

**IRREDUCIBLE COMPONENTS OF HURWITZ SPACES
PARAMETERIZING GALOIS COVERINGS OF CURVES OF
ARBITRARY GENUS**

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*To the memory of my friend and teacher,
Andrey Todorov*

INTRODUCTION

Let Y be a smooth, projective, irreducible complex curve. A G -covering $p : C \rightarrow Y$ is a Galois covering, where C is a smooth, projective, irreducible curve and an isomorphism $G \xrightarrow{\sim} \text{Aut}(C/Y)$ is fixed. Two G -coverings are equivalent if there is a G -equivariant isomorphism between them. We are concerned with the Hurwitz spaces $H_n^G(Y)$ and $H_n^G(Y, y_0)$. The first one parameterizes G -equivalence classes of G -coverings of Y branched in n points. The second one, given a point $y_0 \in Y$, parameterizes G -equivalence classes of pairs $[p : C \rightarrow Y, z_0]$, where $p : C \rightarrow Y$ is a G -covering unramified at y_0 and $z_0 \in p^{-1}(y_0)$. When $G = S_d$ one can equivalently consider coverings $f : X \rightarrow Y$ of degree d with full monodromy group S_d . The Hurwitz spaces are smooth algebraic varieties and associating to a covering its branch divisor yields finite étale morphisms $H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ and $H_n^G(Y) \rightarrow (Y - y_0)^{(n)} - \Delta$, where Δ is the codimension one subvariety of $Y^{(n)}$ whose points correspond to effective non simple divisors of Y .

The main result of the present paper is the explicit calculation of the monodromy action of the fundamental groups of $Y^{(n)} - \Delta$ and $(Y - y_0)^{(n)} - \Delta$ on the fibers of the above topological coverings (see Theorem 2.8 and Theorem 2.10). The connected (=irreducible) components of $H_n^G(Y)$ and $H_n^G(Y, y_0)$ are in one-to-one correspondence with the orbits of these monodromy actions.

The case $Y = \mathbb{P}^1$, $G = S_d$ is classical. Hurwitz, using results of Clebsch [Cl], proved in [Hu] the connectedness of the space which parameterizes equivalence classes of simple coverings of \mathbb{P}^1 branched in n points (see [Vo2, Lemma 10.15] for a modern account). The Hurwitz spaces of Galois coverings of \mathbb{P}^1 were first introduced and studied by Fried in [Fr] in connection with the inverse Galois problem. Fried and Völklein prove in [FV] that $H_n^G(\mathbb{P}^1)$ has a structure of algebraic variety over \mathbb{Q} and if furthermore the center of G is trivial they relate the solution of the inverse Galois problem to the existence of Q -rational points of $H_n^G(\mathbb{P}^1)$. They also address the problem of determining the connected components of the complex variety $H_n^G(\mathbb{P}^1)$. Berstein and Edmonds study in [BE] the Hurwitz spaces of simply branched coverings $X \rightarrow Y$ and address the problem of its connectedness in relation to the topological classification of the generic branched coverings between

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two compact, oriented real surfaces. Harris, Graber and Starr proved in [HGS] the irreducibility of the Hurwitz space of simply branched coverings of degree d of a curve of positive genus, with monodromy group S_d , branched in $\geq 2d$ points. This result was used in the proof of their theorem on existence of sections of morphisms $V \rightarrow Y$ with rationally connected fibers. In our paper [K2] the irreducibility of this Hurwitz space was used in the proof of the unirationality of certain types of Hurwitz spaces of coverings of degree ≤ 5 of a curve Y of positive genus, thus extending to curves Y of arbitrary genus a well known result of Arbarello and Cornalba for coverings of \mathbb{P}^1 . Catanese, Lönne and Perroni achieved in [CLP] the classification of the irreducible components of the locus (in the moduli space of curves) of curves admitting an effective action of the dihedral group $G = D_m$. Identifying certain connected components of $H_n^G(Y)$, when $G = D_m$, is a part of their study.

The present paper is essentially an adaptation to the case of Galois coverings of the first two sections of the unpublished preprint [K1]. In the remaining sections of this preprint we addressed the problem of proving the irreducibility of the Hurwitz spaces of coverings $X \rightarrow Y$ of degree d with full monodromy group S_d in the cases of simple branching or simple branching in all but one point. In particular, fixing the type of the local monodromy at one special point, we obtained the irreducibility provided the number of points of Y of simple branching is $\geq 2d - 2$. The lower bound $2d - 2$ was improved in the paper of F. Vetro [Ve1]. In a series of papers F. Vetro addressed the problem of determining the irreducible components of the Hurwitz spaces of coverings $X \rightarrow Y$ of a curve Y of positive genus, imposing various conditions on the type of branching. She also studied coverings with a covering involution, when the monodromy group is not the full symmetric group, but a Weyl group of type B_ℓ or of type D_ℓ (see [Ve1, Ve2, Ve3, Ve4]). She has used in an essential way the formulas and the results of the first two sections of the preprint [K1] which we include in the present paper.

In the preprint [KhK] Kulikov and Harlamov used an algebraic tool called semi-group over a group in order to determine the connected components of the Hurwitz spaces of coverings of a curve of arbitrary genus with fixed type of branching and fixed monodromy group. When the number of branch points of each of the branching types is sufficiently large, they reduce the problem of finding the number of the irreducible components of the Hurwitz space to a combinatorial problem involving a certain graph.

The paper is organized as follows. In Section 1 we define and give some basic properties of the Hurwitz spaces $H_n^G(Y)$ and $H_n^G(Y, y_0)$. This material is well known when $Y \cong \mathbb{P}^1$ (cf. [FV], [Vo1], [Vo2], [Em]) and the extension to curves Y of arbitrary genus is immediate. We include a detailed account for reader's convenience.

Section 2 is devoted to the explicit calculation of the monodromy action of the braid group $\pi_1((Y - y_0)^{(n)} - \Delta, D)$ on the fiber of $H_n^G(Y, y_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ over a point D . The fiber is identified with the set of Hurwitz systems of G . These are ordered sets of elements $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ of G , which generate G , $t_i \neq 1 \forall i$ and $t_1 t_2 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$. After recalling a result of Birman [Bi] which yields a system of generators of $\pi_1((Y - y_0)^{(n)} - \Delta, D)$ we give in Theorem 2.8 explicit formulas for the monodromy action of these generators on the set of Hurwitz systems. The monodromy action for the covering $H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ is obtained

by passing from Hurwitz systems to equivalence classes of Hurwitz systems with respect to inner automorphisms of G .

In Section 3 we first discuss how to extend the arguments of the previous sections removing the assumption that G is finite. In Proposition 3.1 (this is the Main Lemma of [K1]) we prove a statement which turns out to be useful in determining the connected components of the Hurwitz spaces. Namely, given a Hurwitz system $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ of a group G , such that $t_i t_{i+1} = 1$, let H be the subgroup of G generated by $t_1, \dots, t_{i-1}, t_{i+2}, \dots, t_n, \lambda_1, \mu_1, \dots, \lambda_g, \mu_g$. Let $h \in H$. Then the given Hurwitz system and the one obtained by replacing the pair (t_i, t_{i+1}) by $(h^{-1}t_i h, h^{-1}t_{i+1}h)$ belong to the same $\pi_1((Y - y_0)^{(n)} - \Delta, D)$ -orbit.

Notation and conventions. We assume that the base field $k = \mathbb{C}$. By homotopy of arcs we always mean homotopy which fixes the end points. If G is a group and $t, h \in G$ we denote $h^{-1}th$ by t^h and hth^{-1} by ${}^h t$.

1. PRELIMINARIES ON HURWITZ SPACES

1.1. We consider the product in a fundamental group induced by product of arcs defined as

$$(1) \quad \gamma_1 * \gamma_2 = \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq 1/2, \\ \gamma_2(2t - 1) & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

We recall some standard facts ([Ma, Chapter 5], [Fo, Chapter 1]). Let $f : X \rightarrow Y$ be a surjective morphism of smooth, projective, irreducible curves. Let $D \subset Y$ be the branch locus. Let $Y' = Y - D$ and $X' = f^{-1}(Y')$. Then the restriction $f' : X' \rightarrow Y'$ is a topological (unramified) covering. Let $y_0 \in Y'$. Then $\pi_1(Y', y_0)$ acts on the right on $f^{-1}(y_0)$ as follows: if $[\alpha] \in \pi_1(Y', y_0)$ and $x \in f^{-1}(y_0)$, let $\tilde{\alpha}_x : I \rightarrow X'$ be the lifting of α with initial point x ; then $x \cdot [\alpha] := \tilde{\alpha}_x(1)$.

Consider the particular case of a Galois covering $p : C \rightarrow Y$ whose Galois group is isomorphic to G . Let us fix an isomorphism $C \xrightarrow{\sim} \text{Aut}(C/Y) \cong \text{Deck}(C'/Y')$. Here G acts on C on the left. We call such a data a G -covering of Y . Let $z_0 \in p^{-1}(y_0)$. Then a surjective homomorphism $m_{z_0} : \pi_1(Y', y_0) \rightarrow G$, called *monodromy homomorphism* is defined as follows:

$$(2) \quad m_{z_0}([\alpha]) = g \quad \text{iff} \quad z_0 \cdot [\alpha] = gz_0.$$

One has $\text{Ker}(m_{z_0}) = p_*(\pi_1(C', z_0))$. Suppose $p : C \rightarrow Y$ is branched in $n \geq 1$ points. Let $D = \{b_1, \dots, b_n\}$ be the branch locus. Let us choose neighborhoods $U_i \ni b_i$ such that every \bar{U}_i is homeomorphic to the unit disk, $i = 1, \dots, n$, and $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $\forall i \neq j$. For each $i = 1, \dots, n$ let us choose an arc η_i in $Y - \cup_{i=1}^n U_i$ such that $\eta_i(0) = y_0, \eta_i(1) \in \partial \bar{U}_i$, then let γ_i be the closed arc which starts at y_0 , travels along η_i , makes a counterclockwise loop along $\partial \bar{U}_i$ and returns to y_0 along η_i^- . It is clear that $m_{z_0}([\gamma_i])$, called *local monodromy at b_i along γ_i* , has the property $m_{z_0}([\gamma_i]) \neq 1$ for every $i = 1, \dots, n$. Let $O_i \subset G$ be the conjugacy class of $m_{z_0}([\gamma_i])$. Another choice of U_i and η_i as above results in replacing each $m_{z_0}([\gamma_i])$ by another element in O_i . We conclude that $m = m_{z_0} : \pi_1(Y', y_0) \rightarrow G$ satisfies the following condition:

$$(3) \quad m([\gamma_1]) \neq 1, \dots, m([\gamma_n]) \neq 1,$$

which is furthermore independent of the choice of $\gamma_i, i = 1, \dots, n$ as above.

1.2. Two G -coverings $p : C \rightarrow Y$ and $p_1 : C_1 \rightarrow Y$ are called G -equivalent if there is a G -equivariant isomorphism $f : C \rightarrow C_1$ such that $p_1 = f \circ p$. Fixing $y_0 \in Y'$ let $z_0 \in p^{-1}(y_0)$, $z_1 \in p_1^{-1}(y_0)$. The pairs (C, z_0) , (C_1, z_1) are called G -equivalent if there is an isomorphism f as above, which furthermore satisfies $f(z_0) = z_1$. Such an f is unique, when exists, since every covering automorphism of $C' \rightarrow Y'$ which fixes z_0 is the identity. It is clear that if (C, z_0) and (C_1, z_1) are G -equivalent, then $m_{z_0} = m_{z_1}$.

Proposition 1.3 (Riemann's Existence Theorem). *Let $D \subset Y$ be a finite set, $|D| = n$. Let $Y' = Y - D$ and let $y_0 \in Y'$. Let G be a finite group. The following sets are in one-to-one correspondence:*

(a) *the set of G -equivalence classes of pairs $(p : C \rightarrow Y, z_0)$, where $p : C \rightarrow Y$ is a G -covering, whose branch locus equals D and $z_0 \in p^{-1}(y_0)$;*

(b) *the set of epimorphisms $m : \pi_1(Y', y_0) \rightarrow G$, which satisfy Condition 3. If two pairs $(p : C \rightarrow Y, z_0)$ and $(p_1 : C_1 \rightarrow Y, z_1)$ correspond to one and the same $m : \pi_1(Y', y_0) \rightarrow G$, then the G -equivalence $f : C \rightarrow C_1$, $f(z_0) = z_1$ is unique.*

Proof. One associates to $(p : C \rightarrow Y, z_0)$ as in (a) the epimorphism $m = m_{z_0}$. The statements of the proposition are standard (see [Fo] and [Vo2]). We only mention how one constructs a G -covering from an epimorphism $m : \pi_1(Y', y_0) \rightarrow G$. Let $\Gamma = \text{Ker}(m)$. One lets

$$C' = \{\Gamma[\alpha] \mid [\alpha] \text{ is a homotopy class of arcs with } \alpha(0) = y_0, \alpha(1) \in Y'\}.$$

The map $p' : C' \rightarrow Y'$ is defined by $p'(\Gamma[\alpha]) = \alpha(1)$. If $g \in G$ and $m([\sigma]) = g$ one lets for $z = \Gamma[\alpha] \in C'$

$$gz = \Gamma[\sigma * \alpha].$$

One defines in a natural way topology on C' such that $p' : C' \rightarrow Y'$ is a topological covering map. It is regular and $G \cong \text{Deck}(C'/Y')$. The complex analytic structure on Y' induces a complex analytic structure on C' . Compactifying one obtains a holomorphic map of compact Riemann surfaces $p : C \rightarrow Y$ branched at D and the action of G on C' extends to an action on C . The distinguished point z_0 is $\Gamma[c_{y_0}]$, where $c_{y_0} : I \rightarrow \{y_0\} \subset Y'$. One has $m_{z_0} = m$.

It remains to prove that $p : C \rightarrow Y$ has an algebraic structure. Let $K = \mathcal{M}(C)$ be the field of meromorphic functions of C . It is a Galois extension of the field of rational functions $\mathbb{C}(Y)$ with group isomorphic to G (see [Vo2, Theorem 5.9], where $Y \cong \mathbb{P}^1$ but the arguments apply to Y of arbitrary genus). Let $f : X \rightarrow Y$ be the normalization of Y in K . One has $\mathcal{M}(X^{an}) = \mathbb{C}(X) \cong K = \mathcal{M}(C)$. Choosing a primitive element F of the field extension $K/\mathbb{C}(Y)$ and using [Fo, Proposition 8.9] one concludes that $p : C \rightarrow Y$ is biholomorphically equivalent to $f^{an} : X^{an} \rightarrow Y$. Furthermore this equivalence is G -equivariant and compatible with the action of G on the field K as follows from [Fo, Proposition 8.12]. \square

Let $Y^{(n)}$ be the symmetric product of Y with itself n times and let $\Delta \subset Y^{(n)}$ be the codimension one subvariety whose elements are the effective divisors of degree n which have points of multiplicity > 1 . Abusing notation we identify a point $D \in Y^{(n)} - \Delta$, $D = b_1 + \dots + b_n$ with its support $\text{Supp}(D) = \{b_1, \dots, b_n\}$.

Definition 1. Let $y_0 \in Y$. Let G be a finite group. We denote by $H_n^G(Y, y_0)$ the set of pairs (D, m) where $D \in (Y - y_0)^{(n)} - \Delta$ and $m : \pi_1(Y - D, y_0) \rightarrow G$ is a surjective homomorphism which satisfies Condition 3.

According to Proposition 1.3 the set $H_n^G(Y, y_0)$ is in bijective correspondence with the set of equivalence classes $[p : C \rightarrow Y, z_0]$, where $p : C \rightarrow Y$ is a G -covering branched in n points, unramified at y_0 , $z_0 \in p^{-1}(y_0)$ and the equivalence is the G -equivalence as defined in § 1.2.

Example. Let $f : X \rightarrow Y$ be a covering of smooth, projective, irreducible curves of degree d with branch locus D , $|D| = n$. Let $y_0 \notin D$. Let $\Lambda = f^{-1}(y_0)$, let $S(\Lambda)$ be the symmetric group acting on the right on Λ . Suppose the monodromy homomorphism $\mu : \pi_1(Y - D, y_0) \rightarrow S(\Lambda)$ is surjective. The associated Galois covering is explicitly constructed as follows. The curve

$$C' = \{(x_1, \dots, x_n) \mid \{x_1, \dots, x_n\} = f^{-1}(y), y \in Y - D\}$$

is an irreducible, finite, étale covering of $Y - D$ of degree $n!$. The group S_d of permutations is considered acting on the right on $\{1, \dots, d\}$. For $\sigma \in S_d$ we write i^σ for the image of i . The group S_d acts on the left on C' by $\sigma \cdot (x_1, \dots, x_d) = (x_{1^\sigma}, \dots, x_{d^\sigma})$. The smooth completion $p : C \rightarrow Y$ is the Galois covering associated with $f : X \rightarrow Y$. To fix a point $z_0 \in C$, $z_0 \in p^{-1}(y_0)$ means here to fix a bijection $\phi : f^{-1}(y_0) \rightarrow \{1, \dots, d\}$.

Let $p : C \rightarrow Y$ be a G -covering with branch locus D . Let $Y' = Y - D$. Let $y_0, y_1 \in Y'$, let $z_0 \in p^{-1}(y_0), z_1 \in p^{-1}(y_1)$. Let us connect z_0 and z_1 by an arc in $C' = p^{-1}(Y')$ and let τ be its projection in Y' . Then one has the following relation between $m_{z_0} : \pi_1(Y', y_0) \rightarrow G$ and $m_{z_1} : \pi_1(Y', y_1) \rightarrow G$:

$$m_{z_0}([\tau * \alpha * \tau^-]) = m_{z_1}([\alpha]) \quad \text{for every } [\alpha] \in \pi_1(Y', y_1)$$

Definition 2. Let $y_0, y_1 \in Y' = Y - D$. Two homomorphisms $m_0 : \pi_1(Y', y_0) \rightarrow G$ and $m_1 : \pi_1(Y', y_1) \rightarrow G$ are called arcwise connected if there is an arc $\tau : I \rightarrow Y'$ with $\tau(0) = y_0, \tau(1) = y_1$ such that $m_1([\alpha]) = m_0([\tau * \alpha * \tau^-])$ for every $[\alpha] \in \pi_1(Y', y_1)$. We write $m_1 = m_0^\tau$.

1.4. The following statements are easily verified.

- (i) Arcwise connectedness is a relation of equivalence.
- (ii) Given a G -covering $p : C \rightarrow Y$ and $m_{z_0} : \pi_1(Y', y_0) \rightarrow G$ as in § 1.1 every $m_{z_1} : \pi_1(Y', y_1) \rightarrow G$ arcwise connected with m_{z_0} equals m_{z_1} for some $z_1 \in p^{-1}(Y')$.
- (iii) Given a surjective homomorphism $m : \pi_1(Y', y_0) \rightarrow G$, a homomorphism $m_1 : \pi_1(Y', y_0) \rightarrow G$ is arcwise connected with m if and only if it belongs to the orbit ${}^G m = \{gmg^{-1} \mid g \in G\}$ where $gmg^{-1}([\alpha]) = gm([\alpha])g^{-1}$ for every $[\alpha] \in \pi_1(Y', y_0)$.
- (iv) If $m : \pi_1(Y', y_0) \rightarrow G$ satisfies Condition 3 then every $m_1 : \pi_1(Y', y_1) \rightarrow G$ arcwise connected with m satisfies it.

Proposition 1.5. Let $D \subset Y$ be a finite set, $|D| = n$. Let $Y' = Y - D$, let $y_0 \in Y'$. Let G be a finite group. The following sets are in one-to-one correspondence:

- (a) the set of G -equivalence classes of G -coverings $p : C \rightarrow Y$ whose branch locus equals D .
- (b) the set of arcwise connected families of epimorphisms $\underline{m} = \{m : \pi_1(Y', y) \rightarrow G \mid y \in Y'\}$ which satisfy Condition 3

Suppose the center of the group is trivial: $Z(G) = 1$. In this case if two coverings $p : C \rightarrow Y$ and $p_1 : C_1 \rightarrow Y$ correspond to one and the same \underline{m} , then the G -equivalence $f : C \rightarrow C_1$ is unique. This property holds if and only if $Z(G) = 1$.

Proof. Given $p : C \rightarrow Y$ as in (a) one lets $\underline{m} = \{m_z : \pi_1(Y', y) \rightarrow G \mid p(z) = y \in Y'\}$. The proposition follows from: Proposition 1.3; the statements of § 1.4; the

fact that every covering automorphism of $p : C \rightarrow Y$ is of the form $\sigma(z) = hz$ for some $h \in G$ and from the fact that it is G -equivariant iff $h \in Z(G)$. \square

Definition 3. Let G be a finite group. We denote by $H_n^G(Y)$ the set of pairs (D, \underline{m}) , where $D \in Y^{(n)} - \Delta$ and \underline{m} is a family of arcwise connected epimorphisms $m : \pi_1(Y - D, y) \rightarrow G$, which satisfy Condition 3.

1.6. According to Proposition 1.5 the set $H_n^G(Y)$ is in bijective correspondence with the set of equivalence classes $[p : C \rightarrow Y]$, where $p : C \rightarrow Y$ is a G -covering branched in n points with respect to the G -equivalence defined in § 1.2. Let $y_0 \in Y$ and let $U(y_0) \subset H_n^G(Y)$ be the subset $\{(D, \underline{m}) \mid y_0 \notin D\}$. By § 1.4 (iii) the map $H_n^G(Y, y_0) \rightarrow U(y_0)$ given by $(D, m) \mapsto (D, \underline{m})$ is G -invariant with respect to the action $g(D, m) = (D, gm g^{-1})$ and $U(y_0)$ is bijective to $H_n^G(Y, y_0)/G$.

1.7. Let $b : H_n^G(Y, y_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ be the map $b(D, m) = D$. Following [Fu, § 1.3] we want to define a Hausdorff topology on $H_n^G(Y, y_0)$ which makes b a topological covering map. Let $(D, m) \in H_n^G(Y, y_0)$. Let $D = b_1 + \dots + b_n$. Let $U_i \ni b_i$ be neighborhoods, such that every \bar{U}_i is homeomorphic to the unit disk, $i = 1, \dots, n$, furthermore $\bar{U}_1, \dots, \bar{U}_n$ are disjoint and $y_0 \notin \cup_{i=1}^n \bar{U}_i$. Let $N(U_1, \dots, U_n) \subset (Y - y_0)^{(n)} - \Delta$ be the neighborhood of D consisting of $E = y_1 + \dots + y_n$ with $y_i \in U_i$. Let $U = \cup_{i=1}^n U_i$. The inclusion $Y - U \rightarrow Y - D$ is a deformation retract, so for every epimorphism $m : \pi_1(Y - D, y_0) \rightarrow G$ there is a unique epimorphism $m(E) : \pi_1(Y - E, y_0) \rightarrow G$ such that the following diagram commutes

$$(4) \quad \begin{array}{ccccc} \pi_1(Y - D, y_0) & \xleftarrow{\cong} & \pi_1(Y - U, y_0) & \xrightarrow{\cong} & \pi_1(Y - E, y_0) \\ & \searrow m & \downarrow \mu & \swarrow m(E) & \\ & & G & & \end{array}$$

We let

$$N_m(U_1, \dots, U_n) = \{(E, m(E)) \mid E \in N(U_1, \dots, U_n)\}$$

Proposition 1.8. Let Y be a smooth, projective, irreducible curve. Let $y_0 \in Y$. Let G be a finite group. Let $H_n^G(Y, y_0)$ be as in Definition 1. Suppose $H_n^G(Y, y_0) \neq \emptyset$.

(a) $H_n^G(Y, y_0)$ may be endowed by a Hausdorff topology such that the map $b : H_n^G(Y, y_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ given by $b(D, m) = D$ is a covering map. A basis of open sets of this topology is the family of sets $N_m(U_1, \dots, U_n)$ obtained varying $D \in (Y - y_0)^{(n)} - \Delta$, $m : \pi_1(Y - D, y_0) \rightarrow G$ and U_1, \dots, U_n as above. Every open set $N(U_1, \dots, U_n) \subset (Y - y_0)^{(n)} - \Delta$ is evenly covered: $b^{-1}(N(U_1, \dots, U_n))$ is a disjoint union of open sets homeomorphic to $N(U_1, \dots, U_n)$. The connected components of $b^{-1}(N(U_1, \dots, U_n))$ are in bijective correspondence with the set of epimorphisms $\mu : \pi_1(Y - \cup_{i=1}^n U_i, y_0) \rightarrow G$. The covering b is proper.

(b) $H_n^G(Y, y_0)$ has a unique structure of a complex manifold, such that $b : H_n^G(Y, y_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ is holomorphic. Furthermore b is a finite étale holomorphic map.

(c) $H_n^G(Y, y_0)$ has a unique structure of a \mathbb{C} -scheme of finite type, such that $H_n^G(Y, y_0)^{an}$ is biholomorphic to the complex manifold of Part (b). Furthermore $H_n^G(Y, y_0)$ is smooth, equidimensional of dimension n and $b : H_n^G(Y, y_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ is a finite, étale morphism.

(d) $H_n^G(Y, y_0)$ is an affine variety.

Proof. The statements of (a) are easily verified and are left to the reader.

(b) Starting from a complex analytic atlas of $(Y - y_0)^{(n)} - \Delta$ with charts contained in various $N(U_1, \dots, U_n)$ one constructs a complex analytic atlas of $H_n^G(Y, y_0)$ and with this complex analytic structure $b : H_n^G(Y, y_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ is a finite étale holomorphic map. Let M be this complex manifold. If M' is another complex manifold with underlying topological space $H_n^G(Y, y_0)$, such that $b : M' \rightarrow (Y - y_0)^{(n)} - \Delta$ is holomorphic, then every $x \in M'$ has a complex analytic chart (V, ϕ) , $\phi(V) \subset \mathbb{C}^n$, such that b maps V homeomorphically in some $N(U_1, \dots, U_n)$. It is known that such a map $b : V \rightarrow b(V)$ is biholomorphic (cf. [Nar, p. 86]). Therefore $id : M \rightarrow M'$ is biholomorphic.

(c) These statements follow from Théorème 5.1, Proposition 3.1 and Proposition 3.2 of [SGA1, Exposé XII].

(d) The curve $Y - y_0$ is affine, so $(Y - y_0)^{(n)} - \Delta$ is an affine variety. Therefore the finite cover $H_n^G(Y, y_0)$ is an affine variety. \square

Proposition 1.9. *Let Y be a smooth, projective, irreducible curve. Let G be a finite group. Let $H_n^G(Y)$ be the set of Definition 3. Suppose $H_n^G(Y) \neq \emptyset$.*

(a) $H_n^G(Y)$ may be endowed by a Hausdorff topology such that the map $b : H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ given by $b(D, \underline{m}) = D$ is a proper covering map. For every $D \in Y^{(n)} - \Delta$ any neighborhood $N(U_1, \dots, U_n)$ (cf. § 1.7) is evenly covered.

(b) $H_n^G(Y)$ has a unique structure of a complex manifold such that $b : H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ is holomorphic. Furthermore b is a finite étale holomorphic map.

(c) $H_n^G(Y)$ has a unique structure of a \mathbb{C} -scheme of finite type, such that $H_n^G(Y)^{an}$ is biholomorphic to the complex manifold of Part (b). Furthermore $H_n^G(Y)$ is smooth, equidimensional of dimension n and $b : H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ is a finite, étale morphism.

(d) $H_n^G(Y)$ is a quasiprojective variety. If $Y \cong \mathbb{P}^1$ it is an affine variety.

Proof. (a) Let $y_0 \in Y$. The group G acts on $H_n^G(Y, y_0)$ by $g(D, m) = (D, gmg^{-1})$. This induces a faithful, properly discontinuous action of $G/Z(G)$ on $H_n^G(Y, y_0)$. One endows $U(y_0) \cong H_n^G(Y, y_0)/G$ by the quotient topology (cf. § 1.6). One has that $H_n^G(Y) = \cup_{y \in Y} U(y)$ and for each pair $y_1, y_2 \in Y$ the intersection $U(y_1) \cap U(y_2)$ is open in $U(y_i)$, $i = 1, 2$. This defines a topology on $H_n^G(Y)$. Let $D = b_1 + \dots + b_n \in Y^{(n)} - \Delta$, let $U_i \ni b_i$, $y_0 \notin \cup_{i=1}^n \overline{U}_i$ be as in § 1.7. Let $\underline{m} = \{m : \pi_1(Y - D, y) \rightarrow G\}$ be a family of arcwise connected epimorphisms. It is uniquely determined by a G -orbit ${}^G m = \{gmg^{-1} \mid g \in G\}$ for some $m : \pi_1(Y - D, y_0) \rightarrow G$ (cf. § 1.4). Passing to G -orbits in (4) let $\underline{m}(E)$ correspond to ${}^G m(E)$. We let

$$N_{\underline{m}}(U_1, \dots, U_n) = \{(E, \underline{m}(E)) \mid E \in N(U_1, \dots, U_n)\}.$$

This is an open subset of $U(y_0)$ equal to the image of $N_{\underline{m}}(U_1, \dots, U_n)$ under the quotient map $H_n^G(Y, y_0) \rightarrow U(y_0)$. One has the equality

$$b^{-1}(N(U_1, \dots, U_n)) = \bigsqcup_{\underline{m}} N_{\underline{m}}(U_1, \dots, U_n)$$

and furthermore for every \underline{m} the restriction $b : N_{\underline{m}}(U_1, \dots, U_n) \rightarrow N(U_1, \dots, U_n)$ is a homeomorphism, as it follows from the corresponding statement for the covering $H_n^G(Y, y_0) \rightarrow (Y - y_0)^{(n)} - \Delta$.

The statements of (b) and (c) are proved similarly to those of Proposition 1.9.

(d) It suffices to prove that each connected component H of $H_n^G(Y)$ is quasiprojective. Every such H is irreducible since $H_n^G(Y)$ is nonsingular. Let \overline{H} be the

normalization of $Y^{(n)}$ in the field $\mathbb{C}(H)$. Then \overline{H} is projective since $Y^{(n)}$ is and H is isomorphic to a Zariski open subset of \overline{H} . Therefore H is quasiprojective. If $Y \cong \mathbb{P}^1$, then $Y^{(n)} \cong \mathbb{P}^n$ and Δ is isomorphic to a hypersurface of degree n . So $Y^{(n)} - \Delta$ is an affine variety. Therefore $H_n^G(Y)$ is an affine variety as well. \square

Definition 4. Let O_1, \dots, O_k be conjugacy classes of G , $O_i \neq O_j$ if $i \neq j$. Let $\underline{n} = n_1 O_1 + \dots + n_k O_k$ be a formal sum where $n_i \in \mathbb{N}$. Let $|\underline{n}| := n_1 + \dots + n_k = n$. We denote by $H_{\underline{n}}^G(Y, y_0)$ (resp. $H_{\underline{n}}^G(Y)$) the subset of $H_n^G(Y, y_0)$ (resp. $H_n^G(Y)$) whose points correspond to G -equivalence classes $[p : C \rightarrow Y, z_0]$ (resp. $[p : C \rightarrow Y]$) where the coverings $p : C \rightarrow Y$ have n_i branch points with local monodromies which belong to O_i , $i = 1, \dots, k$.

It is clear that $H_{\underline{n}}^G(Y, y_0)$ is a union of connected components of $H_n^G(Y, y_0)$ if it is nonempty and one has

$$H_n^G(Y, y_0) = \bigsqcup_{|\underline{n}|=n} H_{\underline{n}}^G(Y, y_0).$$

Similarly $H_{\underline{n}}^G(Y)$ is a union of connected components of $H_n^G(Y)$ if it is nonempty and one has

$$H_n^G(Y) = \bigsqcup_{|\underline{n}|=n} H_{\underline{n}}^G(Y).$$

2. BRAID MOVES

2.1. Suppose that $g(Y) \geq 1$. Let us fix the orientation of the real 2-manifold Y considered as a complex manifold. Let $D = \{b_1, \dots, b_n\} \subset Y$. Let $b_0 \notin D$. We describe a standard way of choosing generators of $\pi_1(Y - D, b_0)$. Cutting Y along $2g$ simple closed arcs, which begin at b_0 and do not contain any of b_i , $i \geq 1$, one obtains a standard $4g$ -polygon with sides $\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \dots, \alpha_g, \beta_g, \alpha_g^{-1}, \beta_g^{-1}$ which circle the polygon in counterclockwise direction. We consider a simple closed arc L which begins at b_0 , $L - b_0$ is contained in the interior of the $4g$ -polygon and passes consecutively in counterclockwise direction through the points $\{b_1, \dots, b_n\}$. The closed arc L divides the $4g$ -polygon into two regions R and R' which stay on the left, respectively on the right side of L with respect to its counterclockwise orientation. We choose a simple arc ℓ_1 which lies inside the region R and connects b_0 and b_1 . Then we choose a second simple arc ℓ_2 inside R which connects b_0 with b_2 , has only b_0 as point in common with ℓ_1 , and lies on the left side of ℓ_1 . Continuing in this way we obtain an ordered n -tuple (ℓ_1, \dots, ℓ_n) of simple arcs which do not meet outside b_0 . Let γ_i be a closed arc which begins at b_0 , travels along ℓ_i to a point near b_i , makes a small counterclockwise loop around b_i , and returns to b_0 along ℓ_i . We obtain a $(n + 2g)$ -tuple of closed arcs $(\gamma_1, \dots, \gamma_n; \alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ which we call a *standard system of closed arcs*. The corresponding homotopy classes yield a standard system of generators for $\pi_1(Y - D, b_0)$ which satisfy the only relation

$$(5) \quad \gamma_1 \gamma_2 \cdots \gamma_n \simeq [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$$

Figure 1 illustrates such a standard system. A reader who prefers the clockwise orientation of closed arcs and ordering of the branch points from left to right may look at this and all subsequent figures from the other side of the sheet. Here and in the rest of the paper we will use the same notation, for the sake of its simplification, for a closed arc and for its homotopy class when the correct meaning is clear from the context. Given a G -covering $p : C \rightarrow Y$ with branch locus D , a point $b_0 \in Y - D$

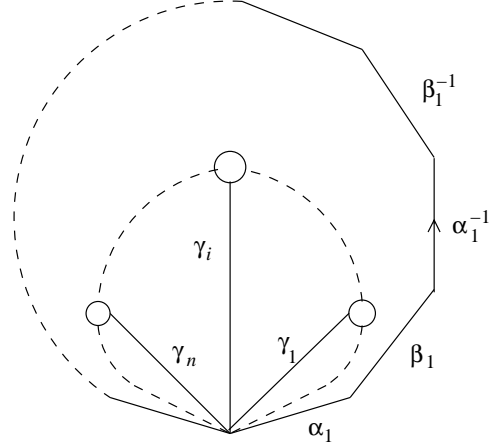


FIGURE 1. Standard system of closed arcs

and a point $z_0 \in p^{-1}(b_0)$ consider the homomorphism $m = m_{z_0} : \pi_1(Y - D, b_0) \rightarrow G$. Let $t_i = m(\gamma_i)$, $\lambda_k = m(\alpha_k)$, $\mu_k = \lambda_{g+k} = m(\beta_k)$.

Definition 5. Let G be a group. An ordered set $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ of elements of G which generate G , has the property $t_i \neq 1$ for $\forall i$ and satisfies the relation

$$t_1 t_2 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$$

is called a *Hurwitz system* (of generators) of the group G . We let $\lambda_{g+k} = \mu_k$.

According to Proposition 1.3, given $D \in Y^{(n)} - \Delta$ and $b_0 \notin D$, and fixing the closed arcs $(\gamma_1, \dots, \gamma_n; \alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ as above, the fiber of $H_n^G(Y, b_0) \rightarrow (Y - b_0)^{(n)} - \Delta$ over D may be identified with the set of all Hurwitz systems, while by Proposition 1.5 and § 1.6 the fiber of $H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ over D may be identified with the set of equivalence classes $[t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ of Hurwitz systems modulo inner automorphisms of G , i.e. (t_1, \dots, μ_g) is equivalent to (t'_1, \dots, μ'_g) if there exists an element $g \in G$ such that $t'_i = g t_i g^{-1}$, $\lambda'_k = g \lambda_k g^{-1}$, $\mu'_k = g \mu_k g^{-1}$ for $\forall i, k$.

An equivalent way of constructing a standard system of closed arcs is the following. One chooses first the $2g$ simple closed arcs $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$. Then one chooses n simple arcs which start at b_0 , lie inside the $4g$ -polygon, do not meet outside b_0 , and have for end points the n points of D . One enumerates these arcs according to the directions of departure in counterclockwise order. The obtained $(n + 2g)$ -tuple $(\ell_1, \dots, \ell_n; \alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ is called an *arc system* (cf. [Lo] p.416). One considers the induced ordering of the points of D . One can take for R a star-like region which contains the union $\ell_1 \cup \dots \cup \ell_n$ and let $L = \partial R$. In this way one obtains all ingredients used to construct a standard system of closed arcs $(\gamma_1, \dots, \gamma_n; \alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$.

2.2. The connected components of $H_n^G(Y)$ are in one-to-one correspondence with the orbits of the full n -strand braid group $\pi_1(Y^{(n)} - \Delta, D)$ acting on the fiber of the topological covering $H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ over D . Similar statement holds about $H_n^G(Y, b_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ where one considers the braid group

$\pi_1((Y - b_0)^{(n)} - \Delta, D)$. The identification of these fibers with the Hurwitz systems reduces the problem of determining the connected components of $H_n^G(Y, b_0)$ and $H_n^G(Y)$ to calculating the action of the respective braid groups on Hurwitz systems and then finding the orbits. Abusing notation we will identify, using Proposition 1.3, the G -equivalence classes $[C \rightarrow Y, z_0]$ with the corresponding points $(D, m) \in H_n^G(Y, b_0)$ and similarly by Proposition 1.5 the G -equivalence classes $[C \rightarrow Y]$ with the corresponding points $(D, \underline{m}) \in H_n^G(Y)$.

Let $\gamma_i, \alpha_k, \beta_k, 1 \leq i \leq n, 1 \leq k \leq g$ be a standard system of closed arcs obtained from an arc system $\ell_i, \alpha_k, \beta_k, 1 \leq i \leq n, 1 \leq k \leq g$ as in § 2.1. Let $D^u, 0 \leq u \leq 1$ be a closed arc in $(Y - b_0)^{(n)} - \Delta$ with $D^0 = D^1 = D$. We show later, in the cases we need, that starting from the given arc system, one can extend the map $u \mapsto D^u$ to a homotopy of arc systems $\ell_i^u, \alpha_k^u, \beta_k^u, 1 \leq i \leq n, 1 \leq k \leq g$ based at b_0 . This yields a corresponding homotopy of closed arcs $\gamma_i^u, \alpha_k^u, \beta_k^u, 1 \leq i \leq n, 1 \leq k \leq g$ which form a standard system for each $u \in [0, 1]$. Let $[C \rightarrow Y, z_0]$ correspond to $(D, m) \in H_n^G(Y, b_0)$ and let $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ be the corresponding Hurwitz system. Then the lifting of the closed arc $D^u, 0 \leq u \leq 1$ starting from $[C \rightarrow Y, z_0]$ equals, according to Proposition 1.8, to $(D^u, m^u : \pi_1(Y - D^u, b_0) \rightarrow G)$ where

$$m^u(\gamma_i^u) = t_i, \quad m^u(\alpha_k^u) = \lambda_k, \quad m^u(\beta_k^u) = \mu_k$$

Let $\gamma_i^1 = \gamma'_i, \alpha_k^1 = \alpha'_k, \beta_k^1 = \beta'_k$ and $m^1 = m'$. We obtain that the end point of the lifting of D^u is $[C' \rightarrow Y, z'_0]$ where $m_{z'_0} = m' : \pi_1(Y - D, b_0) \rightarrow G$ is defined by

$$m'(\gamma'_i) = t_i, \quad m'(\alpha'_k) = \lambda_k, \quad m'(\beta'_k) = \mu_k$$

Evaluating m' at $\gamma_i, \alpha_k, \beta_k$ we obtain

$$(6) \quad m'(\gamma_i) = t'_i, \quad m'(\alpha_k) = \lambda'_k, \quad m'(\beta_k) = \mu'_k.$$

So $(t'_1, \dots, t'_n; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ is the Hurwitz system corresponding to $[C' \rightarrow Y, z'_0]$. One may also consider

$$(7) \quad m(\gamma'_i) = t''_i, \quad m(\alpha'_k) = \lambda''_k, \quad m(\beta'_k) = \mu''_k.$$

The Hurwitz system $(t''_1, \dots, t''_n; \lambda''_1, \mu''_1, \dots, \lambda''_g, \mu''_g)$ corresponds to a pair $[C'' \rightarrow Y, z''_0]$ such that $m_{z''_0} = m'' : \pi_1(Y - D, b_0) \rightarrow G$ satisfies

$$(8) \quad m''(\gamma_i) = t''_i, \quad m''(\alpha_k) = \lambda''_k, \quad m''(\beta_k) = \mu''_k$$

Lemma 2.3. *Given the topological covering $H_n^G(Y, b_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ let us lift the arc $D^{1-u}, 0 \leq u \leq 1$ starting from $[C \rightarrow Y, z_0]$. Then the end point is $[C'' \rightarrow Y, z''_0]$.*

Proof. In terms of $\gamma'_i, \alpha'_k, \beta'_k$ the monodromy homomorphism of $[C \rightarrow Y, z_0]$ is given by $m(\gamma'_i) = t'_i, m(\alpha'_k) = \lambda'_k, m(\beta'_k) = \mu'_k$. Let us consider the homotopy of closed arcs $\gamma_i^{1-u}, \alpha_k^{1-u}, \beta_k^{1-u}, 0 \leq u \leq 1$ and the arc in $H_n^G(Y, b_0)$ given by $n^u : \pi_1(Y - D^{1-u}, b_0) \rightarrow S_d$ where $n^u(\gamma_i^{1-u}) = t'_i, n^u(\alpha_k^{1-u}) = \lambda'_k, n^u(\beta_k^{1-u}) = \mu'_k$. Then $n^0 = m, n^1 = m''$ (cf. Eq.(8)) \square

Definition 6. Given a closed arc $D^u, 0 \leq u \leq 1$ in the configuration space $(Y - b_0)^{(n)} - \Delta$ the transformation of Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) \mapsto (t'_1, \dots, t'_n; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ given by Eq.(6) is called a *braid move* of the first type. The transformation $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) \mapsto (t''_1, \dots, t''_n; \lambda''_1, \mu''_1, \dots, \lambda''_g, \mu''_g)$ given by (8) is called a braid move of the second type.

The braid moves of the first and of the second type are inverse to each other according to Lemma 2.3. It is evident that the braid moves of both types commute with inner automorphisms of G . So the braid moves are well-defined on equivalence classes of Hurwitz systems (cf. § 2.1).

2.4. There is a convenient system of generators of $\pi_1((Y - b_0)^{(n)} - \Delta, D)$. We include here some material borrowed from [Bi] and [Sc] for the sake of convenience of the reader and since our choices differ slightly from theirs. Consider the Galois covering $p : (Y - b_0)^n \rightarrow (Y - b_0)^{(n)}$ with Galois group S_n . Restricting to the complement of Δ one obtains an unramified Galois covering $p : (Y - b_0)^n - p^{-1}(\Delta) \rightarrow (Y - b_0)^{(n)} - \Delta$. Following the notation of [FN] if $Q_1 = \{b_0\}$ one denotes $(Y - b_0)^n - p^{-1}(\Delta)$ by $F_{1,n}Y$. Let $D = \{b_1, \dots, b_n\}$, $\tilde{D} = (b_1, \dots, b_n)$. One has an exact sequence

$$(9) \quad 1 \longrightarrow \pi_1(F_{1,n}Y, \tilde{D}) \longrightarrow \pi_1((Y - b_0)^{(n)} - \Delta, D) \longrightarrow S_n \longrightarrow 1$$

One determines first a system of generators of the pure braid group $\pi_1(F_{1,n}Y, \tilde{D})$ as follows. Consider the closed arcs in $F_{1,n}Y$ defined by $(b_1, \dots, b_{i-1}, r_{ik}(t), b_{i+1}, \dots, b_n)$ and $(b_1, \dots, b_{i-1}, t_{ik}(t), b_{i+1}, \dots, b_n)$, with $t \in [0, 1]$, where r_{ik} and t_{ik} are the closed simple arcs based at b_i and pictured on Figure 2 by a continuous line and by a dotted line respectively. We denote the corresponding homotopy classes by

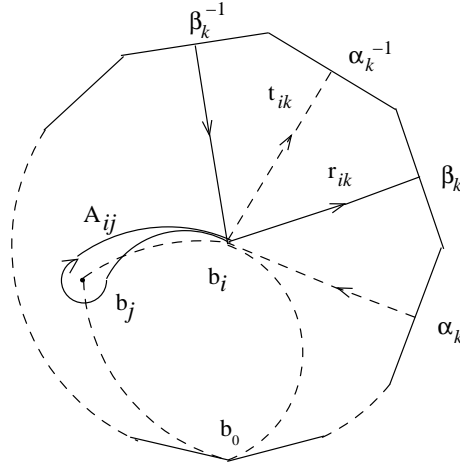


FIGURE 2. Generators of the pure braid group $\pi_1(F_{1,n}, \tilde{D})$

$\rho_{ik}, \tau_{ik} \in \pi_1(F_{1,n}(Y), \tilde{D})$. Informally ρ_{ik} corresponds to a loop of the i -th point along α_k and τ_{ik} corresponds to a loop of the i -th point along β_k . Let us denote by A_{ij} , $i < j$ the element of $\pi_1(F_{1,n}Y, \tilde{D})$ represented by a closed simple arc in Y^n which leaves fixed b_k for $k \neq i$ and moves b_i along the arc pictured on Figure 2. For every i and every $j > i$ the pictured loop is chosen so as to stay on the left of all arcs used to construct ρ_{ik} and τ_{ik} for $k = 1, \dots, g$.

Claim. *The pure braid group $\pi_1(F_{1,n}Y, \tilde{D})$ is generated by ρ_{ik}, τ_{ik} and A_{ij} where $i, j = 1, \dots, n$, $i < j$ and $k = 1, \dots, g$.*

Proof. This is proved by induction on n . When $n = 1$ the claim is obvious. Let $n \geq 2$. Consider the fibration $F_{1,n}Y \rightarrow F_{1,n-1}Y$ defined by $(y_1, y_2, \dots, y_n) \rightarrow$

(y_2, \dots, y_n) . One has an exact sequence

$$1 \rightarrow \pi_1(Y - \{b_0, b_2, \dots, b_n\}, b_1) \rightarrow \pi_1(F_{1,n}Y, (b_1, \dots, b_n)) \rightarrow \pi_1(F_{1,n-1}Y, (b_2, \dots, b_n)) \rightarrow 1$$

since $\pi_2(F_{1,n-1}Y) = 1$ by [FN], Corollary 2.2. The elements $\rho_{ik}, \tau_{ik}, A_{ij}$ with $i \geq 2$ map to the corresponding elements in $\pi_1(F_{1,n-1}Y, (b_2, \dots, b_n))$. The group $\pi_1(Y - \{b_0, b_2, \dots, b_n\}, b_1)$ is freely generated by elements which map into $\rho_{1,k}, \tau_{1,k}, A_{1,j}$ where $j \geq 2, k = 1, \dots, g$. This shows the claim. \square

Following [Fu] p.547 and using the notation of § 2.1 let us denote by R_i the simply connected region of R enclosed by the arcs ℓ_i and ℓ_{i+1} and the arc of L from b_i to b_{i+1} . For every $i = 1, \dots, n-1$ choose simple arcs $\eta_i : [0, 1] \rightarrow Y$ going from b_i to b_{i+1} in R_i and $\eta'_i : [0, 1] \rightarrow Y$ going from b_{i+1} to b_i in R' . Let $s_i : [0, 1] \rightarrow Y^{(n)} - \Delta$ be the closed arc

$$s_i(t) = \{b_1, \dots, b_{i-1}, \eta_i(t), \eta'_i(t), b_{i+2}, \dots, b_n\}$$

The homotopy class of s_i is denoted by σ_i . We may consider $\pi_1(F_{1,n}(Y), \tilde{D})$ as embedded in $\pi_1((Y - b_0)^{(n)} - \Delta, D)$. The following relations are easy to verify (cf. [FB] p.249)

$$(10) \quad \begin{aligned} \rho_{i+1,k} &= \sigma_i \rho_{ik} \sigma_i^{-1}, & \tau_{i+1,k} &= \sigma_i \tau_{ik} \sigma_i^{-1} \\ A_{ij} &= \sigma_i^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_i \end{aligned}$$

Proposition 2.5 (Birman). *Let $a, b \in \mathbb{Z}, 1 \leq a, b \leq n$. The braid group $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ is generated by $\sigma_j, \rho_{ak}, \tau_{bk}$ where $1 \leq j \leq n-1, 1 \leq k \leq g$. The corresponding homotopy classes generate $\pi_1(Y^{(n)} - \Delta, D)$ as well.*

Proof. Let us consider the exact sequence (9). The braids σ_j map to the transpositions $(j j + 1), j = 1, \dots, n-1$ which generate S_n . So $\sigma_j, \rho_{ik}, \tau_{ik}, A_{ij}$ with $i < j$ generate $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ according to the claim proved in § 2.4. The relations (10) show that the generators may be reduced as stated in the theorem. The last statement follows from the surjection $\pi_1((Y - b_0)^{(n)} - \Delta, D) \twoheadrightarrow \pi_1(Y^{(n)} - \Delta, D)$. \square

We described in § 2.4 closed arcs in $(Y - b_0)^{(n)} - \Delta$ based at $D = \{b_1, \dots, b_n\}$ whose homotopy classes form a system of generators $\sigma_j, \rho_{ik}, \tau_{ik}$ for $\pi_1((Y - b_0)^{(n)} - \Delta, D)$. Our aim now is for each of these to construct a homotopy of the standard system of closed arcs $\gamma_i^u, \alpha_k^u, \beta_k^u$ as in § 2.2. This will permit us to calculate eventually the corresponding braid moves of the Hurwitz systems. The calculation of the braid moves $\sigma_j, j = 1, \dots, n-1$ is due to Hurwitz (cf. [Hu] or e.g. [Vo2], Theorem 10.3). We define closed arcs $\delta_k : [0, 1] \rightarrow Y, \delta_k(0) = \delta_k(1) = b_0, k = 0, 1, \dots, g$ as follows. We let $\delta_0(t) = b_0, \forall t \in [0, 1]$. We connect the initial vertex of α_1 with the end vertex of β_1^{-1} in the $4g$ -polygon of Figure 1 by a simple arc which belongs to the region R' on the right of L (cf. 2.1). This yields δ_1 . We connect the initial vertex of α_1 with the end vertex of β_2^{-1} by a simple arc which belongs to the region on the right of L and on the left of δ_1 . We denote the corresponding closed arc of Y by δ_2 . Continuing in this way we obtain $\delta_1, \delta_2, \dots, \delta_g$ (see Figure 3). Clearly $\delta_k \simeq [\alpha_1, \beta_1] \cdots [\alpha_k, \beta_k]$ in $Y - D$ and $\delta_g \simeq \gamma_1 \cdots \gamma_n$ according to (5).

Theorem 2.6. *Let Y be a compact, closed Riemann surface of genus $g(Y) \geq 1$. Let $b_0 \in Y$ and let $\gamma_i, \alpha_k, \beta_k$ with $1 \leq i \leq n, 1 \leq k \leq g$ be a standard system of closed arcs as in 2.1. Let $\delta_k, k = 0, 1, \dots, g$ be the closed arcs defined above. For each of the closed arcs in $(Y - b_0)^{(n)} - \Delta$ constructed in § 2.4 and representing*

$\sigma_j, \rho_{ik}, \tau_{ik}$ there is a homotopy $\gamma_i^u, \alpha_k^u, \beta_k^u, u \in [0, 1]$ of the standard system of closed arcs such that the end system $\gamma_i', \alpha_k', \beta_k', 1 \leq i \leq n, 1 \leq k \leq g$ is homotopic to:

a. for σ_j where $1 \leq j \leq n-1$

$$\begin{aligned} \gamma_i' &\simeq \gamma_i \quad \text{for } \forall i \neq j, j+1, \quad \alpha_k' \simeq \alpha_k, \beta_k' \simeq \beta_k \quad \text{for } \forall k \\ \gamma_j' &\simeq \gamma_{j+1}, \quad \gamma_{j+1}' \simeq \gamma_{j+1}^{-1} \gamma_j \gamma_{j+1} \end{aligned}$$

b. for ρ_{ik} where $1 \leq i \leq n, 1 \leq k \leq g$

$$\begin{aligned} \gamma_j' &\simeq \gamma_j \quad \text{for } \forall j \neq i, \quad \alpha_\ell' \simeq \alpha_\ell \quad \text{for } \forall \ell, \quad \beta_\ell' \simeq \beta_\ell \quad \text{for } \forall \ell \neq k \\ \gamma_i' &\simeq \mu_{ik} \gamma_i \mu_{ik}^{-1}, \quad \beta_k' \simeq (\zeta_{ik} \gamma_i \zeta_{ik}^{-1}) \beta_k, \quad \text{where} \\ \mu_{ik} &\simeq (\gamma_1 \cdots \gamma_{i-1})^{-1} \delta_{k-1} \alpha_k (\delta_k^{-1} \delta_g) (\gamma_{i+1} \cdots \gamma_n)^{-1}, \quad \zeta_{ik} \simeq (\delta_k^{-1} \delta_g) (\gamma_{i+1} \cdots \gamma_n)^{-1} \end{aligned}$$

c. for τ_{ik} where $1 \leq i \leq n, 1 \leq k \leq g$

$$\begin{aligned} \gamma_j' &\simeq \gamma_j \quad \text{for } \forall j \neq i, \quad \alpha_\ell' \simeq \alpha_\ell \quad \text{for } \forall \ell \neq k, \quad \beta_\ell' \simeq \beta_\ell \quad \text{for } \forall \ell \\ \gamma_i' &\simeq \nu_{ik} \gamma_i \nu_{ik}^{-1}, \quad \alpha_k' \simeq (\xi_{ik} \gamma_i^{-1} \xi_{ik}^{-1}) \alpha_k, \quad \text{where} \\ \nu_{ik} &\simeq \gamma_{i+1} \cdots \gamma_n (\delta_k^{-1} \delta_g)^{-1} \beta_k \delta_{k-1}^{-1} \gamma_1 \cdots \gamma_{i-1}, \quad \xi_{ik} \simeq \delta_{k-1}^{-1} \gamma_1 \cdots \gamma_{i-1} \end{aligned}$$

For each of the inverse closed arcs corresponding to $\sigma_j^{-1}, \rho_{ik}^{-1}, \tau_{ik}^{-1}$ there is a corresponding homotopy of the standard system $\gamma_i, \alpha_k, \beta_k, 1 \leq i \leq n, 1 \leq k \leq g$ such that the end system $\gamma_i'', \alpha_k'', \beta_k'', 1 \leq i \leq n, 1 \leq k \leq g$ is homotopic to:

d. for σ_j^{-1} where $1 \leq j \leq n-1$

$$\begin{aligned} \gamma_i'' &\simeq \gamma_i \quad \text{for } \forall i \neq j, j+1, \quad \alpha_k'' \simeq \alpha_k, \beta_k'' \simeq \beta_k \quad \text{for } \forall k \\ \gamma_j'' &\simeq \gamma_j \gamma_{j+1} \gamma_j^{-1}, \quad \gamma_{j+1}'' \simeq \gamma_j \end{aligned}$$

e. for ρ_{ik}^{-1} where $1 \leq i \leq n, 1 \leq k \leq g$

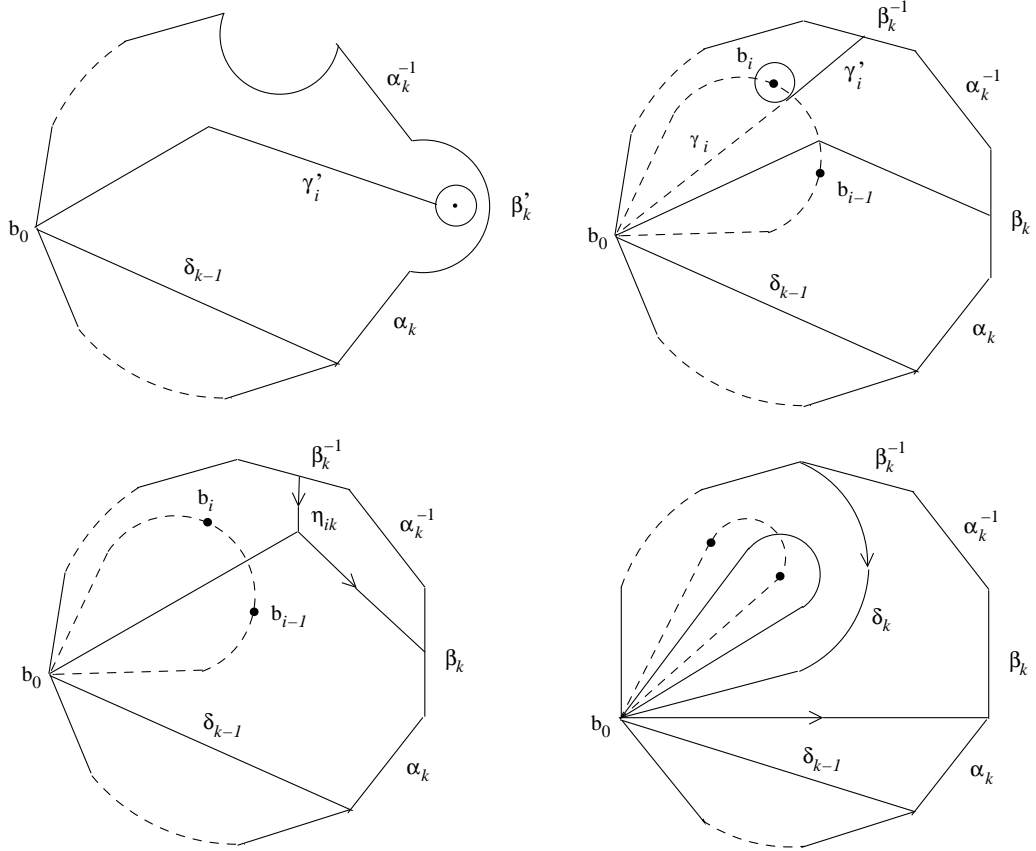
$$\begin{aligned} \gamma_j'' &\simeq \gamma_j \quad \text{for } \forall j \neq i, \quad \alpha_\ell'' \simeq \alpha_\ell \quad \text{for } \forall \ell, \quad \beta_\ell'' \simeq \beta_\ell \quad \text{for } \forall \ell \neq k \\ \gamma_i'' &\simeq \mu_{ik}^{-1} \gamma_i \mu_{ik}, \quad \beta_k'' \simeq (\tilde{\zeta}_{ik}^{-1} \gamma_i^{-1} \tilde{\zeta}_{ik}) \beta_k, \quad \text{where} \\ \mu_{ik} &\text{ is as in (b) and } \tilde{\zeta}_{ik} \simeq (\gamma_1 \cdots \gamma_{i-1})^{-1} \delta_{k-1} \alpha_k \end{aligned}$$

f. for τ_{ik}^{-1} where $1 \leq i \leq n, 1 \leq k \leq g$

$$\begin{aligned} \gamma_j'' &\simeq \gamma_j \quad \text{for } \forall j \neq i, \quad \alpha_\ell'' \simeq \alpha_\ell \quad \text{for } \forall \ell \neq k, \quad \beta_\ell'' \simeq \beta_\ell \quad \text{for } \forall \ell \\ \gamma_i'' &\simeq \nu_{ik}^{-1} \gamma_i \nu_{ik}, \quad \alpha_k'' \simeq (\tilde{\xi}_{ik}^{-1} \gamma_i \tilde{\xi}_{ik}) \alpha_k, \quad \text{where} \\ \nu_{ik} &\text{ is as in (c) and } \tilde{\xi}_{ik} \simeq (\gamma_{i+1} \cdots \gamma_n) (\delta_k^{-1} \delta_g)^{-1} \beta_k \end{aligned}$$

Proof. (a) Here one moves only γ_j and γ_{j+1} . Clearly $\gamma_j' \simeq \gamma_{j+1}$ and $\gamma_j \gamma_{j+1} \simeq \gamma_j' \gamma_{j+1}'$, so $\gamma_{j+1}' \simeq \gamma_{j+1}^{-1} \gamma_j \gamma_{j+1}$.

(b) The effect of the homotopy of $\gamma_i, \alpha_k, \beta_k$ along ρ_{ik} is pictured on Figure 3. One moves the point b_i along the arc r_{ik} pictured on Figure 2 and together with it deforms the closed arc γ_i . At the moment γ_i^u reaches the side β_k one deforms also β_k in order that the condition γ_i^u and β_k^u have no points in common except b_0 remains valid. None of γ_j for $j \neq i$, or α_ℓ for $\forall \ell$, or β_ℓ for $\forall \ell \neq k$ changes in this homotopy. The effect of cutting Y along the closed arcs $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k, \dots, \alpha_g, \beta_g$ is the same as to cut a region containing b_i from the original $4g$ -polygon and glue it along the side β_i as described in Figure 3 (NW) (NW=Northwest). We wish to express γ_i' in terms of the standard system $\gamma_j, \alpha_\ell, \beta_\ell, 1 \leq j \leq n, 1 \leq \ell \leq g$. The

FIGURE 3. Homotopy along ρ_{ik}

closed arc γ'_i is pictured in the original $4g$ -polygon on Figure 3 (NE). It is clear that $\gamma'_i \simeq \eta_{ik}\gamma_i\eta_{ik}^{-1}$ where η_{ik} is the closed arc pictured on Figure 3 (SW). That η_{ik} is homotopic to $(\gamma_1 \cdots \gamma_{i-1})^{-1}(\delta_{k-1}\alpha_k)\delta_k^{-1}(\gamma_1 \cdots \gamma_{i-1})$ is evident from Figure 3 (SE). Furthermore $\eta_{ik}\gamma_i\eta_{ik}^{-1} \simeq (\eta_{ik}\gamma_i)\gamma_i(\eta_{ik}\gamma_i)^{-1}$. Using the relation $\gamma_1 \cdots \gamma_n \simeq \delta_g$ we obtain $\eta_{ik}\gamma_i \simeq (\gamma_1 \cdots \gamma_{i-1})^{-1}\delta_{k-1}\alpha_k(\delta_k^{-1}\delta_g)(\gamma_{i+1} \cdots \gamma_n)^{-1}$. This proves the formula for γ'_i of Part (b). The calculation of β'_k is similar. The closed arc β'_k is pictured in the original $4g$ -polygon on Figure 4 (NW). It is homotopic to $\omega_{ik} \cdot \beta_k$ where ω_{ik} is the closed arc based at b_0 pictured on Figure 4 (NE). We then consider $\delta_k\omega_{ik}\delta_k^{-1}$ (see Figure 4 (SW)). The latter is homotopic to $(\gamma_1 \cdots \gamma_{i-1})\gamma_i(\gamma_1 \cdots \gamma_{i-1})^{-1}$. We thus obtain $\omega_{ik} \simeq (\delta_k^{-1}\gamma_1 \cdots \gamma_{i-1})\gamma_i(\gamma_1 \cdots \gamma_1)^{-1}\delta_k$. We then have

$$\begin{aligned} \omega_{ik} &\simeq \delta_k^{-1}(\gamma_1 \cdots \gamma_{i-1}\gamma_i)\gamma_i(\gamma_1 \cdots \gamma_{i-1}\gamma_i)^{-1}\delta_k \\ &\simeq [\delta_k^{-1}\delta_g(\gamma_{i+1} \cdots \gamma_n)^{-1}]\gamma_i[\gamma_{i+1} \cdots \gamma_n\delta_g^{-1}\delta_k] \end{aligned}$$

since $\gamma_1 \cdots \gamma_n \simeq \delta_g$. This proves the second formula of Part (b).

(c) The arguments here are very similar to those of Part (b). One deforms γ_i and α_k with b_i^u moving along the arc t_{ik} (see Figure 2). When b_i^u returns to b_i

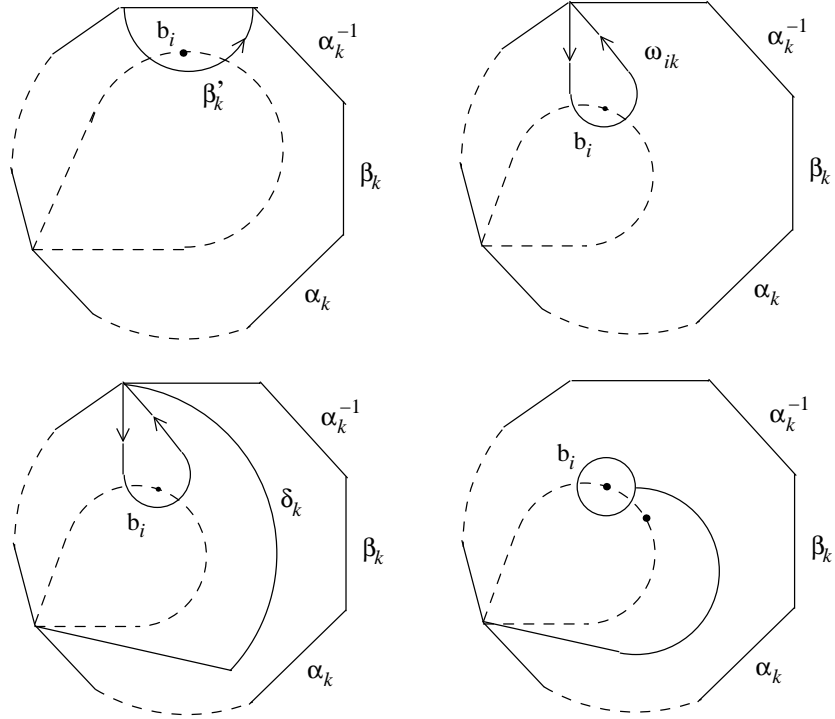


FIGURE 4

one obtains closed arcs γ'_i and α'_k for which $\gamma'_i \simeq \theta_{ik}\gamma_i\theta_{ik}^{-1}$, $\alpha'_k = \varepsilon_{ik}\alpha_k$ where θ_{ik} and ε_{ik} are represented by arcs in the original $4g$ -polygon pictured on Figure 5. We calculate θ_{ik} in the following way. We consider it as a product of four arcs

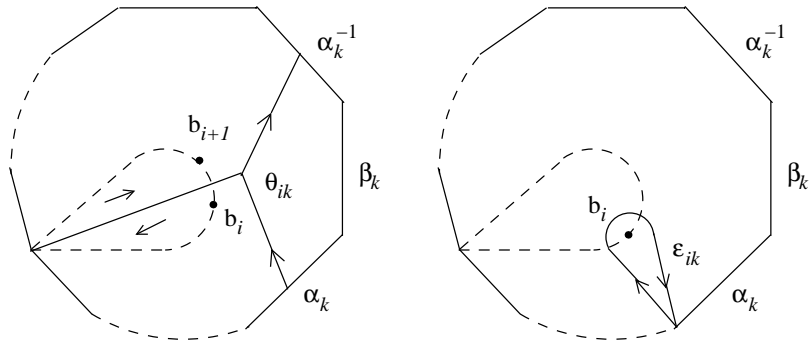


FIGURE 5

according to the picture. We deform θ_{ik} in such a way that the first arc becomes a closed simple arc encircling $\{b_1, \dots, b_i\}$ in clockwise direction, then the second arc goes from the initial point of α_1 to the end point of α_k^{-1} , the third arc goes from

the initial point of α_k to the initial point of α_1 and the fourth arc equals the first one with the opposite orientation. We have accordingly

$$\theta_{ik} \simeq (\gamma_1 \cdots \gamma_i)^{-1} (\delta_k \beta_k) \delta_{k-1}^{-1} (\gamma_1 \cdots \gamma_i)$$

Conjugating γ_i by θ_{ik} we may cancel the last factor γ_i from θ_{ik} and replace $(\gamma_1 \cdots \gamma_i)^{-1}$ by $(\gamma_{i+1} \cdots \gamma_n) \delta_g^{-1}$. We thus obtain $\gamma'_i = \nu_{ik} \gamma_i \nu_{ik}^{-1}$ where

$$\nu_{ik} \simeq (\gamma_{i+1} \cdots \gamma_n) (\delta_k^{-1} \delta_g)^{-1} \beta_k \delta_{k-1}^{-1} (\gamma_1 \cdots \gamma_{i-1})$$

Finally, $\delta_{k-1} \varepsilon_{ik} \delta_{k-1}^{-1} \simeq (\gamma_1 \cdots \gamma_{i-1}) \gamma_i^{-1} (\gamma_1 \cdots \gamma_{i-1})^{-1}$. This proves the last formula of Part (c).

In order to obtain the formulas of (d), (e) and (f) from those of (a), (b) and (c) we notice that if we apply in each case to $(\gamma''_1, \dots, \gamma''_n; \alpha''_1, \dots, \beta''_g)$ the braid moves $\sigma_j, \rho_{ik}, \tau_{ik}$ respectively we obtain $(\gamma_1, \dots, \gamma_n; \alpha_1, \dots, \beta_g)$. For example, in order to verify (d) we have $\gamma_j \simeq \gamma''_{j+1}, \gamma_{j+1} \simeq (\gamma''_{j+1})^{-1} \gamma''_j \gamma''_{j+1}$. Therefore $\gamma''_j \simeq \gamma_j \gamma_{j+1} \gamma_j^{-1}$. In Case (e) applying ρ_{ik} to $(\gamma''_1, \dots, \beta''_g)$ we obtain $\gamma_j \simeq \gamma''_j$ for $\forall j \neq i, \alpha_\ell \simeq \alpha''_\ell$ for $\forall \ell, \beta_\ell \simeq \beta''_\ell$ for $\forall \ell \neq k$. Furthermore $\gamma_i \simeq \mu''_{ik} \gamma''_i (\mu''_{ik})^{-1}$ where μ''_{ik} is expressed by $\gamma''_j, \alpha''_\ell, \beta''_\ell$ as in (b). Since neither γ''_i nor β''_k enter in this expression we may replace $\gamma''_j, \alpha''_\ell, \beta''_\ell$ by $\gamma_j, \alpha_\ell, \beta_\ell$ and we obtain $\mu''_{ik} = \mu_{ik}$. Thus $\gamma''_i \simeq \mu_{ik}^{-1} \gamma_i \mu_{ik}$. Similarly we have $\beta_k = (\zeta_{ik} \gamma''_i \zeta_{ik}^{-1}) \beta''_k$ where ζ_{ik} is as in (b). Replacing γ''_i by $\mu_{ik}^{-1} \gamma_i \mu_{ik}$ and canceling we obtain $\beta''_k = (\zeta_{ik}^{-1} \gamma_i^{-1} \zeta_{ik}) \beta_k$ where ζ_{ik} is as in (e). In a similar manner one deduces (f) from (c). \square

2.7. Recall from Section 1 and § 2.1 that given $D \in Y^{(n)} - \Delta, b_0 \in Y - D$ and fixing a standard system of closed arcs $\gamma_1, \dots, \gamma_n; \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ there is a bijective correspondence between the fiber of $H_n^G(Y, b_0) \rightarrow (Y - b_0)^{(n)} - \Delta$ over D and the set of Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$. Similarly there is a bijective correspondence between the fiber of $U(b_0) \rightarrow (Y - b_0)^{(n)} - \Delta$ and the set of equivalence classes of Hurwitz systems $[t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ modulo inner automorphisms, where $U(b_0)$ is the preimage of $(Y - b_0)^{(n)} - \Delta$ with respect to the map $H_n^G(Y) \rightarrow Y^{(n)} - \Delta$. In the next theorem we calculate the monodromy action of $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ on these fibers. According to Proposition 2.5 it suffices to determine the braid moves which correspond to the generators $\sigma_j, \rho_{ik}, \tau_{ik}$. We denote the corresponding braid moves of the first type (cf. Definition 6) by $\sigma'_j, \rho'_{ik}, \tau'_{ik}$ and we denote the corresponding braid moves of the second type (inverse to those of the first type) by $\sigma''_j, \rho''_{ik}, \tau''_{ik}$.

Theorem 2.8. *Let $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g), \lambda_{g+k} = \mu_k$, be a Hurwitz system of the group G . Let $u_k = [\lambda_1, \mu_1] \cdots [\lambda_k, \mu_k]$ for $k = 1, \dots, g$ and let $u_0 = 1$. The following formulas hold for the braid moves $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) \mapsto (t'_1, \dots, t'_n; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ of the first type.*

a. For σ'_j where $1 \leq j \leq n-1$

$$(11) \quad \begin{aligned} t'_i &= t_i \quad \text{for } \forall i \neq j, j+1, \quad \lambda'_\ell = \lambda_\ell, \quad \mu'_\ell = \mu_\ell \quad \text{for } \forall \ell \\ (t_j, t_{j+1}) &\mapsto (t'_j, t'_{j+1}) = (t_j t_{j+1} t_j^{-1}, t_j). \end{aligned}$$

b. For ρ'_{ik} where $1 \leq i \leq n$, $1 \leq k \leq g$

$$t'_j = t_j \text{ for } \forall j \neq i, \quad \lambda'_\ell = \lambda_\ell \text{ for } \forall \ell, \quad \mu'_\ell = \mu_\ell \text{ for } \forall \ell \neq k$$

$$(t_i, \mu_k) \mapsto (t'_i, \mu'_k) = (a_1^{-1} t_i a_1, (b_1^{-1} t_i^{-1} b_1) \mu_k) \text{ where}$$

$$a_1 = (t_1 \cdots t_{i-1})^{-1} u_{k-1} \lambda_k (u_k^{-1} u_g) (t_{i+1} \cdots t_n)^{-1}, \quad b_1 = (t_1 \cdots t_{i-1})^{-1} u_{k-1} \lambda_k.$$

c. For τ'_{ik} where $1 \leq i \leq n$, $1 \leq k \leq g$

$$t'_j = t_j \text{ for } \forall j \neq i, \quad \lambda'_\ell = \lambda_\ell \text{ for } \forall \ell \neq k, \quad \mu'_\ell = \mu_\ell \text{ for } \forall \ell$$

$$(t_i, \lambda_k) \mapsto (t'_i, \lambda'_k) = (c_1^{-1} t_i c_1, (d_1^{-1} t_i d_1) \lambda_k) \text{ where}$$

$$c_1 = t_{i+1} \cdots t_n (u_k^{-1} u_g)^{-1} \mu_k (u_{k-1})^{-1} t_1 \cdots t_{i-1}, \quad d_1 = t_{i+1} \cdots t_n (u_k^{-1} u_g)^{-1} \mu_k.$$

The following formulas hold for the braid moves $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) \mapsto (t'_1, \dots, t'_n; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ of the second type.

d. For σ''_j where $1 \leq j \leq n-1$

$$t''_i = t_i \text{ for } \forall i \neq j, j+1, \quad \lambda''_\ell = \lambda_\ell, \quad \mu''_\ell = \mu_\ell \text{ for } \forall \ell$$

$$(12) \quad (t_j, t_{j+1}) \mapsto (t''_j, t''_{j+1}) = (t_{j+1}, t_{j+1}^{-1} t_j t_{j+1}).$$

e. For ρ''_{ik} where $1 \leq i \leq n$, $1 \leq k \leq g$

$$t''_j = t_j \text{ for } \forall j \neq i, \quad \lambda''_\ell = \lambda_\ell \text{ for } \forall \ell, \quad \mu''_\ell = \mu_\ell \text{ for } \forall \ell \neq k$$

$$(t_i, \mu_k) \mapsto (t''_i, \mu''_k) = (a_2^{-1} t_i a_2, (b_2^{-1} t_i b_2) \mu_k) \text{ where}$$

$$a_2 = t_{i+1} \cdots t_n (u_k^{-1} u_g)^{-1} \lambda_k^{-1} (u_{k-1})^{-1} t_1 \cdots t_{i-1}, \quad b_2 = t_{i+1} \cdots t_n (u_k^{-1} u_g)^{-1}.$$

f. For τ''_{ik} where $1 \leq i \leq n$, $1 \leq k \leq g$

$$t''_j = t_j \text{ for } \forall j \neq i, \quad \lambda''_\ell = \lambda_\ell \text{ for } \forall \ell \neq k, \quad \mu''_\ell = \mu_\ell \text{ for } \forall \ell$$

$$(t_i, \lambda_k) \mapsto (t''_i, \lambda''_k) = (c_2^{-1} t_i c_2, (d_2^{-1} t_i^{-1} d_2) \lambda_k) \text{ where}$$

$$c_2 = (t_1 \cdots t_{i-1})^{-1} u_{k-1} \mu_k^{-1} (u_k^{-1} u_g) (t_{i+1} \cdots t_n)^{-1}, \quad d_2 = (t_1 \cdots t_{i-1})^{-1} u_{k-1}.$$

Proof. Formulas (d), (e) and (f) are obtained from formulas (a), (b) and (c) of Theorem 2.6 respectively applying the homomorphism $m : \pi_1(Y - D, b_0) \rightarrow G$ and equalities (7) and (8). By Lemma 2.3 the braid moves of the first type are inverse to the braid moves of the second type. We may thus obtain formulas (a), (b) and (c) from formulas (d), (e) and (f) of Theorem 2.6 respectively applying the homomorphism m . \square

Corollary 2.9. *Using the notation of Theorem 2.8 the following formulas hold.*

(i) For ρ'_{ik} :

$$\rho'_{nk} : t_n \mapsto t'_n = a_1^{-1} t_n a_1, \quad \text{where } a_1 = (u_k^{-1} u_g)^{-1} [\mu_k, \lambda_k] \lambda_k (u_k^{-1} u_g)$$

$$\rho'_{1k} : \mu_k \mapsto \mu'_k = (b_1^{-1} t_1^{-1} b_1) \mu_k, \quad \text{where } b_1 = u_{k-1} \lambda_k$$

$$\rho'_{nk} : \mu_k \mapsto \mu'_k = (b_1^{-1} t_n^{-1} b_1) \mu_k, \quad \text{where } b_1 = (u_k^{-1} u_g)^{-1} [\mu_k, \lambda_k] \lambda_k.$$

(ii) For τ'_{ik} :

$$\tau'_{1k} : t_1 \mapsto t'_1 = c_1^{-1} t_1 c_1, \quad \text{where } c_1 = u_{k-1} [\lambda_k, \mu_k] \mu_k u_{k-1}^{-1}$$

$$\tau'_{1k} : \lambda_k \mapsto \lambda'_k = (d_1^{-1} t_1 d_1) \lambda_k, \quad \text{where } d_1 = u_{k-1} [\lambda_k, \mu_k] \mu_k$$

$$\tau'_{nk} : \lambda_k \mapsto \lambda'_k = (d_1^{-1} t_n d_1) \lambda_k, \quad \text{where } d_1 = (u_k^{-1} u_g)^{-1} \mu_k.$$

(iii) For ρ''_{ik} :

$$\rho''_{1k} : t_1 \mapsto t_1'' = a_2^{-1} t_1 a_2, \quad \text{where } a_2 = u_{k-1} [\lambda_k, \mu_k] \lambda_k^{-1} u_{k-1}^{-1}$$

$$\rho''_{1k} : \mu_k \mapsto \mu_k'' = (u_k^{-1} t_1 u_k) \mu_k,$$

$$\rho''_{nk} : \mu_k \mapsto \mu_k'' = (b_1^{-1} t_n b_1) \mu_k, \quad \text{where } b_1 = (u_k^{-1} u_g)^{-1}.$$

(iv) For τ''_{ik} :

$$\tau''_{nk} : t_n \mapsto t_n'' = c_2^{-1} t_n c_2, \quad \text{where } c_2 = (u_k^{-1} u_g)^{-1} [\mu_k, \lambda_k] \mu_k^{-1} (u_k^{-1} u_g)$$

$$\tau''_{1k} : \lambda_k \mapsto \lambda_k'' = (u_{k-1}^{-1} t_1^{-1} u_{k-1}) \lambda_k,$$

$$\tau''_{nk} : \lambda_k \mapsto \lambda_k'' = (d_2^{-1} t_n^{-1} d_2) \lambda_k, \quad \text{where } d_2 = (u_k^{-1} u_g)^{-1} [\mu_k, \lambda_k].$$

In particular

$$(13) \quad \rho''_{ng} : \mu_g \mapsto t_n \mu_g, \quad \tau''_{11} : \lambda_1 \mapsto t_1^{-1} \lambda_1.$$

Proof. Let us prove the first formula of Part (ii). The other formulas can be either proved similarly or are restatements of particular cases of Theorem 2.8. We have $\tau'_{1k} : t_1 \mapsto t_1' = c_1^{-1} t_1 c_1$ where $c_1 = t_2 \cdots t_n (u_k^{-1} u_g)^{-1} \mu_k u_{k-1}^{-1}$. Since $t_1 \cdots t_n = u_g$ it holds $(t_2 \cdots t_n)^{-1} t_1 (t_2 \cdots t_n) = u_g^{-1} t_1 u_g$. Hence

$$\begin{aligned} t_1' &= (u_{k-1} \mu_k^{-1} u_k^{-1}) t_1 (\cdots)^{-1} = (u_{k-1} \mu_k^{-1} [\lambda_k, \mu_k]^{-1} u_{k-1}^{-1}) t_1 (\cdots)^{-1} \\ &= (u_{k-1} [\lambda_k, \mu_k] \mu_k u_{k-1}^{-1})^{-1} t_1 (u_{k-1} [\lambda_k, \mu_k] \mu_k u_{k-1}^{-1}) \end{aligned}$$

□

Summing up the discussion made so far in this section we obtain the following result.

Theorem 2.10. *Let n, g be integers such that $n > 0, g \geq 1$. Let us consider the set of all Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ of the group G (cf. Definition 5). Let F be the free group generated by the symbols $\sigma_j, \rho_{ik}, \tau_{ik}$ where $1 \leq j \leq n-1, 1 \leq i \leq n, 1 \leq k \leq g$. Let us consider the action of F on the right on the set of Hurwitz systems of G defined by the formulas for $\sigma'_j, \rho'_{ik}, \tau'_{ik}$ of Theorem 2.8 (a)–(c). Then the connected components of $H_n^G(Y, b_0)$ (cf. Proposition 1.8) correspond bijectively to the orbits of this action and the connected components of $H_n^G(Y)$ (cf. Proposition 1.9) correspond bijectively to the orbits of the associated action of F on the set of equivalence classes of Hurwitz systems modulo inner automorphisms of G . The same statements hold if we consider instead the action of F on the left on the set of Hurwitz systems of G defined by the formulas for $\sigma''_j, \rho''_{ik}, \tau''_{ik}$ of Theorem 2.8 (d)–(f).*

Proof. $H_n^G(Y, b_0) \rightarrow (Y - b_0)^{(n)} - \Delta$ is a topological covering map. According to the definition of product of arcs (cf. (1)) the monodromy action of the fundamental group $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ on the fiber over D is a right action. The identification of this fiber with the set of Hurwitz systems, Proposition 2.5 and the calculation of the braid moves of the first type $\sigma'_j, \rho'_{ik}, \tau'_{ik}$ in Theorem 2.8 yield the statement about the orbits of the right action of F .

Consider the associated left action $gx = xg^{-1}$. The orbits are the same and according to Lemma 2.3 the braid moves of the second type $\sigma''_j, \rho''_{ik}, \tau''_{ik}$ are inverse to those of the first type $\sigma'_j, \rho'_{ik}, \tau'_{ik}$ respectively. This shows the statement about the orbits of the left action of F .

Let $b_0 \in Y$. The inclusion $Y - b_0 \hookrightarrow Y$ induces an epimorphism $\pi_1((Y - b_0)^{(n)} - \Delta, D) \rightarrow \pi_1(Y^{(n)} - \Delta, D)$, so the monodromy action of these fundamental groups on the fiber over D of $U(b_0) \rightarrow (Y - b_0)^{(n)} - \Delta$ (cf. § 2.7) and $H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ respectively yields the same monodromy group. This shows the statement about $H_n^G(Y)$. \square

Corollary 2.11. *Let O_1, \dots, O_k be k different conjugacy classes of G . Let $\underline{n} = n_1 O_1 + \dots + n_k O_k$, let $|\underline{n}| = n_1 + \dots + n_k = n$ and let $H_n^G(Y, y_0)$ be as in Definition 4. Then $H_n^G(Y, y_0)$ is connected (hence irreducible) if and only if F acts transitively on the set of Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ which have the property that n_i of the elements t_1, \dots, t_n belong to O_i , $i = 1, \dots, k$. Similarly $H_n^G(Y)$ is connected (hence irreducible) if and only if F acts transitively on the set of equivalence classes of Hurwitz systems $[t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ which have the above property.*

3. HURWITZ SYSTEMS OF ARBITRARY GROUPS

The definitions and the arguments of Section 1 and Section 2 may be applied to arbitrary, possibly infinite, groups. If G is infinite the sets $H_n^G(Y, y_0)$ and $H_n^G(Y)$ are non empty if and only if G has at least one Hurwitz system of generators. In this case each one of $H_n^G(Y, y_0)$ and $H_n^G(Y)$ has a structure of a complex manifold, the maps $H_n^G(Y, y_0) \rightarrow (Y - y_0)^{(n)} - \Delta$ and $H_n^G(Y) \rightarrow Y^{(n)} - \Delta$ given by $(D, m) \mapsto D$ and $(D, \underline{m}) \mapsto D$ respectively, are topological covering maps and are holomorphic. The set $H_n^G(Y, y_0)$ corresponds bijectively to the set of G -equivalence classes $[p' : C' \rightarrow Y - D, z_0]$ where:

- $D \in Y^{(n)} - \Delta$, $y_0 \notin D$;
- $p' : C' \rightarrow Y - D$ is a holomorphic map of Riemann surfaces, which is a regular topological covering map;
- an isomorphism $G \xrightarrow{\sim} \text{Deck}(C'/Y - D)$ is fixed;
- if $D = b_1 + \dots + b_n$, the local monodromies of p' at the points b_1, \dots, b_n are all nontrivial;
- $z_0 \in p'^{-1}(y_0)$.

Similarly, the set $H_n^G(Y)$ corresponds bijectively to the set of G -equivalence classes $[p' : C' \rightarrow Y - D]$ as above with the conditions $y_0 \notin D$ and $z_0 \in p'^{-1}(y_0)$ skipped. In no point of the proofs of Theorem 2.8 and Theorem 2.10 the condition that G is finite is used. So they hold under the only assumption that the group G has at least one Hurwitz system of generators $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$.

Definition 7. We call two Hurwitz systems of G braid-equivalent if one is obtained from the other by a finite sequence of braid moves $\sigma'_j, \rho'_{ik}, \tau'_{ik}, \sigma''_j, \rho''_{ik}, \tau''_{ik}$ where $1 \leq j \leq n - 1$, $1 \leq i \leq n$, $1 \leq k \leq g$. We denote the braid equivalence by \sim .

Theorem 2.10 shows that two elements (D, m) and (D, m') of $H_n^G(Y, y_0)$ correspond to braid-equivalent Hurwitz systems if and only if they belong to one and the same connected component of $H_n^G(Y, y_0)$.

The following statement is a useful tool for determining if two Hurwitz systems are braid-equivalent.

Proposition 3.1. *Let $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ be a Hurwitz system of the group G . Suppose that $t_i t_{i+1} = 1$. Let H be the subgroup of G generated by*

$\{t_1, \dots, t_{i-1}, t_{i+2}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g\}$. Then for every $h \in H$ the given Hurwitz system is braid-equivalent to $(t_1, \dots, t_{i-1}, t_i^h, t_{i+1}^h, t_{i+2}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$

Proof. Let us fix $t_1, \dots, t_{i-1}, t_{i+2}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g$. Let $H_1 \subseteq H$ be the subset consisting of elements h such that the statement of the lemma holds for an arbitrary pair $(t_i, t_{i+1}) = (\tau, \tau^{-1})$.

Step 1. We claim H_1 is a subgroup of H . Let $h \in H_1$ and let $t = ht_i h^{-1}$. Then $t_i = t^h$, so by assumption $(\dots, t^h, (t^{-1})^h, \dots)$ can be obtained from $(\dots, t, t^{-1}, \dots)$ by a sequence of braid moves. Then one can obtain $(\dots, t, t^{-1}, \dots)$ from $(\dots, t_i, t_i^{-1}, \dots)$ by the inverse sequence of braid moves. Thus $h^{-1} \in H_1$. If $h_1, h_2 \in H_1$, then $(\dots, t_i, t_i^{-1}, \dots) \sim (\dots, t_i^{h_1}, (t_i^{-1})^{h_1}, \dots) \sim (\dots, t_i^{h_1 h_2}, (t_i^{-1})^{h_1 h_2}, \dots)$, so $h_1 h_2 \in H_1$.

Step 2. For every $\ell \neq i, i+1$ the element t_ℓ belongs to H_1 . Applying a sequence of braid moves σ_j', σ_j'' we can move the adjacent pair (t_i, t_{i+1}) wherever we want among the first n elements of the Hurwitz system without changing the other elements. So move (t_i, t_{i+1}) to the left side of t_ℓ . Then we have $(t_i, t_{i+1}, t_\ell) \sim (t_i, t_\ell, t_\ell^{-1} t_{i+1} t_\ell) \sim (t_\ell, t_\ell^{-1} t_i t_\ell, t_\ell^{-1} t_{i+1} t_\ell)$. We then move the pair (t_i^h, t_{i+1}^h) with $h = t_\ell$ back to the initial position.

Step 3. For every $k = 1, \dots, g$ the element $h = u_{k-1} \lambda_k u_k^{-1}$ belongs to H_1 . First suppose that $i = 1$. In this case $t_2 = t_1^{-1}$. Let us perform a braid move ρ'_{1k} . One obtains $(t_1, t_1^{-1}, \dots, \lambda_k, \mu_k, \dots) \sim (t'_1, t_1^{-1}, \dots, \lambda_k, \mu'_k, \dots)$ where $t'_1 = a_1^{-1} t_1 a_1$ with $a_1 = u_{k-1} \lambda_k (u_k^{-1} u_g) (t_2 \cdots t_n)^{-1}$ and $\mu'_k = (b_1^{-1} t_1^{-1} b_1) \mu_k$ with $b_1 = u_{k-1} \lambda_k$. We have $t_1 \cdots t_n = u_g$, so $a_1 = u_{k-1} \lambda_k u_k^{-1} t_1$. Let us move $t_2 = t_1^{-1}$ to the first place using σ''_1 . One obtains $(t_1^{-1}, t_1 t'_1 t_1^{-1}, \dots, \lambda_k, \mu'_k, \dots)$. Here $t_1 t'_1 t_1^{-1} = t_1^h$ where $h = u_{k-1} \lambda_k u_k^{-1}$. Let us perform again ρ'_{1k} . The elements u_{k-1}, λ_k have not changed with respect to the original Hurwitz system, so $\mu'_k \mapsto [(u_{k-1} \lambda_k)^{-1} (t_1^{-1})^{-1} (u_{k-1} \lambda_k)] \mu'_k = \mu_k$. Moving the second element t_1^h to the first place by σ''_1 we obtain $(t_1^h, \tilde{t}_2, t_3, \dots, \lambda_k, \mu_k, \dots)$. Since $t_1 t_2 t_3 \cdots t_n = u_g = t_1^h \tilde{t}_2 t_3 \cdots t_n$ we must have $\tilde{t}_2 = (t_1^{-1})^h$. This proves $(t_1, t_1^{-1}, t_3, \dots) \sim (t_1^h, (t_1^{-1})^h, t_3, \dots)$ with $h = u_{k-1} \lambda_k u_k^{-1}$. One extends this braid equivalence to every adjacent pair (t_i, t_{i+1}) with $t_i t_{i+1} = 1$ by moving first the pair to the front, applying the braid equivalence we have just proved and moving the obtained pair back to the original position.

Step 4. For every $k = 1, \dots, g$ the element $h = u_{k-1} \mu_k^{-1} u_k^{-1}$ belongs to H_1 . The proof is the same as that of Step 3. One uses the braid move τ''_{1k} instead of ρ'_{1k} .

Step 5. By the preceding steps it remains to verify that λ_k, μ_k for $\forall k$ belong to the subgroup $H_2 \subseteq H_1$ generated by $t_j, u_{k-1} \lambda_k u_k^{-1}, u_{k-1} \mu_k^{-1} u_k^{-1}$ where $j = 1, \dots, i-1, i+2, \dots, n$ and $k = 1, \dots, g$. We prove this by induction on k . We have $\lambda_1 u_1^{-1} \in H_2$, $u_1 \mu_1 = (\mu_1^{-1} u_1^{-1})^{-1} \in H_2$, so $\lambda_1 \mu_1 \in H_2$. Furthermore $H_2 \ni u_1 \mu_1 = [\lambda_1, \mu_1] \mu_1 = \lambda_1 \mu_1 \lambda_1^{-1}$. So $\lambda_1^{\pm 1} \in H_2$ and $\mu_1^{\pm 1} \in H_2$. Let $k \geq 2$. Suppose, by inductive assumption, that $\lambda_1, \mu_1, \dots, \lambda_{k-1}, \mu_{k-1}$ belong to H_2 . Then $u_{k-1} = \prod_{\ell=1}^{k-1} [\lambda_\ell, \mu_\ell] \in H_2$. We have $u_{k-1} \lambda_k u_k^{-1} \in H_2$, $u_k \mu_k u_{k-1}^{-1} = (u_{k-1} \mu_k^{-1} u_k^{-1})^{-1} \in H_2$, so $u_{k-1} \lambda_k \mu_k u_{k-1}^{-1} \in H_2$ and therefore $\lambda_k \mu_k \in H_2$. Furthermore $H_2 \ni u_k \mu_k u_{k-1}^{-1} = u_{k-1} [\lambda_k, \mu_k] \mu_k u_{k-1}^{-1}$. Therefore $\lambda_k \mu_k \lambda_k^{-1} \in H_2$, so $\lambda_k^{\pm 1} \in H_2$ and $\mu_k^{\pm 1} \in H_2$.

The proposition is proved. \square

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