Class notes on Cooperative Games[1]

Part I

Basic Core and Shapley

1 The Core

Recall the main definitions from OR chapter 13. Cooperative games model situations in which players may cooperate to achieve their goals. It is assumed that every set of players can form a coalition; the maximal amount that a coalition $S \subseteq N$ can generate through cooperation is called the *worth* of the coalition and is denoted by $v(S)$. The function v is called the characteristic function of the game. We will always assume that the game is cohesive as in definition 258.1.[2] For a payoff profile $x = (x_i)_{i \in N}$ the total payoff to members of S is denoted by $x(S)$, that is $x(S) = \sum_{i \in S} x_i$; a profile with $x(N) = v(N)$ i.e. which allocates $v(N)$ among the players will be called an *allocation* (OR calls it a feasible payoff profile). The Core of the game is the set of allocations x robust to deviations by coalitions, that is such that $x(S) \geq v(S)$ for all S. We let $n = \#N$.

We shall start by finding the Core in simple cases. We write v_1 for $v(\{1\})$, v_{12} for $v(\{1, 2\})$ etc.

1.1 The basic two-person example

This is *the* fundamental example. There are two partners $N = \{1, 2\}$, $0 < v_1, v_2 < 1, v_1 + v_2 < 1$ but $v_{12} = 1$. How to share the surplus $1 - (v_1 + v_2)$ arising from cooperation? In this game the Core gives no hints because the only restrictions on allocations (x_1, x_2) which it imposes are $x_1 + x_2 = 1, x_1 \geq v_1$ and $x_2 \geq v_2$, that is $v_1 \leq x_1 \leq 1 - v_2$. In other words $x_i = v_i + \sigma$ for any $0 \leq \sigma \leq 1 - (v_1 + v_2)$, with $x_j = 1 - x_i$.

1.2 Find the Core

Owner of resource (player 1) with two potential partners

$$n = 3, v(N) = 3, v_1 = 1, v_2 = v_3 = 0, v_{12} = 2, v_{13} = 3, v_{23} = 0.$$

Core: $3 \leq x_1 + x_3 \leq 3$ so $x_2 = 0$; now $2 \leq x_1 + x_2 = x_1$ so the Core is

$$2 \leq x_1 \leq 3, x_2 = 0, x_3 = 3 - x_1$$

[1] S Modica, Game Theory LM-77. Based mostly on Osborne-Rubinstein

[2] That is $v(N) \geq \sum_{k=1}^{K} v(S_k)$ for any partition $\{S_1, \ldots, S_K\}$ of N.

Shapley mechanically (for later), $Sh = 1/6 * (13, 1, 4)$. Here $x_2 > 0$ so not in Core.

Remark: Core is really an index of power

1.3 Find the Core

There are three fund managers, first has 300, second 100 third 200; returns are 8% below 200, 9% from 200 to < 500 and 10% with at least 500. So

$$n = 3, v(N) = 60, v_1 = 27, v_2 = 8, v_3 = 18, v_{12} = 36, v_{13} = 50, v_{23} = 27.$$

From $18 \le x_3 = 60 - x_1 - x_2$ since $x_1 + x_2 \ge 36$ you get $36 \le x_1 + x_2 \le 42, x_1 \ge 27, x_2 \ge 8$; from $x_2 + x_3 \ge 27$ get $x_1 \le 33$ and from $x_1 + x_3 \ge 50$ get $x_2 \le 10$. In the (x_1, x_2) plane it is a square with SW and NE corners chopped off.

Shapley mechanically, $Sh = (30, 9, 21)$. Here $Sh \in Core$; also notice that $Sh \neq$ proportional $(30, 10, 20)$ (which is a Core element).

1.4 The Core may be empty: expedition (OR 259.2)

Any 2 players can carry 1 piece, $n \ge 3$

$$v(S) = \begin{cases} \#S/2 & \#S \text{ even} \\ (\#S - 1)/2 & \#S \text{ odd} \end{cases}$$

We show that the Core empty if n is odd, if n is even then it is the singleton $(1/2, \ldots, 1/2)$. Suppose first n is odd; here $0 < vN = v(N \setminus j)$ so $\sum_{i \neq j} x_i = vN$ whence $x_j = 0$, for all j, contradiction. Consider n even (then ≥ 4); if $x_i + x_j > 1$ then $N \setminus \{i, j\}$ is blocking, so $x_i + x_j = 1\ \forall i, j$. Now fix i and take $j, k \neq i$; it must be $x_i + x_j = x_i + x_k = x_j + x_k = 1$; 2nd and 3rd give $x_i = x_j$, and then 1st gives $x_i = 1/2$.

Exercise. Find the Core if $n = 2$.

1.5 The Core may be empty: 2-player coalitions too strong (OR 259.1)

In this game there are 3 players; single players are powerless, 2-player coalitions can make $\alpha \in (0, 1)$ and $v(N) = 1$. So

$$n = 3, v(N) = 1, v_i = 0, v_{ij} = \alpha \in (0, 1).$$

Core is $x(N) = 1, x(S) \ge \alpha$ if $\#S = 2$. Must be $x_3 = 1 - x_1 - x_2$; for (x_1, x_2) it must be $\alpha \le x_1 + x_2 \le 1$; from $\alpha \le x_2 + x_3 = x_2 + 1 - x_1 - x_2$ get $x_1 \le 1 - \alpha$, and similarly

$x_2 \leq 1 - \alpha$; so the Core is

$$\alpha \leq x_1 + x_2 \leq 1$$
$$0 \leq x_1, x_2 \leq 1 - \alpha$$
$$x_3 = 1 - x_1 - x_2.$$

If $\alpha > 2/3$, for x in Core if $x_1 = x_2 = 1 - \alpha$ then $x_1 + x_2 = 2 - 2\alpha < 2 - 2 * 2/3 = 2/3 < \alpha$ so the Core is empty.

Shapley: by symmetry we get $x_i = 1/3$ all i - for all α. Equal to Core if $\alpha = 2/3$.

1.6 Empty Core, majority game: OR 260.3

Here there are $n \geq 3$ players with n odd, and

$$v(S) = \begin{cases} 1 & \#S \geq n/2 \\ 0 & \text{otherwise} \end{cases}$$

Core is empty by the usual argument - do it before you read the footnote.[3]

1.7 The apex game (Core empty again)

In this game there is a "big player" - the apex; the others are small players. Specifically we take $N = \{1, 2, \ldots, 6\}$ where 1 is the apex and

$$v(S) = \begin{cases} 1 & 1 \in S \;\&\; \#S \geq 2 \\ 1 & S = N \setminus \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

Note in particular that $v(N) = 1$. **Exercise:** show that the Core is empty.

1.8 Outside option in exchange

There is one seller, player 1, of a good which is worth nothing to her, and two potential buyers, players 2 and 3 who value the good at b and $100 > b$ respectively. Thus $v(12) = b, v(13) = 100 = v(N)$ and the other coalitions are worthless. Clearly to realize the value of 100 the horse should be sold at some price p to player 3, in other words the candidate allocations for the Core are of the form $(p, 0, 100 - p)$, because at any allocation $(p, b - p, 0)$ with $p \leq b$ we have $x_1 + x_3 \leq b < 100 = v_{13}$. But player 2 is fundamental to determine the lower bound of p, intuitively because the seller can use her as a "threat" to the actual buyer. Indeed since $x_2 = 0$ a core allocation must be such that $x_1 = x_1 + x_2 \geq v(12) = b$ which means that $p \geq b$.

[3] it must be $\sum x_i = 1$ and also $\sum_{j \neq i} x_j = 1$ so $x_i = 0$ for all i which contradicts $\sum x_i = 1$.

The Core is then the set of $(p, 0, 100-p)$ with $b \leq p \leq 100$. It puts a lower bound on p but does not eliminate the indeterminacy on p which reflects the relative bargaining power of seller and buyer.

1.9 An example from politics: veto players (this is OR Exercise 261.1)

Consider a situation where $v(S)$ can only be zero or one - just losing or winning - and $v(N) = 1$. Such games are called *simple* games. Say that i is a *veto player* if $S \not\ni i \Rightarrow v(S) = 0$ - no coalition can win without him. Then the situation is as follows:

(a) If there are no veto players the Core is empty. To show this suppose $x \in \text{Core}(v)$; fix i and take S with $i \notin S$ and $v(S) = 1$ (such an S exists since i is not a veto player); then $1 \geq \sum_{j \in S} x_j \geq v(S) = 1$ which implies $x_i = 0$; this should be true for all i, contradicting $\sum x_i = 1$.

(b) If there is a non-empty set V of veto players then for any x in the Core $\sum_{i \in V} x_i = 1$ (the others get nothing). To prove this we show that these allocations are in the Core and there are no others. So suppose first $x(V) = 1$; is $x(S) \geq v(S)$ for all S? Certainly so if $v(S) = 0$; and if $v(S) = 1$ then S must contain all the veto players that is $S \supseteq V$ which implies $x(S) \geq x(V) = 1 = v(S)$. Suppose on the contrary that $x_i > 0$ for some $i \notin V$; then there is a winning coalition $S \not\ni i$ getting $x(S) < 1 = v(S)$ - so x is not in the Core.

Moral of the story here: if no player has any "power" the Core (being empty) predicts chaos; if there are few veto players it predicts dire consequences for the others; if there are many it has little predictive power. A richer example we shall see is the *weighted majority game.*

1.10 The Core may be unreasonable: the gloves games

Consider the game with $n = 2m+1$ players where m players possess a left glove and $m+1$ a right one. Each pair of players with different gloves has a pair of gloves, worth 1. Two right or two left gloves are worth nothing. Letting N_L, N_R the sets of players with left and right gloves - where $N_L \cap N_R = \emptyset$ and $N_L \cup N_R = N$ - we then have $v(N) = m$ and

$$v(S) = \min\{\#(S \cap N_L), \#(S \cap N_R)\}$$

The Core consists of the unique allocation where $x_i = 1\ \forall i \in N_L$ and $x_j = 0\ \forall j \in N_R$. The latter part follows because $\sum_{h \in N \setminus j} x_h = v(N)$ for any $j \in N_R$. Exercise: show that $x_i = 1$ for all $i \in N_L$.

So all players in the "long" side of the market (the right gloves in this case) get zero in the Core. The unsatisfactory aspect of this is that if we add two left gloves then all the left glovers get zero; so the Core changes dramatically, whereas if m is large the "economics" of the situation is essentially unchanged (there are approximately as many left as right gloves).

Shapley. The marginal contribution of a player in a coalition is either zero - if his glove type is redundant - or one - if there are fewer gloves of his type in the coalition. The Shapley value is expected marginal contribution hence it assigns a left-glover the probability that in a random coalition there are fewer left gloves than right ones; similarly for right glovers. These probabilities can be shown to converge to $1/2$ as $n \to \infty$.

1.11 Buyers and sellers - OR ex 260.1

In similar vein to the gloves game.

1.12 Production - OR ex. 259.3

We use the following property of a concave function. For $0 < h < h'$

$$\frac{f(x) - f(x-h)}{h} \leq \frac{f(x) - f(x-h')}{h'}.$$

First note, assuming the answer is correct, that by concavity $x_c \geq f(w) - w\,[f(w) - f(w-1)] > 0$ so the capitalist gets a positive amount in any Core allocation. Next to prove the result: first if $x_i > f(w) - f(w-1)$ the others would deviate since they can make $f(w-1)$ without him. If on the other hand $x_i \leq f(w) - f(w-1)$ for all i we have to show that no coalition can deviate. This is clear if $S \not\ni c$ or $S = \{c\}$ (since the alternative is to get zero). Suppose $S = Z \cup \{c\}$ with $|Z| > 0$; then $x(S) = f(w) - \sum_{i \notin S} x_i \geq f(w) - (w - |Z|)\,[f(w) - f(w-1)] \geq f(|Z|) = v(S)$, the last inequality by concavity of f. So S would not deviate.

1.13 OR exercise 261.3 pollute the lake

The main concern here is essentially collective choice. The fact that $b \leq nc$ implies that it is optimal for the group that all members treat their waste. On the other hand since $b \geq c$ an isolated member would find it optimal not to threat her waste. The problem is how to share the cost of treating the waste among members. The not so surprising result is that a Core way of doing this is to have everyone pay b.

You can do part a of the exercise (which asks to set up the game), and for part b just show that the allocation $(-b, \ldots, -b)$ is in the Core. This is its unique element when $b = nc$, but proving this is a little harder.

2 The Shapley Value

The definitions are given in OR chapter 14 page 291. As before $n = \#N$. Recall that a *value* is a function ψ which assigns to each game $\langle N, v\rangle$ an allocation; so $\psi(N, v) = (\psi_1(N, v), \ldots, \psi_n(N, v))$ and $\sum_i \psi_i(N, v) = v(N)$.

The Shapley value φ is defined by the average marginal contribution property

$$\varphi_i(N,v) = \frac{1}{n!}\sum\nolimits_{R\in\mathcal{R}} \Delta_i(S_i(R))$$

where $\mathcal{R}$ is the set of all $n!$ orderings of N, $S_i(R)$ is the set of players preceding i in R, and for $S \not\ni i$ $\Delta_i(S) = v(S\cup\{i\}) - v(S)$. OR Proposition 291.3 says that it is the only value satisfying the balanced contribution condition that the contribution of j to i should be equal to i's contribution to j:

$$\psi_i(N,v) - \psi_i(N\setminus j, v^{N\setminus j}) = \psi_j(N,v) - \psi_j(N\setminus i, v^{N\setminus i}).$$

Note that the definition involves subgames of $\langle N,v\rangle$, where for $S\subseteq N$ a subgame $\langle S,v^S\rangle$ is defined by $v^S(T) = v(T)$ for $T\subseteq S$.

Of course one has to check that $\sum_i \varphi_i(N,v) = v(N)$. But

$$\sum\nolimits_i \varphi_i(N,v) = \frac{1}{n!}\sum\nolimits_{R\in\mathcal{R}}\sum\nolimits_i \Delta_i(S_i(R)),$$

and $\sum_i \Delta_i(S_i(R)) = v(N)$ for any R so the displayed sum is $(1/n!)\cdot n!\,v(N)$.[4]

There are alternative characterizations of the Shapley value. One which is particularly appealing is the following, due to Peyton Young, who has shown that the Shapley value is characterized by the following two axioms. The set of players is fixed at N.

1 (symmetry). If for all $S\not\ni i,j$ it is $\Delta_i(S) = \Delta_j(S)$ then $\psi_i(v) = \psi_j(v)$.

2 (marginalism). For games v,w, if for all $S\not\ni i$ $v(S\cup\{i\}) - v(S) = w(S\cup\{i\}) - w(S)$ then $\psi_i(v) = \psi_i(w)$.

2.1 The Shapley value in two-player games

We return to the basic two-person example of Section 1.1. Recall that in that case the Core criterion does not yield restrictions on the way the surplus should be split. On the other hand, it is easy to see that Shapley prescribes sharing the surplus equally: $\varphi_1 = v_1 + [1-(v_1+v_2)]/2, \varphi_2 = v_2 + [1-(v_1+v_2)]/2$. You should verify this by applying the definition. In this case there is no doubt this is the "fair" solution. Also the nucleolus gives the same result.

2.2 The Shapley allocation in the apex game

We go back to the apex game of Section 1.7, which we know to have empty Core. For Shapley we may observe that the value of the small players must be the same by symmetry, so the

[4]To see that $\sum_i \Delta_i(S_i(R)) = v(N)$ fix $R = (i_1,\dots,i_n)$; then

$$\sum\nolimits_i \Delta_i(S_i(R)) = v(i_1) - v(\emptyset) + v(i_1 i_2) - v(i_1) + \cdots + v(N) - v(i_1\dots i_{n-1}) = v(N).$$

whole vector can be computed once we know the value of one of them, say player 2. She makes a positive contribution (of value 1) only in two cases: either she comes after 1; or after all the other small players. So

$$\phi_2 = \frac{1!(6-1-1)!}{6!} \cdot 1 + \frac{4!(6-4-1)!}{6!} \cdot 1 = \frac{1}{15}.$$

Therefore $\phi = 1/15 \cdot (10, 1, 1, 1, 1, 1) = 1/45 \cdot (30, 3, 3, 3, 3, 3)$. Alternatively we can compute the big player's payoff. Her marginal contribution is 0 if he arrives first or last, which happens with probability 2/6, and 1 otherwise; so he gets $4/6 = 30/45$. By symmetry the others get 3/45.

Nucleolus (for later). We may assume it is of the form $(1-5\alpha, \alpha, \dots, \alpha)$ with $0 \leq \alpha \leq 1/5$. The coalitions with positive value are $\{1, j\}$ with $j > 1$ and $N \setminus 1$. In the candidate vector the excess of the former is $1 - (1 - 5\alpha) = 4\alpha$, of the latter $1 - 5\alpha$. These are the highest ones, and the nucleolus has them equalized; so from $4\alpha = 1 - 5\alpha$ we get $\alpha = 1/9$; so we get $1/9 \cdot (4, 1, 1, 1, 1, 1) = 1/45 \cdot (20, 5, 5, 5, 5, 5)$.

2.3 The Shapley allocation in the gloves game does not belong to the Core

Even in cases where the Core is not empty the Shapley allocation may not belong to it. Consider a variant of the gloves game seen in Section 1.10, with $n + m$ players where n have a left glove and $m > n$ have a right glove. Again a pair of gloves is worth 1 (a single glove is worthless of course). A coalition S with n_1 right gloves and n_2 left ones has value $v(S) = \min\{n_1, n_2\}$. In particular $v(N) = n$ (the number of pairs of gloves).

The Core contains only the allocation where the owners of the left gloves get 1 and the others (in excess supply) get zero: any S including all players except some $m - n$ owners of a right glove must get n hence those owners of a right glove must get zero. That all left glovers get 1 follows as before.

Now take the case of $n = 1, m = 2$; the Shapley allocation is $\varphi = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6}) \neq (1, 0, 0)$: the left glover's contribution is 0 if he is first and 1 otherwise - so he gets 2/3, and the two others share the rest by symmetry.

Note that the Shapley allocation is not in the Core. Indeed $\varphi_1 + \varphi_2 = 1 - 1/6$ so so since $v_{12} = 1$ player 1 can team with 2 and agree for example on the splitting $x_1 = 2/3 + 1/12, x_2 = 1/6 + 1/12$; both are better off that in Shapley. But Shapley, arguing on fairness grounds, could tell player 1 "You only have a left glove; without a right one it's nothing, so do not complain if you get a little less than 1." And even more compellingly to player 2, "If player 1 approached 3 instead of you, you would certainly hope that 3 would not team with 1 and leave you with zero. So do not team with 1 yourself."

2.4 Shapley is not equal to voting shares

Consider a game with 4 people holding $10, 30, 30, 40$ votes (or shares in a company) respectively where the value $v(N) = 1$ can be obtained by any simple majority. All coalitions S not reaching a majority have $v(S) = 0$. The "voting shares" proportional allocation is $(1/11, 3/11, 3/11, 4/11)$. The Core is empty by Exercise 261.1 (see Section 1.9 above) since this is a simple game with no veto player. To find the Shapley allocation we may observe that the marginal contribution of player 1 is always zero so $\psi_1 = 0$; and each of the others have the same marginal contribution in any coalition. Therefore $\psi(v) = (0, 1/3, 1/3, 1/3)$.

2.5 Same story, different shares

Again 4 players as above but now with shares holdings $10, 20, 30, 40$. The Shapley allocation has

$$\varphi_i = \sum\nolimits_T \frac{(t-1)!(n-t)!}{n!}$$

the sum extending over all coalitions T such that T is winning but $T \setminus \{i\}$ is not winning. In each such T there are $t-1$ players preceding i and $n-t$ are following her, and they can be reordered in all possible ways to form orderings. As an exercise, write down the winning coalitions and show that the Shapley allocation here is $(\frac{1}{12}, \frac{3}{12}, \frac{3}{12}, \frac{5}{12})$.

2.6 Weighted majority game

This is a slight generalization of the apex game. There are 5 shareholders; the first has $w_1 = 8$ shares, and the $i = 2, \ldots, 5$ have $w_i = 3$; so total of 20 shares, majority 10 (reached by either 1 and at least one $i > 1$ or by $N \setminus \{1\}$). A majority may choose to undertake a project worth w_i to each. Then the game is $v(S) = \sum_{i \in S} w_i$ if $\sum_{i \in S} w_i \geq 10$, zero otherwise; $v(N) = 20$. It can be shown that the Core is non-empty.

We compute the Shapley value. By symmetry $\phi_i = \alpha$ equal for all $i > 1$; and then $4\alpha + \phi_1 = 20$ so it suffices to find ϕ_1. In the OR notation we have, letting $s = \#S_1(R)$,

$$\Delta_1(S_1(R)) = \begin{cases} 0 & S_1(R) = \emptyset \\ 8 + 3s & 1 \leq s \leq 3. \\ 8 & s = 4 \end{cases}$$

There are $\binom{n-1}{s} s!(n-1-s)!$ orders R with s players preceding any player; and $\binom{n-1}{s} s!(n-1-s)!/n! = 1/n$. Therefore $\phi_1 = 1/5 \cdot [\sum_{1 \leq s \leq 3}(8 + 3s) + 8] = 50/5 = 10$. In conclusion $\phi = (10, 2.5, 2.5, 2.5, 2.5)$. Note the difference with respect to the shares vector $(8, 3, 3, 3, 3)$.

2.7 The production economy of OR exercise 259.3

In this game the Shapley value assigns $\sum_{i=1}^{w} f(i)/(w+1)$ to the capitalist (to be proved); by symmetry the workers share the rest equally. Suppose $w = 2, f(1) = 1, f(2) = 1 + \epsilon$. Show that for $0 < \epsilon < 1/4$ the Shapley allocation is not in the Core.

3 For convex games Shapley is in the Core

The aim here is to show that for convex games (a class which contains for example the bankruptcy game and the airport game, both described below) the Core is non-empty and contains the Shapley allocation. First, a set $A \subseteq \mathbb{R}^n$ is *convex* if $x, y \in A \Rightarrow \alpha x + (1-\alpha)y \in A$ for $0 \leq \alpha \leq 1$. The point $\alpha x + (1-\alpha)y$ is in the segment joining x and y. It is easy to show by induction that if A is convex and $x_i \in A$ for $i = 1, \ldots, n$ then the convex combination $\sum_i \alpha_i x_i$ where the α_i's are non-negative and sum up to 1 also belongs to A. First we observe

Lemma 1. *The Core allocations of a game v form a convex set.*

Proof. If for all S we have $\sum_{i \in S} x_i \geq v(S)$ and $\sum_{i \in S} y_i \geq v(S)$ then $\alpha \sum_{i \in S} x_i + (1 - \alpha) \sum_{i \in S} y_i \geq v(S)$. $\square$

A game v over N is *convex* if for all $S, T \subseteq N$ we have $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. We always use the convention $v(\emptyset) = 0$. To interpret this take disjoint sets: it says that cooperation is beneficial. The next lemma gives an equivalent definition of convexity - reminding of convexity in the sense of increasing marginal contribution - which is usually easier to check.

Lemma 2. *v is convex if and only if for all $S \subset T$ not containing i*

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

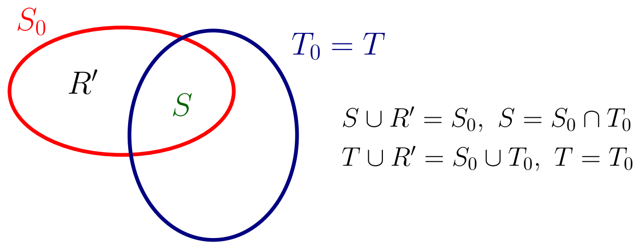
Proof. Suppose $v(S)+v(T) \leq v(S \cup T)+v(S \cap T)$ for all coalitions, equivalently $v(S_0)-v(S_0 \cap T_0) \leq v(S_0 \cup T_0) - v(T_0)$ for all $S_0, T_0 \subseteq N$. For $S \subset T$ not containing i, with $S_0 = S \cup \{i\}$ and $T_0 = T$ we get the inequality in the lemma. Conversely, assume the inequality. Take first $S \subset T$, let $R = N \setminus T \equiv \{i_1, \ldots, i_k\}$ and consider $R' = \{i_1, i_2\} \subset R$. By hypothesis

$$v(S \cup \{i_1\}) - v(S) \leq v(T \cup \{i_1\}) - v(T)$$
$$v(S \cup \{i_1, i_2\}) - v(S \cup \{i_1\}) \leq v(T \cup \{i_1, i_2\}) - v(T \cup \{i_1\})$$

so by summation we get

$$v(S \cup R') - v(S) \leq v(T \cup R') - v(T).$$

Since we can do the same with any subset of R the above inequality holds for any $R' \subseteq R$. Now take arbitrary sets $S_0, T_0 \subseteq N$, set $S = S_0 \cap T_0$ and $T = T_0$ and apply the inequality just proved. The figure below



makes it clear that we get $v(S_0) - v(S_0 \cap T_0) \leq v(S_0 \cup T_0) - v(T_0)$ as wanted. $\square$

Now we can prove the result we were after:

Proposition. *If v is convex the Core is non-empty and contains the Shapley allocation.*

Proof. We show that for any given order $\{i_1, \ldots, i_n\}$ of N the corresponding vector of marginal contributions

$$x_{i_k} = v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\}), \; k = 1, \ldots, n$$

is in the Core. This implies that the Shapley allocation is in the Core since it is a convex combination of marginal contribution vectors. We have to show that for any S we have $\sum_{i_k \in S} x_{i_k} \geq v(S)$. This is most easily seen in a special case; the general argument then only involves setting up the appropriate notation. Suppose $\#N = 7$ and that the order and S are as in the figure below:

$$\begin{array}{lccccccc} N: & 5 & 1 & 4 & 7 & 3 & 2 & 6 \\ S: & & & 4 & & 3 & & 6 \end{array}$$

If the game is convex then by the inequality in Lemma 2 we get

$$\begin{aligned} &v(\{4\}) - v(\emptyset) \leq v(\{5,1,4\}) - v(\{5,1\}) = x_4 \\ &v(\{4,3\}) - v(\{4\}) \leq v(\{5,1,4,7,3\}) - v(\{5,1,4,7\}) = x_3 \\ &v(\{4,3,6\}) - v(\{4,3\}) \leq v(\{5,1,4,7,3,2,6\}) - v(\{5,1,4,7,3,2\}) = x_6 \end{aligned}$$

and by summing the three inequalities we obtain $v(S) \leq x_4 + x_3 + x_6$. The result follows because the argument is valid for any S. $\square$

Example 1 (The airport/elevator game)**.** In this case the game is most naturally defined in terms of costs: the cost to the ith floor is c_i, with $c_1 < c_2 < \cdots < c_n$, and $c(S) = c_i$ for the highest i in S (Mr. i lives on the ith floor). To check convexity we have to define the value of a coalition, and this is just the opposite of cost: $v(S) := -c(S)$. Then convexity of v (in the formulation of Lemma 2) is equivalent to $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$ for all $S \subset T$ not containing i. It is left as exercise to check convexity for the case $n = 3$ where we compute all quantities of interest, for $i = 1, 2, 3$. For example for $i = 2$: a pair $S \subset T$ with T

not containing 2 must have $\#T \leq 2$ so $\#S \leq 1$; thus we have to check the cases listed in the table below:

	S	T
a	$\emptyset$	$\{1\}, \{3\}, \{1,3\}$
b	$\{1\}$	$\{1,3\}$
c	$\{3\}$	$\{1,3\}$

for example in case b we have to check that $c(\{1,2\}) - c(\{1\}) \geq c(N) - c(\{1,3\})$; this is $c_2 - c_1 \geq 0$ which we know to be true. The other cases are analogous.

Example 2 (The bankruptcy game)**.** This is defined by an estate E and n creditors with claims c_i such that $\sum c_i > E$. Letting $c(S) = \sum_{i \in S} c_i$ the value is defined as

$$\begin{aligned} v(S) &= \max\{0, E - c(N \setminus S)\} \\ &= \max\{0, \alpha + c(S)\}, \quad \alpha = E - c(N) < 0. \end{aligned}$$

Observe that $\max\{0,a\} + \max\{0,b\} = \max\{0, a, b, a+b\}$. Using Lemma 2, to show convexity of the game it suffices to show that for $i \notin S \subset T$ we have $v(S \cup \{i\}) + v(T) \leq v(T \cup \{i\}) + v(S)$. But using the above observation we have

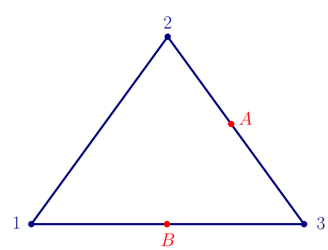
$$\begin{aligned} v(S \cup \{i\}) + v(T) &= \max\{0, \alpha + c(S) + c_i, \alpha + c(T), 2\alpha + c(S) + c(T) + c_i\} \\ v(T \cup \{i\}) + v(S) &= \max\{0, \alpha + c(T) + c_i, \alpha + c(S), 2\alpha + c(S) + c(T) + c_i\} \end{aligned}$$

which clearly implies the wanted inequality since $\alpha + c(T) + c_i$ is larger than both $\alpha + c(S) + c_i$ and $\alpha + c(T)$.

4 Other examples/exercises

4.1 A street lights game

Three persons (families, communities...) $1, 2, 3$ living at the vertices of a triangle can build one or two street lights A, B, as in the figure below:



Each light costs 40 Euros. Utilities are as follows: one adjacent light gives 30, two give 45, none gives zero. So for example if only light A is built then the utility vector is $(0, 30, 30)$, which net of cost gives $60 - 40 = 20$; if both lights are built utilities are $(30, 30, 45)$ and net surplus is $105 - 80 = 25$. Since no lights give no utility the group should build both lights. The problem is how much each should pay.

We can formulate the situation as a cooperative game, where from the above we know that $v(N) = 25$. Complete the specification of the value function (for example $v(2, 3) = 20$; solution in footnote, do it before you look at it).[5] Note that a payoff vector (x_1, x_2, x_3) with sum $v(N) = 25$ is a vector of net surpluses, so the implied cost allocation (c_1, c_2, c_3) with sum 80 is given by $u_i - c_i = x_i$ where $u_1 = u_2 = 30$ and $u_3 = 45$. For example $(x_1, x_2, x_3) = (2, 2, 21)$ means $(c_1, c_2, c_3) = (28, 28, 24)$.

Compute the set of Core allocations and the Shapley allocation in terms of x_i and then translate them in terms of c_i (*Hint:* for the Core the answer is that 1 and 2 should pay between 25 and 30 each, with 3 left to pay between 20 and 30; the Shapley allocation in this case is an extreme point of the Core).

4.2 Sharing the cost of a public good[6]

A community of individuals $i = 1, 2, \ldots, n$ may acquire a public good at cost K (Euros, say). To fix ideas we shall imagine that the public good is a bridge. Individual i has W_i Euros and $\sum_i W_i > K$ so they can build the bridge if they want. Also, i derives utility U_i from the bridge and $\sum_i U_i > K$ - so it would be in the common interest to build it. The problem is to determine how much each must pay, that is to find a vector $t = (t_1, \ldots, t_n)$ of taxes such that $\sum_i t_i = K$. Given such a vector the individual surplus is $U_i - t_i$, and the total surplus is $\sum_i (U_i - t_i) = \sum_i U_i - K$.

We specify the characteristic function of this game as

$$v(S) = \begin{cases} \max\{\sum_{i\in S} U_i - K, 0\} & \textit{if } \sum_{i\in S} W_i > K \\ 0 & \textit{otherwise}. \end{cases}$$

This is obviously monotone in the sense that larger coalitions have greater worth. Note that $v(N) = \sum_i U_i - K$. We may use the Shapley value as a fairness criterion to determine the tax system. Player i's contribution to coalition $S \not\ni i$ is

$$v(S \cup \{i\}) - v(S) = \begin{cases} U_i & \textit{if } v(S) > 0 \\ \sum_{j\in S\cup\{i\}} U_j - K & \textit{if } v(S) = 0 \textit{ and } v(S \cup \{i\}) > 0 \\ 0 & \textit{if } v(S \cup \{i\}) = 0. \end{cases}$$

Letting φ denote the Shapley allocation, since $\sum_i \varphi_i = v(N) = \sum_i (U_i - t_i)$ we may write $\varphi_i = U_i - t_i^*$, where the asterisk denotes that $t_i^* = U_i - \varphi_i$ is determined by the Shapley

[5] $v(i) = v(1, 2) = 0$; $v(1, 3) = v(2, 3) = 20$

[6] From Aliprantis and Chakrabarti, *Games and decision making*

allocation.

To see what this involves consider the case where there are 4 players, with $K = 10,000$ and wealth end utilities given by

$$W_1 = 50,000 \quad W_2 = 75,000 \quad W_3 = 100,000 \quad W_4 = 200,000$$
$$U_1 = 5,000 \quad U_2 = 4,000 \quad U_3 = 6,000 \quad U_4 = 8,000.$$

Here $v(N) = 13,000$. As an exercise, write down the characteristic function and compute the Shapley allocation via an online software. For the characteristic function note that the condition $\sum_{i \in S} W_i > K$ has strict inequality. You should get

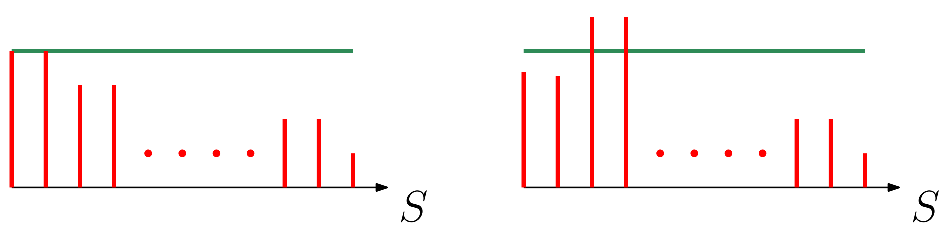
$$\varphi = (2833.33, 2166.67, 3333.33, 4666.67)$$
$$t^* = (2166.67, 1833.33, 2666.67, 3333.33)$$

Part II

Shapley vs Nucleolus Examples

The definition of the Nucleolus, due to David Schmeidler, is in OR page 286; the formal statement we read from there. As in OR we let $e(S,x) = v(S) - x(S)$; this is the "sacrifice" S makes at x. The idea of the Nucleolus is based on minimizing the maximum sacrifice. Since coalitions with maximum sacrifice are presumably those who make the highest pressure against x, the nucleolus represents a "stability" criterion: nucleolus allocations are "the least unstable". Figure 1 illustrates the minimization involved.

Figure 1: In the horizontal axis there are *coalitions;* in the vertical axis the height of a red line measures the sacrifice of the corresponding coalition; the sacrifices are in descending order. If the allocation giving rise to the diagram on the left is the nucleolus then if you try to lower the highest sacrifice some other coalition will end up with an even higher sacrifice, as in the left panel.



5 Nucleolus starting from Shapley

5.1 A three-person game

We compute the nucleolus starting from the Shapley allocation. This three-person game is defined by[7]

$$v(\emptyset) = 0,\ v(1) = -6, v(2) = 0, v(3) = 6,\ v(12) = 18, v(13) = 24, v(23) = 12,\ v(123) = 30$$

So player 1 has negative value on her own but 2 and 3 get the most if they team with her. The Shapley value is easily computed to be *Shapley* $= (10, 7, 13)$. To find the Nucleolus let us again start from the Shapley allocation. The usual table is the following, where in the third column there is the excess vector at $x =$ *Shapley* and the step from that to the next column is explained after the table:

S	$e(x,S) = v(S) - \sum_{i\in S} x_i$	*Shapley*	$(16,4,10)$
1	$-6 - x_1$	-16	-22
2	$-x_2$	-7	-4
3	$6 - x_3$	-7	-4
12	$18 - (x_1 + x_2) = x_3 - 12$	1	-2
13	$24 - (x_1 + x_3) = x_2 - 6$	1	-2
23	$12 - (x_2 + x_3) = x_1 - 18$	-8	-2

Clearly we should lower the excesses of $\{1, 2\}$ and $\{1, 3\}$ and we can do it together by lowering x_3 and x_2 by the same amount. Note that by so doing $\Delta x_1 = -2\Delta x_i, i = 2, 3$ since the sum is constant 30. Excesses of $2, 3, 23$ go up but that of 23 goes faster. Equality of excesses of 12, 13 and 23 is reached when $\Delta x_2 = \Delta x_3 = -3$ (and $\Delta x_1 = 6$). At this point the highest excesses are all equal and $x = (16, 4, 10)$ is fully determined. This is then the Nucleolus.

5.2 4-player weighted majority game (OR exercise 289.2)

We consider the case of Example 294.1: $w = (1, 1, 1, 2)$ and $q = 3$. We will start from the Shapley allocation $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$. The first three players play identical roles in any coalition so they must have the same payoff. Any one or two of them will be denoted by i or i, j. In obvious notation we then have

$$v(i) = v(ij) = 0, \quad v(i4) = v(ij4) = v(123) = v(N) = 1$$

therefore their excesses at the Shapley allocation are

$$e(i) = -\frac{1}{6},\ e(4) = -\frac{1}{2},\ e(i4) = \frac{1}{3} = \frac{10}{30},\ e(ij4) = \frac{1}{6},\ e(123) = \frac{1}{2} = \frac{15}{30}.$$

[7]Adapted from Thomas Ferguson *Game Theory*

We must lower the last one by raising the three x_i's by the same amount - thereby lowering x_4 by three times as much as x_i since $\sum \Delta x_i = 0$ (the sum $\sum x_i$ is fixed at 1) - so that $\Delta e(4) = -\Delta x_4 = 3\Delta x_i = -\Delta e(123)$. $\Delta e(4)$ goes up from $-1/2$, but the excess which halts the fall in $e(123)$ is $e(i4)$. We have $\Delta e(i4) = -\Delta(x_i + x_4) = 2\Delta x_i$ and that goes up from $1/3$. For $\Delta x_i = 1/30$ we have $\Delta e(123) = -3/30$ and $\Delta e(i4) = 2/30$ so that $e(123) + \Delta e(123) = e(i4) + \Delta e(i4) = 12/30$. And we have no more freedom since Δx_i determines the whole Δx and we get

$$x + \Delta x = (\frac{1}{6} + \frac{1}{30}, \frac{1}{6} + \frac{1}{30}, \frac{1}{6} + \frac{1}{30}, \frac{1}{2} - \frac{3}{30}) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}).$$

This is the Nucleolus of the game. Of course it is a special case of the general formula $x_i = w_i / \sum w_j$. Compared to the Shapley allocation the Nucleolus takes something away from the strong player 4 to distribute it to the other, weaker players in the game. It seems that Shapley is more in the spirit of a measure of "power" and the Nucleolus a "fair" allocation of $v(N)$.

For the sake of curiosity, observing that $\Delta e(ij4) = \Delta x_i$, the resulting excess vector in the Nucleolus is

$$e(i) = -\frac{1}{5},\ e(4) = -\frac{2}{5},\ e(i4) = \frac{2}{5},\ e(ij4) = \frac{1}{5},\ e(123) = \frac{2}{5}.$$

6 The airport Game

We compute the Shapley allocation, and for $n = 3$ Nucleolus and Core.

The game is usually described in terms of the cost of runaways of various lengths in an airport. We phrase it in terms of the more familiar elevator cost. The cost to first floor is c_1, cost to i-th floor is c_i and so on up to the cost c_n to last n-th floor; so $c_1 < c_2 \cdots < c_n$ and the elevator costs c_n. The problem is how this cost is to be shared among the families living in the building (one for each floor), that is we look for cost allocations $x = (x_1, \ldots, x_n)$ with $\sum_i x_i = c_n$.

Shapley

A "reasonable" solution is that since every family uses the stretch from ground to first floor then everybody should share that; families from 2 to n use the second stretch so they should share the increment $c_2 - c_1$, and so on up to the last stretch from $n-1$ to n which only the uppermost family should pay for.

That is: letting x_i the share payed by i, where $\sum_i x_i = c_n$, and letting also $\delta_i = c_i - c_{i-1}$ (with $c_0 \equiv 0$), the proposal is $x_1 = \frac{\delta_1}{n}, x_2 = \frac{\delta_1}{n} + \frac{\delta_2}{n-1}$ and so on, that is

$$x_i = \sum\nolimits_{j=1}^{i} \frac{\delta_j}{n-(j-1)} = \frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_i}{n-(i-1)} \quad i = 1, \ldots, n$$

where notice that $n-(j-1) = \#\{j, j+1, \ldots, n\}$ (the number of families who share δ_j). Since all terms contain δ_1/n, all except the first (that is a total of $n-1$ terms) contain $\delta_2/(n-1)$ and so on we get $\sum_i x_i = \sum_i \delta_i = c_n$ as it should be.

We next show that this is actually the Shapley allocation of the corresponding cooperative game. This is defined in terms of costs, where for $S \subseteq N$ we have $c(S) = c_i$ with i the highest index in S. The argument is the following (adapted from a book by H. Moulin).[8] Consider a randomly ordered N, say $(2, 1, 3, 5, 7, \ldots)$; here 2 pays $\delta_1 + \delta_2$, 1 pays nothing, 3 pays δ_3, 5 pays $\delta_4 + \delta_5$ and 7 pays $\delta_6 + \delta_7$; looking at 5, if 3 was not before it as in $(2, 1, 5, 7, \ldots)$ then 5 should pay δ_3 as well (with a total of $\delta_3 + \delta_4 + \delta_5$); in general, for $j \leq 5$ family 5 must pay δ_j if in the given order there is nobody in $\{j, j+1, \ldots, n\}$ appearing before it. So in $(2, 1, 3, 5, 7, \ldots)$ it pays δ_4 since it is the first family in $\{4, \ldots, n\}$ appearing in the order and δ_5 since it is the first family in $\{5, \ldots, n\}$; and in $(2, 1, 5, 7, \ldots)$ it also pays δ_3 because it is also the first in $\{3, \ldots, n\}$ appearing in the order. Since in a random order family i is the first in $\{3, \ldots, n\}$ with probability $1/(n-2)$, and in general it is the first in $\{j, \ldots, n\}$ with probability $1/[n-(j-1)]$ we get that i pays δ_j with probability $1/[n-(j-1)]$, for $j \leq i$. This is the formula displayed above.

Nucleolus ($n = 3$)

For $n = 3$ the Shapley allocation is

$$x_1 = \frac{\delta_1}{3},\ x_2 = \frac{\delta_1}{3} + \frac{\delta_2}{2},\ x_3 = \frac{\delta_1}{3} + \frac{\delta_2}{2} + \delta_3.$$

The "problem" with this is that coalition 23 has a total net gain of $c(23) - (x_2 + x_3) = \delta_1/3$ while family 1 gets more: $c(1) - x_1 = 2\delta_1/3$. As we shall see by avoiding this potential "complaint" we arrive at the Nucleolus.

To work with positive numbers we let $\eta(x, S) = c(S) - \sum_{i \in S} x_i$ (the gain of S) and successively try to maximize the lowest value. We start from the Shapley allocation $x_{Sh} = (\frac{\delta_1}{3}, \frac{\delta_1}{3} + \frac{\delta_2}{2}, \frac{\delta_1}{3} + \frac{\delta_2}{2} + \delta_3)$, as in the table below.

S	$c(S)$	$\eta(x, S)$	$\eta(x_{Sh}, S)$	$(\frac{1}{2}\delta_1, \frac{1}{4}\delta_1 + \frac{1}{2}\delta_2, \frac{1}{4}\delta_1 + \frac{1}{2}\delta_2 + \delta_3)$
1	c_1	$\delta_1 - x_1$	$\frac{2}{3}\delta_1$	$x_1 = \frac{1}{2}\delta_1$
2	c_2	$\delta_1 + \delta_2 - x_2$	$\frac{2}{3}\delta_1 + \frac{1}{2}\delta_2$	$\frac{3}{4}\delta_1 + \frac{1}{2}\delta_2$
3	c_3	$\delta_1 + \delta_2 + \delta_3 - x_3$	$\frac{2}{3}\delta_1 + \frac{1}{2}\delta_2$	$\frac{3}{4}\delta_1 + \frac{1}{2}\delta_2$
12	c_2	$\delta_1 + \delta_2 - x_1 - x_2$	$\frac{1}{3}\delta_1 + \frac{1}{2}\delta_2$	$\frac{1}{4}\delta_1 + \frac{1}{2}\delta_2$
13	c_3	$\delta_1 + \delta_2 + \delta_3 - x_1 - x_3$	$\frac{1}{3}\delta_1 + \frac{1}{2}\delta_2$	$\frac{1}{4}\delta_1 + \frac{1}{2}\delta_2$
23	c_3	$\delta_1 + \delta_2 + \delta_3 - x_2 - x_3$	$\frac{1}{3}\delta_1$	$\frac{1}{2}\delta_1$

Coalition 23 has the lowest gain so we raise it by lowering $x_2 + x_3$ - which means raising x_1 - until the gain of 23 becomes equal to that of 1; this gives $x_1 = \delta_1/2$. We are left with an

[8] *Axioms for Cooperative Decision Making*, Cambridge University Press 1988

the extra $\delta_1/6$ to divide between 2 and 3; and we divide it evenly between between them (by lowering their x_i) since both have the same excesses; this results in the allocation in the 5*th* column. And that is the nucleolus, since there 2 and 3 have the same excesses so we cannot touch either. Therefore the difference compared to the Shapley allocation is in this case that the cost to the first floor is shared equally between 1 and 23. Thus the Nucleolus is given by

$$x_1 = \frac{\delta_1}{2},\ x_2 = \frac{\delta_1}{4} + \frac{\delta_2}{2},\ x_3 = \frac{\delta_1}{4} + \frac{\delta_2}{2} + \delta_3.$$

Core ($n = 3$)

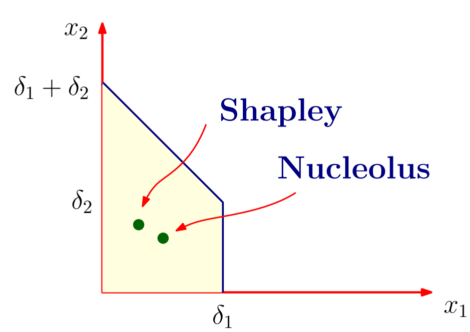
Observe that $c_i = \sum_{j=1}^{i} \delta_j$ for all i. The inequalities defining the Core are the following:

$$\sum_i x_i = \sum_i \delta_i \quad x_i \leq \sum_{j=1}^{i} \delta_j$$
$$x_1 + x_2 \leq \delta_1 + \delta_2 \quad x_1 + x_3,\ x_2 + x_3 \leq \sum_i \delta_i$$

Now given $\sum_i x_i = \sum_i \delta_i$: $x_2 + x_3 \leq \sum_i \delta_i$ implies $x_1 \geq 0$, and $x_1 + x_3 \leq \sum_i \delta_i$ implies $x_2 \geq 0$. Since x_3 may be computed as the difference $c_3 - (x_1 + x_2)$ we can draw the Core in (x_1, x_2) space, where it is characterized by the inequalities $0 \leq x_1 \leq \delta_1$, $0 \leq x_1 + x_2 \leq \delta_1 + \delta_2$. Note that the Core is large, in particular it contains the allocation where $x_1 = x_2 = 0$ and 3 pays the whole cost and at the other extreme also the point where 3 pays only δ_3; and also the distribution of costs between 1 and 2 has virtually no restrictions.

Comparison in a diagram

We visualize the three solutions in the figure below (for x_1 and x_2). The Core is the yellow region; Shapley and Nucleolus allocations are the marked points.



7 The bankruptcy game

The estate is E and claims are $c_i,\ i = 1, \ldots, n$ with $\sum c_i > E$. The game is defined by

$$v(S) = \max\{0, E - \sum_{i \notin S} c_i\}.$$

Notice that $v(N) = E$. We assume to fix ideas that $c_1 < c_2 < \cdots < c_n$.

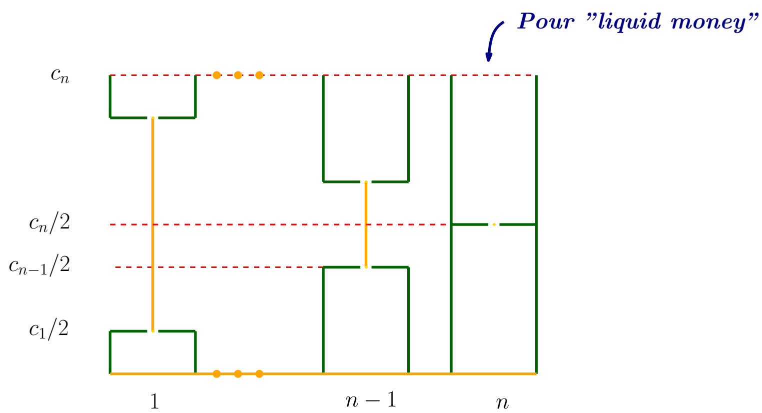
Aumann and Maschler[9] report that the Babylonian Talmud consider the case of 3 players with claims fixed at $c_1 = 100, c_2 = 200$ and $c_3 = 300$ and three possible values of E, namely $100, 200, 300$. The allocations prescribed in the Talmud are in the following table, and the surprising fact is that these prescriptions are exactly the nucleolus allocations in the three cases.

		Claims		
		100	200	300
	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
E	200	50	75	75
	300	50	100	150

The Nucleolus of the general bankruptcy game can be computed as follows:[10]

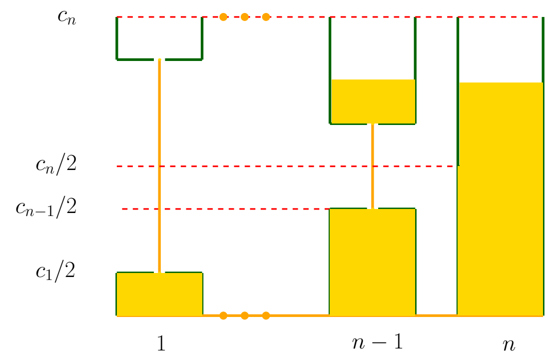
- If $E \leq \sum c_i/2$ then give equal incremental increases to all i such that $x_i < c_i/2$
- If $E > \sum c_i/2$ then above $\sum c_i/2$ distribute equally to all i such that $c_i - x_i$ is highest.

This is most easily done with the help of the following picture:



these are supposed to be containers of width 1; the rightmost has height c_n so to fill it up it takes a quantity c_n of "water", and it is divided in two equal parts; the next one on its left has total height c_{n-1} and it is also divided in two equal parts. Observe that the total area of the cylinders is $\sum c_i$. Water poured from above freely passes through the different halves of the containers, as indicated by the orange lines and the vertical pipes are supposed to have width zero. So if you pour water from above into any container the desired allocation is realized. An example is visualized in the figure below:

[9] Journal of economic Theory 1985
[10] Aumann-Maschler JET 1985

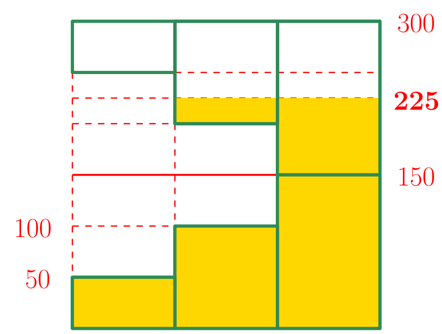


We consider the simple case of $n = 3$ players. The Shapley value is computed as usual. So we can compare allocations for different values of E (for the Nucleolus you should draw the appropriate pictures). We do so for a few cases in the table below; claims are fixed at $c_1 = 100, c_2 = 200, c_3 = 300$.[11]

[11]For exemplification we find the Shapley value for $E = 200$; in this case $v(N) = 200, v(2,3) = 100$ and otherwise $v(S) = 0$. Then the usual procedure gives the following table:

orderings			$\Delta_i(S_i(R))$		
			$i=1$	$i=2$	$i=3$
1	2	3	0	0	200
1	3	2	0	200	0
2	1	3	0	0	200
2	3	1	100	0	100
3	1	2	0	200	0
3	2	1	100	100	0
Shapley			$33\frac{1}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$

We compute the Nucleolus for $E = 400$ using the appropriate picture. Note that the colored area is 400.



		x_1	x_2	x_3
$E = 200$	Shapley	$33\frac{1}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$
	Nucleolus	50	75	75
$E = 300$	Shapley	50	100	150
	Nucleolus	50	100	150
$E = 350$	Shapley	$58\frac{1}{3}$	$108\frac{1}{3}$	$183\frac{1}{3}$
	Nucleolus	50	100	200
$E = 400$	Shapley	$66\frac{2}{3}$	$116\frac{2}{3}$	$216\frac{2}{3}$
	Nucleolus	50	125	225
$E = 500$	Shapley	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$
	Nucleolus	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$

There is not much we can learn from these numbers really. That is why axioms are needed.

One more point about bankruptcy and Nucleolus

There is a a fairly strong argument in favor of the Nucleolus in the bankruptcy game, which we now present. Letting $v_i = v(\{i\})$ we first show the following

Lemma 3. $\sum v_i \leq E$.

Proof. First observe that $v_i < c_i$. For if $\sum_{j \neq i} c_j \geq E$ then $v_i = 0 < c_i$; otherwise $v_i = E - \sum_{j \neq i} c_j < \sum c_i - \sum_{j \neq i} c_j = c_i$. Next proceed by induction. Take $n = 2$. If $c_i \geq E$ then $v_1 + v_2 = v_i \leq E$; if on the other hand $c_1, c_2 \leq E$ then $v_1 + v_2 = 2E - (c_1 + c_2) < 2E - E = E$. Now assume the inequality is true for $n - 1$; then for n: if $c_1 \geq E$ then $\sum v_i = v_1 \leq E$; if $c_1 < E$ then

$$\sum_2^n v_i = \sum_2^n \max\{0, (E - c_1) - \sum_{j \neq i, \geq 2}^n c_j\} \leq E - c_1$$

by the induction hypothesis (on the game with the $n-1$ players from 2 to n and estate $E - c_1$); therefore $\sum v_i < c_1 + \sum_2^n v_i \leq E$ (using $v_1 < c_1$). □

Now consider the special case of $n = 2$ and the following so-called "Contested Garment Rule" on estate E:

$$x_i = v_i + \frac{1}{2}\left[E - (v_1 + v_2)\right] \tag{CGR}$$

This is pretty compelling: each player is granted her value and the non-negative excess $E - (v_1 + v_2)$ is split equally between the two.[12] It is easy to check directly from the definitions that both Shapley and the Nucleolus agree with the rule in the two-player case. For Shapley the usual table is

orderings	allocations	
12	v_1	$E - v_1$
21	$E - v_2$	v_2
Shapley	$\frac{v_1 + E - v_2}{2}$	$\frac{v_2 + E - v_1}{2}$

[12]Note that $x_i \geq v_i$ by the previous lemma.

and the solution is the one we want; for the Nucleolus, the excesses of the two players at the given allocation are equal: $e(i,x) = v_i - x_i = -\left[E - (v_1 + v_2)\right]/2$.

In the general n-player game the two solutions may differ. We say that the allocation x is *consistent with the Contested Garment Rule* if for any pair of players i, j the rule over estate $x_i + x_j$ (and credits c_i and c_j) gives (x_i, x_j). And the result is the following:

Theorem (Aumann and Maschler (JET 1985))**.** *In the bankruptcy game the only allocation consistent with the Contested Garment Rule is the nucleolus.*

This is a fairly strong point in favor of the Nucleolus in this game. In fact, with Shapley there is also another problem: it is not *"population monotonic"*. The idea is that if each player is replicated and the estate is doubled then in the resulting game the allocation should give each the same payoff as in the original game; and Shapley (but not the Nucleolus as is easy to check) fails this test. Consider this example[13]: $c_1 = 200, c_2 = 300, E = 300$. Then Shapley is $x = (100, 200)$; but if there are two players with credit 100 and two with credit 200 and the estate is 600 the Shapley allocation is not $(100, 100, 200, 200)$ - in fact you can check that it is $(116\frac{2}{3}, 116\frac{2}{3}, 183\frac{1}{3}, 183\frac{1}{3})$. The Nucleolus give in both cases 100 to the first type and 200 to the second type.

Remark. Going back to the two-player case: the model also applies to the division of surplus $v(N) = v(12)$ between two partners, each of which can make $v_i \geq 0$ on her own and $v_1 + v_2 \leq v(12)$. In this case the (CGR) formula - here $x_i = v_i + [v(12) - (v_1 + v_2)]/2$ - can be useful in practice if v_i can be reliably assessed.

8 The heritage game[14]

There are "sisters" $i = 1, \dots n$ who inherit "houses" $h_j, j = 1, \dots m$. We assume for simplicity that $m \leq n$. Sister i has utility $u_i(h_j) \geq 0$ for house h_j. If $m < n$ some sisters are assigned no house; in this case we say they are assigned house h_0 and take $u_i(h_0) = 0$ for all i. Money transfers $t_1, \dots, t_n$ with $t_1 + \dots + t_n = 0$ are allowed. Suppose sister i is assigned house $\varsigma(i)$; then overall utility of i is $u_i(h_{\varsigma(i)}) + t_i$.

8.1 One house, n sisters

Here we can simplify notation and let u_i sisters i's utility for the house. Assume $u_1 < u_2 < \dots < u_n$. The cooperative game is defined by $v(S) = u_i$ for the highest $i \in S$. The value $v(N) = u_n$ is obtained by assigning the house to the sister who values it most. Of course she will have to compensate her sisters as we shall see. Observe that the game has the same structure as the airport game, the only difference being that it is formulated in terms of

[13] Taken from Young, *Equity,* p.71

[14] Adapted from H. Moulin *Cooperative Microeconomics.*

utilities instead of costs. Therefore letting $u_0 = 0$ and $\delta_i = u_i - u_{i-1}$ we know that the Shapley allocation is given by

$$x_i = \sum\nolimits_{j=1}^{i} \frac{\delta_j}{n-(j-1)} = \frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_i}{n-(i-1)} \quad i = 1, \ldots, n$$

that is $x_1 = \frac{\delta_1}{n}, x_2 = \frac{\delta_1}{n} + \frac{\delta_2}{n-1}$ and so on up to the highest utility sister who gets $\frac{\delta_1}{n} + \cdots + \frac{\delta_{n-1}}{2} + \delta_n$. Since the house goes to the n-th sister the x_i's for $i < n$ are to be interpreted as transfers t_i from her, while since $\sum_i x_i = v(N) = u_n$ sister n has $x_n = u_n - \sum_1^{n-1} x_i$ that is her utility from getting the house minus the compensations paid to her sisters. Using the more natural notation $x_i = t_i$ for $i < n$ we may and will write any allocation in this game as

$$(t_1, \ldots, t_{n-1}, u_n - \sum\nolimits_{i=1}^{n-1} t_i).$$

The case $n = 3$: Shapley versus Nucleolus

In the $n = 3$ case Shapley prescribes that sister 3 pays

$$t_1 = \frac{u_1}{3},\ t_2 = \frac{u_1}{3} + \frac{u_2 - u_1}{2}$$

while we know from the airport game that the Nucleolus prescribes

$$t_1 = \frac{u_1}{2},\ t_2 = \frac{u_1}{4} + \frac{u_2 - u_1}{2}$$

that is more to the sister who value the house the least and less to the other. The total transfer from sister 3 is higher in the Nucleolus.

8.2 Envy

As it happens, a common problem in heritage situation is envy. Are the allocations we have considered envy-free? The answer for $n > 2$ is clearly no because in both cases sisters 1 and 2 get no house and in general $t_1 \neq t_2$. In particular Shapley has $t_2 > t_1$ so sister 1 surely envies at least sister 2. In the Nucleolus envy is less pronounced. In the case $n = 2$, in an allocation $(t_1, u_2 - t_1)$ sister 1 does not envy 2 if she prefers t_1 to having the house and paying t_1 that is $t_1 \geq u_1 - t_1$ or $t_1 \geq u_1/2$; and similarly sister 2 does not envy 1 if $u_2 - t_1 \geq t_1$ that is $t_1 \leq u_2/2$. So the no-envy conditions in this case are

$$\frac{u_1}{2} \leq t_1 \leq \frac{u_2}{2}.$$

In the $n = 2$ case Shapley has $t_1 = u_1/2 < u_2/2$ so no-envy obtains; and since excesses in this allocation, $v_1 - x_1 = u_1/2$ and $v_2 - x_2 = u_2 - (u_2 - u_1/2) = u_1/2$, are equal the Nucleolus coincides with Shapley.

Which allocations $(t_1, \ldots, t_{n-1}, u_n - \sum_1^{n-1} t_i)$ are envy-free in the general case? Sisters

1 to $n-1$ get (no house and) t_i so to avoid envy among them we must impose $t_1 = \cdots = t_{n-1} \equiv t$. Avoiding envy between sister n and the others then gives $u_n - (n-1)t \geq t$ and $t \geq u_{n-1} - (n-1)t$ (if $n-1$ does not envy n nor will the others since u_{n-1} is highest among them). Thus we get the no-envy conditions

$$t_1 = \cdots = t_{n-1} = t \quad and \quad \frac{u_{n-1}}{n} \leq t \leq \frac{u_n}{n}.$$

Consider for example the case $n = 3$ with $u_1 = 0, u_2 = \alpha, u_3 = 100$ with $0 < \alpha < 100$. In this simple example Shapley and the Nucleolus coincide because $t_1 = 0$. In both cases 1 envies both 2 and 3. No envy of 3 on the part of 2 gives $t_2 \geq u_2 - t_2 - t_1$ that is $t_2 \geq \alpha/2$ which is satisfied with equality. The condition that 3 should not envy 2 is $u_3 - t_2 - t_1 \geq t_2$ that is $t_2 \leq u_3/2$ which is satisfied strictly. Notice that in this case full no envy gives $t_1 = t_2$ and $\frac{\alpha}{3} \leq t \leq 33\frac{1}{3}$ which is not very plausible. We should mention that in practice the utilities u_i are not easy to pin down, so that the model may be difficult to apply.

8.3 General case

Generalizing the case of one house we define $v(N)$ to be the highest sum of utilities over all possible assignments ς:

$$v(N) = \max_{\varsigma} \sum\nolimits_i u_i(h_{\varsigma(i)}).$$

To take a concrete case consider the following case where $n = m = 3$ and utilities are in the following table and the optimal assignment is in boldface and gives total utility of $v(N) = 27$ (check that it is indeed the highest possible sum):

	u_1	u_2	u_3
h_1	3	9	**9**
h_2	**12**	6	6
h_3	9	**6**	3

An allocation x is determined by an optimal assignment ς and a vector of zero-sum transfers $t = (t_1, \ldots, t_n)$ giving $x_i = u_i(h_{\varsigma(i)}) + t_i$. No envy can be defined for any (not necessarily optimal) assignment and transfer set: *a pair* (ς, t) *is envy-free if no* i *envies any* i', that is if $u_i(h_{\varsigma(i)}) + t_i \geq u_i(h_{\varsigma(i')}) + t_{i'}$ for any i, i'. It is remarkable that in any envy-free pair (ς, t) the assignment ς is automatically optimal. The proof is in footnote.[15]

Given an optimal assignment ς we can find the set of transfers which make the pair (ς, t)

[15] Consider $n = m$ first, assume (ς, t) is envy free and consider any other assignment $\tilde{\varsigma}$; sister i in ς does not envy the i' who gets her house in $\tilde{\varsigma}$ - that is the i' such that $\varsigma(i') = \tilde{\varsigma}(i)$. So $u_i(h_{\varsigma(i)}) + t_i \geq u_i(h_{\tilde{\varsigma}(i)}) + t_{i'}$ - and by summing up these inequalities we get $\sum u_i(h_{\varsigma(i)}) \geq \sum u_i(h_{\tilde{\varsigma}(i)})$ for the transfers sum to zero; this shows that ς is optimal. If $m < n$ this argument breaks down because if i gets the empty house in $\tilde{\varsigma}$ there are more than one who get this in ς; but the fix is just to consider the lowest i' who gets the empty house in ς, and the rest of the argument is the same.

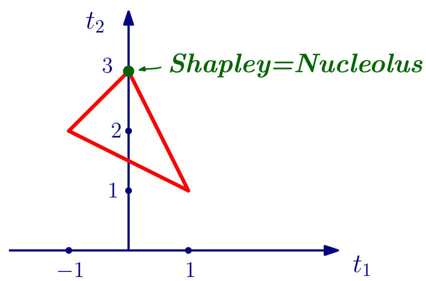
envy free. In the above example we get the following inequalities:

$$\begin{array}{lll} \textit{no envy by } 1: & 12+t_1 \geq 9+t_2 & 12+t_1 \geq 3+t_3 \\ \textit{no envy by } 2: & 6+t_2 \geq 9+t_3 & 6+t_2 \geq 6+t_1 \\ \textit{no envy by } 3: & 9+t_3 \geq 6+t_1 & 9+t_3 \geq 3+t_2 \end{array}$$

You can check that the three inequalities in the last column are implied by the other three, and since $t_3 = -(t_1+t_2)$ we end up with the following conditions on (t_1, t_2):

$$t_2 \leq 3+t_1 \quad 3 \leq t_1+2t_2 \quad 2t_1+t_2 \leq 3$$

which we can draw in the (t_1, t_2) plane. It is the triangular region in the figure below:



We have also drawn the transfers implied by Shapley allocation, which we can easily compute. In the same spirit as in the bankruptcy game the v function is obtained here by taking for any $S \subseteq N$ the sum of the utilities resulting from an optimal assignment within the coalition *after* the others are given their optimal assignment; for example sister 1 may leave houses 1 and 3 to the others (who could do no better on their own) so that $v(1) = u_1(2) = 12$; the other values are obtained similarly and the result is the following (besides $v(N) = 27$):

$$\begin{array}{ccc} v(1)=12 & v(2)=6 & v(3)=3 \\ v(12)=18 & v(13)=15 & v(23)=15 \end{array}$$

The Shapley allocation of this game is $x = (12, 9, 6)$. These are the overall utilities, which means the implied transfers are $t_1 = 0, t_2 = 3, t_3 = -3$ (sister 3 pays 3 to sister 2). In this case it happens that Shapley prescribes an envy-free allocation of houses and transfers. By writing down the excesses it is easy to see that the Shapley allocation is in this case also equal to the Nucleolus.

The Core of this game also is easily computed: it is the set of x such that

$$x_1 = 12 \quad 6 \leq x_2 \leq 12 \quad 3 \leq x_3 \leq 9 \quad x_2 + x_3 = 15$$

In terms of transfers this is $t_1 = 0,\ 0 \leq t_2 \leq 6,\ t_3 = -t_2$. Shapley and Nucleolus select the midpoint of the admissible (x_2, x_3) pairs in the Core.