K over $k(\mathfrak{m})$. We include for reader's convenience a proof of the following known fact.

Lemma 1.2. *Let $(A, \mathfrak{m}) \to (R, \mathfrak{n})$ be a flat local homomorphism of local Noetherian rings. Let $k = k(\mathfrak{m})$, $K = k(\mathfrak{n})$. Suppose K is separable over k. Suppose $R \otimes_A k$ is a regular ring. Let $(A, \mathfrak{m}) \to (B, J)$ be a flat local homomorphism as above: $\mathfrak{m}B = J$, $k(J)$ is k-isomorphic to K. Then there is an A-isomorphism $\hat{R} \cong \hat{B}[[T_1, \ldots, T_n]]$, where $\hat{R}$ is the $\mathfrak{n}$-adic completion of R and $\hat{B}$ is the J-adic completion of B.*

Proof. Let us first consider the case when $k(\mathfrak{m}) \to k(\mathfrak{n})$ is an isomorphism and $(B, J) = (A, \mathfrak{m})$. We have a commutative diagram of faithfully flat homomorphisms (cf. [14, Theorem 22.4])

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & & \downarrow \\ \hat{A} & \longrightarrow & \hat{R} \end{array} \tag{4}$$

Let $t_1, \ldots, t_n \in \mathfrak{n}$ be elements such that $x_i = t_i (mod\, \mathfrak{m}R), i = 1, \ldots, n$ generate the maximal ideal of the regular local ring $R/\mathfrak{m}R \cong R \otimes_A k$. Let $\varphi : \hat{A}[[T_1, \ldots, T_n]] \longrightarrow$

$\hat{R}$ be the homomorphism of $\hat{A}$-algebras such that $\varphi(T_i) = t_i$ (cf. [5, Theorem 7.16]). Let $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$ and $\hat{\mathfrak{n}} = \mathfrak{n}\hat{R}$ be the maximal ideals of $\hat{A}$ and $\hat{R}$. The ring $A' = \hat{A}[[T_1, \ldots, T_n]]$ is local and complete with maximal ideal $M = (\hat{\mathfrak{m}}, T_1, \ldots, T_n)$ (cf. [4, Ch. III, § 2.6]). One has $\varphi(M) = \hat{\mathfrak{n}}$, so $\hat{R}/M\hat{R} \cong \hat{R}/\hat{\mathfrak{n}} \cong R/\mathfrak{n} \cong k(\mathfrak{n}) \cong k$. Hence $\varphi : A' \to \hat{R}$ is surjective (cf. [14, Theorem 8.4]). In order to prove that φ is injective, applying [14, Theorem 22.5], we need to verify that $\overline{\varphi} : A' \otimes_{\hat{A}} k \to \hat{R} \otimes_{\hat{A}} k$ is injective. One has $\hat{R} \otimes_{\hat{A}} k \cong \hat{R}/\hat{\mathfrak{m}}\hat{R} \cong \widehat{R/\mathfrak{m}R}$. By assumption the composition $k = k(\mathfrak{m}) \to R/\mathfrak{m}R \to R/\mathfrak{n} = k(\mathfrak{n})$ is an isomorphism. Hence $\widehat{R/\mathfrak{m}R} \cong k[[x_1, \ldots, x_n]]$ (cf. [3, p. 124, Remark 2]). Therefore the composition

$$k[[T_1, \ldots, T_n]] \xrightarrow{\sim} A'/\hat{\mathfrak{m}}A' \xrightarrow{\overline{\varphi}} \hat{R}/\hat{\mathfrak{m}}\hat{R} \xrightarrow{\sim} k[[x_1, \ldots, x_n]]$$

which transforms T_i in x_i is an isomorphism. This implies that $\overline{\varphi} : A' \otimes_{\hat{A}} k \to \hat{R} \otimes_{\hat{A}} k$ is an isomorphism. We conclude that $\varphi : \hat{A}[[T_1, \ldots, T_n]] \to \hat{R}$ is an isomorphism provided $k(\mathfrak{m}) \to k(\mathfrak{n})$ is an isomorphism.

Let us consider now the general case when $k = k(\mathfrak{m}) \to k(\mathfrak{n}) = K$ is an arbitrary separable extension. By [14, Theorem 26.9] K is 0-smooth over k. Hence by [14, Theorem 28.10] B is J-smooth over A. This implies that the homomorphism $B \to K = \hat{R}/\hat{\mathfrak{n}}$ has a lifting $\varphi : B \to \hat{R}$ which is a local homomorphism of A-algebras (see [14, p. 214]). Applying [13, 20.G] to $A \to B \to \hat{R}$, taking into account that $B \otimes_A k \to \hat{R} \otimes_A k$ is flat since $B \otimes_A k \cong K$ is a field, we conclude that $B \to \hat{R}$ is flat. Furthermore $k(J) = B/J \to \hat{R}/\hat{\mathfrak{n}} \cong R/\mathfrak{n} = k(\mathfrak{n})$ is an isomorphism and

$$\hat{R} \otimes_B k(J) \cong \hat{R} \otimes_B (B \otimes_A k) \cong \hat{R} \otimes_A k \cong \hat{R}/\mathfrak{m}\hat{R} \cong \widehat{R/\mathfrak{m}}$$

is a regular local ring of dimension n. By the first part of the proof one concludes that $\hat{R} \cong \hat{B}[[T_1, \ldots, T_n]]$ □

Proposition 1.3. *Let $A \to R$ be a flat local homomorphism of Noetherian local rings. Set $\mathfrak{m} = rad(A)$, $\mathfrak{n} = rad(R)$. Suppose $k = k(\mathfrak{m}) \to k(\mathfrak{n}) = K$ is a separable extension. Suppose $R \otimes_A k$ is a regular ring. Let $I \subset R$ be a proper ideal such that:*

(a) *none of the prime ideals of the set $A \cap V(I) \subset Spec\,A$ is contained in an associated prime ideal of A;*
(b) *$I \not\subset \mathfrak{m}R$.*

Then $depth(I, R) \geq 2$.

Proof. We may replace I by its radical and thus assume that $I = rad(I)$. Indeed, $depth(I, R) = depth(rad(I), R)$ (see [5, Corollary 17.8]), $V(I) = V(rad(I))$ and $I \not\subset \mathfrak{m}R$ if and only if $rad(I) \not\subset \mathfrak{m}R$ since the condition that $R \otimes_A k$ is regular implies that $\mathfrak{m}R$ is a prime ideal. Let $I = P_1 \cap \cdots \cap P_r$, where $P_i, i = 1, \ldots, r$ are the minimal prime ideals which contain I. We claim that there exists $a_1 \in A \cap I$, such that a_1 is not a zero divisor of A. If this were not the case, then $A \cap I \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$, where $\mathfrak{p}_i, i = 1, \ldots, s$ are the associated primes of A. Then $A \cap I \subset \mathfrak{p}_j$ for some j and consequently $A \cap P_i \subset \mathfrak{p}_j$ for some i (cf. [3, Proposition 1.11]). This contradicts Condition (a).

Let $\hat{I}$, $\hat{R}$ be the $\mathfrak{n}$-adic completions of I and R. The equality $depth(I, R) = depth(\hat{I}, \hat{R})$ holds. Indeed, let $I = (x_1, \ldots, x_m)$. Consider the Koszul complex $K^\bullet = K^\bullet(x_1, \ldots, x_m)$, where $K^i(x_1, \ldots, x_m) = \Lambda^i N, N = \oplus_{i=1}^m Re_i$ and $d^i : K^i \to K^{i+1}$ is $d^i(v) = x \wedge v$, where $x = \sum_{i=1}^m x_i e_i$. Then $depth(I, R) = r$ iff $H^i(K^\bullet) = 0$

for $i < r$ and $H^r(K^\bullet) \neq 0$ (cf. [5, Theorem 17.4]). Since $\hat{I} = I\hat{R}$ the images x_i' of x_i in $\hat{R}$, $i = 1, \ldots, m$, generate $\hat{I}$. The corresponding Koszul complex is $K^{'\bullet} = K'(x_1', \ldots, x_m') \cong K^\bullet(x_1, \ldots, x_m) \otimes_R \hat{R}$. Since $R \to \hat{R}$ is faithfully flat one has that $H^i(K^{'\bullet}) \cong H^i(K^\bullet) \otimes_R \hat{R}$. By the above criterion $depth(I, R) = depth(\hat{I}, \hat{R})$.

Let $(A, \mathfrak{m}) \to (B, J)$ and $\hat{R} \cong \hat{B}[[T_1, \ldots, T_n]]$ be as in Lemma 1.2. The hypothsis $I \not\subset \mathfrak{m}R$ implies $\hat{I} = I\hat{R} \not\subset (\mathfrak{m}R)\hat{R} = \mathfrak{m}\hat{R}$ since $R \to \hat{R}$, being a faithfully flat homomorphism, has the property that $\mathfrak{a}\hat{R} \cap R = \mathfrak{a}$ for every ideal $\mathfrak{a}$ in R. Furthermore the ring extensions $A \to B \to \hat{B} \to \hat{R}$ yield $\mathfrak{m}\hat{R} = (\mathfrak{m}B)\hat{R} = J\hat{R} = \hat{J}\hat{R}$. Therefore $\hat{I} \not\subset \hat{J}\hat{R}$.

Let $a_1 \in A \cap I$ be a non zero divisor of A as above. Let $a_1' \in \hat{I}$ be its image in $\hat{R}$. Let $f \in \hat{I} \setminus \hat{J}\hat{R}$. We claim that a_1', f is an $\hat{R}$-regular sequence. This implies that $depth(\hat{I}, \hat{R}) \geq 2$. Abusing notation we identify $\hat{R}$ with $\hat{B}[[T_1, \ldots, T_n]]$. Let $\overline{f} = f(mod\, \hat{J}\hat{R}) \in K[[T_1, \ldots, T_n]]$. There exist positive integers $u_1, u_2, \ldots, u_n$ such that the automorphism s of $K[[T_1, \ldots, T_n]]$ defined by $s(T_i) = T_i + T_n^{u_i}$ for $1 \leq i \leq n-1$ and $s(T_n) = T_n$ transforms $\overline{f}$ in $\overline{g}(T_1, \ldots, T_n)$ with $\overline{g}(0, \ldots, 0, T_n) \neq 0$ (cf. [4, Ch.VII § 3 no.7] Lemma 3). The same substitution yields a $\hat{B}$-automorphism φ of $\hat{B}[[T_1, \ldots, T_n]]$. Let $g = \varphi(f)$. Let $C = \hat{B}[[T_1, \ldots, T_{n-1}]]$. This is a complete local ring with maximal ideal $\mathfrak{M} = (\hat{J}, T_1, \ldots, T_{n-1})$ and $\hat{R} \cong C[[T_n]]$. We claim that a_1', g is a regular $\hat{R}$-sequence. Indeed, the injectivity of $A \xrightarrow{\cdot a_1} A$ implies the injectivity of $\hat{R} \xrightarrow{\cdot a_1} \hat{R}$ since the composition $A \to R \to \hat{R}$ is flat. One has $g(mod\, \mathfrak{M}) = \overline{g}(0, \ldots, 0, T_n) \neq 0$ and $\hat{R}/a_1'\hat{R} \cong (\hat{B}/a_1'\hat{B})[[T_1, \ldots, T_n]] \cong \overline{C}[[T_n]]$, where $\overline{C} = (\hat{B}/a_1'\hat{B})[[T_1, \ldots, T_{n-1}]]$ is a complete local ring with maximal ideal $(\hat{J}/a_1'\hat{J}, T_1, \ldots, T_{n-1})$. Applying [4, Ch.VII § 3 no.8] Proposition 5 to the image of $g(mod\, a_1'\hat{R})$ in $\overline{C}[[T_n]]$ we conclude that $g(mod\, a_1'\hat{R})$ is not a zero divisor in $\hat{R}/a_1'\hat{R}$. Therefore a_1', g is a regular $\hat{R}$-sequence. Since $\varphi(a_1') = a_1', \varphi(f) = g$, the same holds for a_1', f with $a_1', f \in \hat{I}$. We thus obtain that $depth(I, R) = depth(\hat{I}, \hat{R}) \geq 2$. □

1.4 (*End of proof of Theorem 1.1*)**.** Recall that we have reduced the proof of the theorem to the case of reduced $X = Spec\, B$ and we have assumed by way of contradiction that $I = I(Y \setminus Y') \not\subset R$. We prove that $depth(I, R) \geq 2$ applying Proposition 1.3. The condition $I \not\subset \mathfrak{m}R$ is fulfilled since $\mathfrak{m}R = \zeta \in Y'$. We want to verify that Condition (a) of Proposition 1.3 holds. By hypothesis A is reduced, so one needs to prove that $A \cap V(I)$ contains none of the minimal prime ideals of A. Let $X_i = V(P_i), i = 1, \ldots, n$ be the irreducible components of X. Assumption (i) and Assumption (v) imply, using that $\eta \in X'$ and $\zeta = f(\eta) \in Y'$, that there is a bijective correspondence between the irreducible components of X and those of Y given by

$$X_i \mapsto X_i \cap X' \mapsto f(X_i \cap X') \mapsto \overline{f(X_i \cap X')} = f(X_i) = Y_i.$$

Let us give to $X_i \subset X$ and $Y_i \subset Y$ the structure of reduced closed subschemes and let $f_i : X_i \to Y_i$ be the morphism induced by f, $i = 1, \ldots, n$. For every i the affine schemes X_i, Y_i are integral, the morphism f_i is finite and if $x_i \in X_i$, $y_i \in Y_i$ are the generic points, the homomorphism

$$(f_i^\sharp)_{y_i} : k(y_i) = \mathcal{O}_{Y_i, y_i} \longrightarrow \mathcal{O}_{X_i, x_i} = k(x_i)$$

is an isomorphism since it coincides with $\mathcal{O}_{Y, y_i} \to \mathcal{O}_{X, x_i}$ and $y_i \in Y', x_i = f^{-1}(y_i) \in X'$. Let $Y^{reg} = \{u \in Y | \mathcal{O}_{Y,y} \text{ is a regular ring}\}$. We claim that $Y^{reg} \subset Y'$. Let $y \in Y^{reg}$. One has that $y \in Y_i \setminus \cup_{j \neq i} Y_j$ for some i. The regular ring $\mathcal{O}_{Y,y} = \mathcal{O}_{Y_i, y}$ is integrally closed in its field of fractions $k(y_i)$. The finite injective homomorphism of integral domains $f_i^\sharp(Y_i) : \mathcal{O}_{Y_i}(Y_i) \to \mathcal{O}_{X_i}(X_i)$ induces an isomorphism of the fields of fractions $k(y_i) \xrightarrow{\sim} k(x_i)$, hence $f_i^{-1}(y)$ consists of a unique point $x \in X_i$ and $(f_i^\sharp)_y : \mathcal{O}_{Y_i,y} \to \mathcal{O}_{X_i,x}$ is an isomorphism. One has $f^{-1}(y) = f_i^{-1}(y) = \{x\}$ since $y \in Y_i \backslash \cup_{j \neq i} Y_j$. Furthermore $\mathcal{O}_{X_i,x} = \mathcal{O}_{X,x}$ since X is reduced and $x \in X_i \backslash \cup_{j \neq i} X_j$. Therefore $(f^\sharp)_y : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism, so $y \in Y'$. The claim that $Y^{reg} \subset Y'$ is proved. Suppose now that $\mathfrak{q} \in V(I) = Y \setminus Y'$ and $A \cap \mathfrak{q} = \mathfrak{p}$ is a minimal prime ideal of A. Let $S = A \setminus \mathfrak{p}$. One has by Assumption (b) that $R \otimes_A k(\mathfrak{p}) = R \otimes_A A_{\mathfrak{p}} = S^{-1}R$ is a regular ring. Hence $R_{\mathfrak{q}}$ is a regular ring and $\mathfrak{q} \in Y^{reg}$ which contradicts the inclusion $Y^{reg} \subset Y'$ proved above. We thus prove that Condition (a) of Proposition 1.3 holds, therefore $depth(I, R) \geq 2$. Theorem 1.1 is proved.

2. Some corollaries of Theorem 1.1

Lemma 2.1. *Let $g : X \to S$ be a morphism of finite type of Noetherian schemes. Suppose there is a $d \in \mathbb{N}$ such that every fiber $g^{-1}(y)$ is irreducible of dimension d. Then every irreducible component of X is a union of fibers of $g : X \to S$. If g is closed, or if g is flat at the generic point of every irreducible component of X, then there is a bijective correspondence between the irreducible components of X and those of S given by $X_i \mapsto \overline{g(X_i)} = S_i$ as well as by $S_i \mapsto g^{-1}(S_i) = X_i$ provided g is closed.*

Proof. Let $X = \cup_{i=1}^n X_i$ be an irredundant union of irreducible components of X. Let $i \in [1, m]$. Let $Z = \overline{g(X_i)}$. Let $g^{-1}(Z) = T_1 \cup T_2 \cup \cdots \cup T_p$ be an irredundant union of closed irreducible sets where $T_1 = X_i$. Let η be the generic point of Z and let $\zeta_1, \zeta_2, \ldots, \zeta_p$ be the generic points of $T_1, T_2, \ldots, T_p$ respectively. Then $\overline{g(\zeta_1)} = \eta$ and for $i \geq 2$ one has $g(\zeta_i) \neq \eta$ since $g^{-1}(\eta)$ is irreducible. Therefore $\overline{g(\zeta_i)} \subsetneq Z$, so $g^{-1}(\eta) \subset T_1 = X_i$. Let $g_i = g|_{X_i}$. Let $y \in g(X_i)$. According to [10, Lemme 13.1.1] $\dim g_i^{-1}(y) \geq \dim g_i^{-1}(\eta) = d$. Furthermore $g_i^{-1}(y)$ is closed in $g^{-1}(y)$ and $g^{-1}(y)$ is irreducible of dimension d by hypothesis. Therefore $g_i^{-1}(y) = g^{-1}(y)$, so $g^{-1}(y) \subset X_i$. This proves the claim that $X_i = \cup_{y \in g(X_i)} g^{-1}(y)$. If g is closed then $Y_i = g(X_i)$ is closed in Y and $X_i = g^{-1}(Y_i)$ as shown above. Hence $Y = \cup_{i=1}^m Y_i$ is an irredundant union of irreducible closed subsets. Let ξ_i be the generic point of $X_i, i = 1, \ldots, m$. If $g : X \to S$ is flat at ξ_i then $g(X_i)$ contains an open subset of S. This implies that $Y_i = \overline{g(X_i)}$ is an irreducible component of Y. As proved above X_i is the unique irreducible component of X_i which dominates Y_i. Therefore if $g : X \to S$ is flat at every ξ_i, then $Y = \cup_{i=1}^m Y_i$ is an irredundant union of irreducible components. □

Theorem 2.2. *Let X, Y and S be schemes over a field k of characteristic 0. Suppose S is reduced and locally Noetherian. Let*

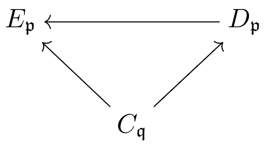
(5)
$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & S & \end{array}$$

be a commutative diagram of morphisms over k such that:

(a) *h is locally of finite type, flat, and every* $y \in Y$ *is a regular point of the fiber* $Y_{h(y)}$*;*
(b) f *is finite and surjective;*
(c) *every nonempty fiber of* g *is irreducible, generically reduced, and* g *is flat at its generic point;*
(d) *if* $s \in h(Y)$ *and* $x_s \in X_s$*,* $y_s \in Y_s$ *are the generic points, then* $f^\sharp(y_s) : k(y_s) \to k(x_s)$ *is an isomorphism.*

Then $f \circ i : X_{red} \to Y$ *is an isomorphism and the closed subset of nonreduced points of* X *contains no fibers* $g^{-1}(g(x)), x \in X$*.*

Proof. Every fiber $Y_s, s \in h(Y)$ is smooth since $char(k) = 0$ (see [11, Corollaire II.5.10]), so h is a smooth morphism. The statements of the teorem are of local character, so we may suppose that X, Y and S are affine Noetherian schemes over k, X and Y are of finite type over S and there is an integer $d \in \mathbb{N}$ such that $dimY_s = d$ for every $s \in h(Y)$ [1, Proposition VII.1.4]. Let $W \subset S$ be the open set $W = h(Y)$. One has $dimX_s = d$ for every $s \in W$ since f is finite and surjective. By Lemma 2.1 every irreducible component of X is a union of fibers of $X \to W$. Let $S = Spec\, C$, $Y = Spec\, D$, $X = Spec\, E$. In order to prove that $D = \mathcal{O}(Y) \to \mathcal{O}(X_{red}) = E/N$ is an isomorphism it suffices to prove that for every $\mathfrak{p} \in Spec\, D$ the localization $D_\mathfrak{p} \to E_\mathfrak{p}/N_\mathfrak{p}$ is an isomorphism. Let $\mathfrak{p} \in Spec\, D$, $\mathfrak{q} = h(\mathfrak{p}) = \mathfrak{p} \cap C$. Consider the diagram obtained from (5) after localization



We claim the assumptions of Theorem 1.1 are fulfilled for $A = C_\mathfrak{q}$, $R = D_\mathfrak{p}$, $Spec\, E_\mathfrak{p} \to Spec\, D_\mathfrak{p}$. Assumption (a) and Assumption (b) hold since $Spec\, D \to Spec\, C$ is a smooth morphism and $char(k) = 0$. Assumption (i) holds for $Spec\, E_\mathfrak{p} \to Spec\, D_\mathfrak{p}$ since by hypothesis $D \to E$ is a finite homomorphism and $Spec\, E \to Spec\, D$ is surjective.

In order to verify Assumption (ii) we observe that the hypothesis that $Spec\, E \otimes_C k(\mathfrak{q}) \cong Spec\, E_\mathfrak{q}/\mathfrak{q}E_\mathfrak{q}$ is irreducible and generically reduced is equivalent to the statement that the ideal $\mathfrak{q}E_\mathfrak{q}$ of $E_\mathfrak{q}$ has an irredundant primary decomposition $\mathfrak{q}E_\mathfrak{q} = Q_1 \cap Q_2 \cap \cdots Q_m$, where $Q_1 = P_1$ is a prime ideal and $P_1 \subsetneqq rad(Q_i)$ for $i \geq 2$. Let $P_1 = P_\mathfrak{q}$ where $P \in Spec\, E$. We claim that $P \cap D \subset \mathfrak{p}$. Indeed, by the surjectivity of $Spec\, E \to Spec\, D$ there exists a prime ideal $P' \in Spec\, E$ such that $P' \cap D = \mathfrak{p}$. One has $P' \cap C = \mathfrak{q}$ since $\mathfrak{p} \cap C = \mathfrak{q}$. The ideal P is the minimal prime ideal of E among those which intersect C in $\mathfrak{q}$, hence $P' \supset P$ and consequently $P \cap D \subset \mathfrak{p}$. Let $S = C \setminus \mathfrak{p}$. Localizing one obtains a primary decomposition $\mathfrak{q}E_\mathfrak{p} = S^{-1}P \cap (\cap_{i \geq 2} S^{-1}Q_i)$, where $S^{-1}P = S^{-1}P_1 \subsetneqq S^{-1}rad(Q_i) = rad(S^{-1}Q_i)$ for $i \geq 2$. This proves that $Spec\,(E_\mathfrak{p}/\mathfrak{q}E_\mathfrak{p})$ is irreducible and generically reduced, which is what Assumption (ii) requires.

Assumption (iii) for $Spec\, E_\mathfrak{p} \to Spec\, C_\mathfrak{q}$ and Assumption (iv) for $Spec\, E_\mathfrak{p} \to Spec\, D_\mathfrak{p}$ follow from Assumption (c) and Assumption (d) of the theorem. It remains to verify Assumption (v). Let $V(S^{-1}P_0)$ be an arbitrary irreducible component of $Spec\, E_\mathfrak{p}$. Here P_0 is a minimal prime ideal of E such that $P_0 \cap D \subset \mathfrak{p}$. By the going-up theorem, there exists a prime ideal $\mathfrak{p}_1$ of E such that $\mathfrak{p}_1 \supset P_0$ and $\mathfrak{p}_1 \cap D = \mathfrak{p}$. One has $\mathfrak{p}_1 \cap C = \mathfrak{q}$. The irreducible component $V(P_0)$ of $Spec\, E = X$ is a union of fibers of g by Lemma 2.1, hence every $\mathfrak{p}_x \in Spec\, E$ such that $\mathfrak{p}_x \cap C = \mathfrak{q}$ contains P_0. In particular this holds for those $\mathfrak{p}_x$ with $\mathfrak{p}_x \cap D \subset \mathfrak{p}$. The latter subset of $Spec\, E$ corresponds bijectively to the closed fiber of $Spec\, E_{\mathfrak{p}} \to Spec\, C_{\mathfrak{q}}$, hence this fiber is contained in $V(S^{-1}P_0)$.

Applying Theorem 1.1 we conclude that $D_{\mathfrak{p}} \to E_{\mathfrak{p}}/N_{\mathfrak{p}}$ is an isomorphism, where N is the nilradical of E. This holds for every $\mathfrak{p} \in Spec\, D$, therefore $D \to E/N$ is an isomorphism. Let $\mathfrak{q} \in g(Spec\, E) \subset Spec\, C$. Let P be the minimal prime ideal of E with the property that $P \cap C = \mathfrak{q}$ (Assumption (c)). Let $P \cap D = \mathfrak{p}$. As we have shown above if $S = D \setminus \mathfrak{p}$, then $S^{-1}P$ is the generic point of the closed fiber of $Spec\, E_{\mathfrak{p}} \to Spec\, C_{\mathfrak{q}}$. According to Theorem 1.1 the local ring $(E_{\mathfrak{p}})_{S^{-1}P}$ is reduced. Since $(E_{\mathfrak{p}})_{S^{-1}P} = E_P$ we conclude that $P \notin Supp\, N = V(Ann(N))$. The statements of Theorem 2.2 that $f \circ i : X_{red} \to Y$ is an isomorphism and the open set of reduced points of X intersects every nonempty fiber of $g : X \to S$ are proved. □

If X is a scheme of finite type over a field k and $W \subset X$ is a locally closed subset we denote by W_0 the set of closed points of W. This is a dense subset of W [6, Proposition 3.35]. A morphism $f : X \to Y$of schemes of finite type over a field k is surjective if and only if $f|_{X_0} : X_0 \to Y_0$ is surjective [6, Exercise 10.6].

In the next proposition we use an argument we have found in [15].

Proposition 2.3. *Let X and S be schemes of finite type over a field k and let $g : X \to S$ be a morphism over k. Suppose there is a $d \in \mathbb{N}$ such that for every closed point $s \in S$ the fiber X_s is irreducible and* $\dim X_s = d$*. Then every irreducible component of S is dominated by a unique irreducible component of X and every irreducible component of X is a union of fibers of g. In each of the following three cases there is a bijective correspondence between the irreducible components of X and those of S given by $X_i \mapsto \overline{g(X_i)}$ as well as by $S_i \mapsto g^{-1}(S_i)$ in Case (a).*

(a) *g is a closed morphism.*
(b) *X is an equidimensional scheme.*
(c) *Every irreducible component of X contains a point in which g is flat.*

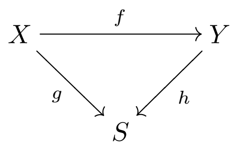
Proof. Let us first suppose that S is irreducible. Let $X = X_1 \cup \cdots \cup X_n$ be an irredundant union of irreducible components of X. Let $y \in S$ be a closed point. By hypothesis $g^{-1}(s)$ is irreducible, so $g^{-1}(s) \subset X_j$ for some j. The hypothesis implies that $g|_{X_0} : X_0 \to S_0$ is surjective, consequently $S = \overline{f(X_i)}$ for some i. Renumbering we may suppose that $i = 1$. Let $U = X_1 \setminus \cup_{i \geq 2} X_i$. If x is a closed point of U and $s = g(x)$, then X_1 is the unique irreducible component of X which contains $g^{-1}(s)$. By a theorem of Chevalley [13, 6.E and 6.C] $g(U)$ contains an open subset $V \subset S, V \neq \emptyset$ and one has $g^{-1}(s) \subset X_1$ for every $s \in V_0$. We claim that $X_i \subset g^{-1}(S \setminus V)$ for $i \geq 2$. Let $x \in (X_i \setminus X_1)_0$. Then $s = g(x) \notin V$. The inclusion $(X_i \setminus X_1)_0 \subset g^{-1}(S \setminus V)$ implies $X_i \subset g^{-1}(S \setminus V)$ since $(X_i \setminus X_1)_0$ is dense in X_i. Therefore X_1 is the unique irreducible component of X which dominates S. Let $\eta \in S$ be the generic point. Then $X_\eta = (X_1)_\eta$ is irreducible, moreover $\dim X_\eta = d$ since the fibers of $g|_{X_1} : X_1 \to S$ over the closed points of V are of dimension d (cf. [10, Theorem 13.1.3]).

Let now S be arbitrary and let $y \in S$. Let $Z = \overline{\{y\}}$. Applying the above argument to $Z \times_S X \to Z$ we conclude that Z is dominated by a unique irreducible component of $g^{-1}(Z)$, X_y is irreducible and $\dim X_y = d$. In particular these statements hold for every irreducible component $Z = S_i$ of S. By Lemma 2.1 every irreducible component of X is a union of fibers of $g : X \to S$.

In Case (a) and Case (c) the bijective correspondence between the irreducible components of X and those of S was proved in Lemma 2.1. In Case (b) it suffices to prove that if S is irreducible, then so is X. Applying to $X_i \to \overline{f(X_i)}$ the same arguments as those relative to $X_1 \to S$ we conclude that $\dim X_i = d + \dim \overline{f(X_i)}$. If X were not irreducible, then $\overline{f(X_i)} \subset S \setminus V$ for $i \geq 2$ would imply that $\dim X_i < \dim X_1$. Therefore $X = X_1$ is irreducible. □

Theorem 2.4. *Let X, Y and S be schemes of finite type over a perfect field k. Suppose S is reduced. Let*

(6)



be a commutative diagram of morphisms over k such that:

(a) *every closed point $y \in Y$ is a regular point of the fiber $Y_{h(y)}$ and h is flat at y;*
(b) *f is finite and $f|_{X_0} : X_0 \to Y_0$ is surjective;*
(c) *for every closed point $s \in h(Y)$ the fiber X_s is irreducible, generically reduced and g is flat at some point of X_s;*
(d) *for every closed point $s \in h(Y)$ the $k(s)$-morphism of integral schemes*

$$(f \otimes k(s))_{red} : (X_s)_{red} \to Y_s$$

is birational.

Then $f \circ i : X_{red} \to Y$ is an isomorphism and the closed subset of nonreduced points of X contains no fiber $g^{-1}(g(x)), x \in X$.

Proof. If $y \in Y$ is a closed point and $s = h(y)$ then $k(y)$ is a separable extension of $k(s)$ since $k(y)$ and $k(s)$ are finite extensions of the perfect field k. Assumption (a) implies that h is smooth at every closed point $y \in Y$. Since smoothness is an open condition, $h : Y \to S$ is a smooth morphism. Reasoning as in Theorem 2.2 we reduce the proof to the case $S = Spec\, C$, $Y = Spec\, D$, $X = Spec\, E$ where C, D and E are finitely generated algebras over k. Furtermore we may assume that there is a $d \in \mathbb{N}$ such that $\dim Y_s = d = \dim X_s$ for every $s \in h(Y)$. If s is a closed point of $h(Y)$ then by Assumption (c) X_s is irreducible. By Proposition 2.3 every irreducible component of X is a union of fibers of $g : X \to S$. We proceed in the same way as in the proof of Theorem 2.2 taking an arbitrary maximal ideal $\mathfrak{p}$ of D. Then $\mathfrak{q} = h(\mathfrak{p}) = \mathfrak{p} \cap C$ is a maximal ideal of C. The rest of the proof is a repetition of that of Theorem 2.2 observing that: the set of points of X where g is flat is open; $D \to E/N$ is an isomorphism if and only if $D_\mathfrak{p} \to (E/N)_\mathfrak{p}$ is an isomorphism for every maximal ideal $\mathfrak{p}$ of D. □

2.5 (*Proof of Theorem 0.1*)**.** The map $f : X \to Y$ is proper since $g : X \to S$ is proper. By hypothesis f is quasifinite, so by Zariski's main theorem f is a finite map [6, Corollary 12.89]. Assumption (c) means that g is smooth at x. Smoothness is an open condition, so g is smooth at the scheme-theoretic generic point $\eta \in g^{-1}(s)$.

This means in particular that the scheme-theoretic fiber X_s is generically reduced, so Assumption (c) of Theorem 2.4 holds. By Theorem 2.4 the map $f : X \to Y$ is an isomorphism.

3. A flatness criterion of Kollár

In this section we discuss Theorem 9 of [12] and point out two questionable points in its proof, which are the reason of writing the present paper. The theorem claims the following.

Theorem 3.1 (J. Kollár). *Let $\varphi : A \to B$ be a local homomorphism of Noetherian local rings. Set $\mathfrak{m} = rad(A)$, $\mathfrak{n} = rad(B)$. Suppose A is excellent. Set $X = Spec\, B$, $x = \mathfrak{n}$, $Y = Spec\, A$, $y = \mathfrak{m}$, $f = Spec\, \varphi : X \to Y$, $Z = X_y = Spec\, B \otimes k(y)$. Assume that*

(a) $\dim Z = 1$ *and the closed subscheme $(Z, \mathcal{O}_Z/torsion\ at\ x)$ of $(Z, \mathcal{O}_Z)$ is smooth;*
(b) *Y is reduced and every primary component of X contains Z;*
(c) *f is flat along $Z \setminus \{x\}$;*
(d) *if $char(k) > 0$ then assume in addition that $k(x)$ is finitely generated over $k(y)$.*

Then f is flat and Z is smooth.

The first questionable point is in the claim at the bottom of page 718 of [12], that it suffices to prove the theorem assuming that A and B are complete. Passing to completions $\hat{A} \to \hat{B}$ it is unclear however why the second part of Assumption (b) (this is Assumption (9.2) in ibid.) should continue to hold. We do not know whether this is true or not under the assumptions of Theorem 3.1, but the following example shows that the validity of this statement is not immediate. The surface $V = V(x^2 - y^2 - zy^2) \subset Spec\, \mathbb{C}[x, y, z]$ is irreducible and contains the curve $C = V(x - y, z)$. Let $o = (0, 0, 0)$, $X = Spec\, \mathcal{O}_{V,o}$, $Z = Spec\, \mathcal{O}_{C,o}$. Consider the completions $\hat{X} = Spec\, \widehat{\mathcal{O}_{V,o}} \subset Spec\, \mathbb{C}[[x, y, z]]$ and $\hat{Z} = Spec\, \widehat{\mathcal{O}_{C,o}}$. Then $\hat{X} = V(x - yg(z)) \cup V(x + yg(z))$, where $1 + z = g(z)^2$ and $\hat{Z} \subset V(x - yg(z))$ while $\hat{Z} \not\subset V(x + yg(z))$.

Another problem is in the claim on page 719, lines 19 − 20, of [12] that $R = Spec\, \mathcal{O}_S[[t]]$ satisfies the condition (S_2). A counterexample to this statement is obtained using [9, Corollaire (5.10.9)] by taking any Y which is not equidimensional.

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