Examples of Normal Form Games [1]

Contents

1 Finite Action Spaces

1.1 A (stupid) Bus Game

The passenger (player 1) can Pay or Not Pay the ticket, the firm (player 2) can Check or Not Check the passenger. So we have a simple 2×2 game. Problem is what are payoffs. We rule

[1] Salvatore Modica, 2023. Of course I did not invent any of them. It is just a collection that I use in class.

out the (realistic) case where the passenger always pays the ticket because she is inherently honest. A possible payoff matrix is the following:

	C	NC
P	$0,0$	$-1,1$
NP	$-M,M$	$1,0$

Then the (unique) mixed equilibrium has - with p and q as usual -

$$p = \frac{M}{M+1} \quad q = \frac{2}{2+M}$$

For instance with $M = 10$ we get $p = 10/11, q = 1/6$. With $M = 20$ the passenger almost always pays, the firm almost never checks - which is not entirely satisfactory. We would like to get that the equilibrium entails that the passenger almost always pays but the firm checks with probability not too far from 1/2.

Here are the payoffs. Since $1 - p = \frac{1}{M+1}, 1 - q = \frac{M}{2+M}$, the firm payoff is

$$p(1-q) + M(1-p)q = \frac{M}{M+1}$$

while the passenger gets $-p(1-q) - M(1-p)q + (1-p)(1-q) = -\frac{M}{M+2} < 0$.

A possible alternative payoff matrix that you can explore is the following, with $a > 0$:

	C	NC
P	$a,0$	$-1,1$
NP	$-M,M$	$1,0$

1.2 A Hiring Game

There are two firms hiring, each needs one worker. Firm $i = 1,2$ offers wage w_i with $w_i > w_j/2, i \neq j$ (equivalently $w_1/2 < w_2 < 2w_1$). Two workers are playing: if they apply to different firms they both get the job; if they apply to the same firm each gets the job with probability 0.5, otherwise he gets 0. Draw the (2 by 2) game between the workers and compute the equilibria. To fix notation denote by i the action "apply to firm i", for each worker. So for example the profile $(2,1)$ gives payoffs w_2, w_1. There are two pure equilibria and one mixed; for the latter let p (resp. q) the probability that player 1 (resp 2) plays 1.

Solution ...

1.3 Rock Paper Scissors

The game is in the matrix below, where only the payoff of player 1 is reported since that of player 2 is just the opposite:

	R	***P***	***S***
R	0	-1	1
P	1	0	-1
S	-1	1	0

Denote a mixed strategy of 1 by (p_1, p_2, p_3) and of 2 by (q_1, q_2, q_3). To find the equilibrium:

1. Show there is no pure strategy equilibrium.
2. Show there is no equilibrium where one of the players mixes over two strategies only.
3. Then show (easy) that the only equilibrium involves uniform randomization over the three alternatives.

Solution ...

1.4 The Penalty Game

The game (with two players) is very simple: kicker kicks the ball and goalie keeps the goal. Actions are for both left, center, right (kick and dive). Kicker plays rows, goalie columns. Payoffs are scoring probability for kicker and the complement for goalie, all multiplied by 100 to ease reading. Scoring probabilities are given in the following table (the payoffs for goalie are the complements to 100)

	L	C	R
l	65	95	95
c	95	0	95
r	95	95	65

Thus, under l, L and r, R (dive same side of kick) kicker has 65% probability of scoring, hence goalie has a 35% chance to save; with c, C the keeper saves for sure; under any profile where the two make different choices kicker scores with 95% probability and kicks out with probability 5%.[2] (a) Find the equilibria of the game. (b) Compute equilibrium probability of scoring. *Hints.* (a) First observe that there are no pure strategy equilibria and no partially

[2]You may object that if the kicker kicks to the center she has a lower probability of kicking out, in which case the non-zero elements of the center row should be higher than 95. This is a fair objection, you may analyze the game with those values replaced by say 99.

mixed equilibria (easy really). So first conclusion is that equilibrium is fully mixed. Now show that for both players the equilibrium probability of playing left or right must be the same (each should be indifferent between left and right...). Conclude then that the only numbers you need to compute are probabilities p, q of playing left. You have shown that indifference between left and right gives $p_l = p_r = p$ and $q_L = q_R = q$, so p and q are determined by indifference between left and center. (b) You may use the payoffs and then divide by 100 (*Answer:* the kicker has equilibrium payoff $\approx .82$).

Solution ...

1.5 Maxmin Strategies in Matching Pennies

Consider the mixed extension of the Matching Pennies game[3]

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

A mixed strategy of player 1 may be represented by the probability p of playing H, and analogously a mixed strategy of player 2 is characterized by the probability q with which she plays H. Letting $U = U_1$, the (expected) utility of player 1, one then has $U = U(p, q)$. We know that $U_2 = -U$.

(i) Compute $U(p, q)$. *Answer:* $U(p, q) = (2p - 1)(2q - 1)$.[4]

(ii) Show that for this game the only conservative profile (hence equilibrium from Proposition 22.2) is given by $p = q = 1/2$.

Solution We have

$$\max_p \min_q (2p-1)(2q-1) = \begin{cases} \max_p (2p-1) & 2p-1 < 0 \\ \max_p [-(2p-1)] & 2p-1 \geq 0 \end{cases} = \max_p [-|2p-1|].$$

The only solution to this problem is clearly $p = 1/2$. The analogous argument for player 2 gives $q = 1/2$. As we see in the zero-sum file this is not a coincidence.

[3] This is taken from the zero-sum game lecture.

[4]

$$p[q - (1-q)] + (1-p)[-q + (1-q)] = (2p-1)(2q-1)$$

1.6 Another zero-sum game

Consider the following zero-sum game:

	L	R
T	$-2, 2$	$3, -3$
B	$3, -3$	$-4, 4$

Obviously there are no pure equilibria, and clearly $\underline{v} = -2$ while $\overline{v} = 2$. It is immediate to find the mixed equilibrium: indifference for 1 gives $-2q + 3(1-q) = 3q - 4(1-q)$ that is $q = 7/12$, and indifference for 2 is $-2p + 3(1-p) = 3p - 4(1-p)$ which gives $p = q$. In equilibrium player 1 gets $-2 \cdot 7/12 + 3 \cdot 5/12 = 1/12$ so $v(G^{\Delta}) = 1/12$.

Derive $v(G^{\Delta})$ directly as $\max_p \min_q U(p,q)$ as an exercise (it is the minimum of two lines). Solution in footnote.[5]

2 Infinite Action Spaces

2.1 A Serious Prisoners Dilemma: the Tragedy of the Commons

There are n players with strategy sets $x_i \in [0,1]$ and payoff

$$u_i(x) = r \sum_j x_j - x_i$$

where $x = (x_1, \ldots, x_n)$ is a profile and $1/n < r < 1$. This is the simplest example of provision of a public good: by contributing x_i each player enjoys part of the total contribution $\sum_j x_j$. Compute the Nash equilibrium x^{Nash} and the profile x^{eff} that maximizes $\sum_i u_i(x)$.

Solution ...

Analysis of Pareto Optima An allocation x is Pareto Optimal if there is no x' such that $u_i(x') \geq u_i(x)$ for all i with inequality strict for some i. Let $m < n$ be such that $mr < 1 \leq (m+1)r$. The following holds:

Proposition. *x is Pareto optimal if and only if $x_i < 1$ for at most m players.*

[5] We know that

$$\min_q U(p,q) = \min\{U(p,L), U(p,R)\} = \min\{-2p + 3(1-p), 3p - 4(1-p)\} = \min\{3 - 5p, 7p - 4\}$$

The first line is downward sloping and the second goes upwards, so (draw!) the minimum of the two (as a function of p) goes up to the intersection along the second then goes down along the first, hence its maximum is at the intersection: $3 - 5p = 7p - 4$, that is $12p = 7$, where it has value $3 - 5p = 3 - 35/12 = 1/12 = v(G^{\Delta})$, confirming what we found above.

Proof. Suppose the first $m+1$ players have $x_i < 1$ (the argument is the same for any subset of that size or larger). Then take y with $y_i = x_i$ for $i > m+1$ and $y_i = x_i + \epsilon \leq 1$ for $i \leq m+1$. Then clearly $u_j(y) > u_j(x)$ for $j > m+1$, and for $i \leq m+1$ we have $u_i(y) - u_i(x) = (m+1)r\epsilon - \epsilon \geq 0$. Hence x is not Pareto optimal.

Suppose conversely that at x the first $n-m$ players have $x_i = 1$ and consider an alternative allocation y. Let $y_i = x_i + \alpha_i$ and $\alpha_j = \max \alpha_i$. We show that $u_j(y) < u_j(x)$. If $\alpha_j < 0$ then

$$u_j(y) - u_j(x) = r \sum\nolimits_i \alpha_i - \alpha_j \leq (nr-1)\alpha_j < 0.$$

If on the other hand $\alpha_j \geq 0$ then, since $\alpha_i \leq 0$ for $i \leq n-m$

$$u_j(y) - u_j(x) = r \sum\nolimits_i \alpha_i - \alpha_j \leq (mr-1)\alpha_j \leq 0$$

and one of the two inequalities must be strict, because: if $\alpha_j = 0$ then for at least one i it must be $\alpha_i < 0$ (otherwise $y = x$) hence the first inequality is strict; if $\alpha_j > 0$ the second one is strict since $mr - 1 < 0$. □

2.2 The Cournot Oligopoly Model

There are $n \geq 2$ firms $i = 1, \ldots, n$ choosing quantity to produce $q_i \geq 0$, with outcome $q = (q_1, \ldots, q_n)$. We assume that that productions cost is linear with unit cost equal across firms: $c_i(q_i) = cq_i$. Denoting by $Q = \sum q_i$ the total quantity produced, market demand is also assumed linear, given by $p(q) = a - Q$.[6] Firms are therefore in the game where $A_i = \mathbb{R}_+$ and i's payoff is its profit

$$\pi_i(q) = q_i p(q) - cq_i$$

We shall compute the Nash equilibrium profile q^{eq} and then look at the economics of the equilibrium.

Computing the equilibrium For each firm q_i^{eq} maximizes profit π_i given choices q_{-i}^{eq} of the other firms. Letting $\sigma = a - c$ we have $\pi_i(q) = q_i(a-Q) - cq_i = q_i[\sigma - Q]$. If $\sigma \leq 0$, for any q_{-i} it is $\pi_i(0, q_{-i}) > \pi_i(q_i, q_{-i})$ for each $q_i > 0$, so the unique equilibrium is $q_i = 0$ for all i - which is uninteresting. We then assume $\sigma > 0$.

The sum of the profits is $\sum \pi_i = Q(\sigma - Q)$; if $Q > \sigma$ this is negative, which implies that $\pi_i < 0$ for some i - which cannot be the case in equilibrium since any firm can always choose $q_i = 0$ which guarantees $\pi_i(q_i, q_{-i}) = 0$ for each q_{-i}. Thus in equilibrium it must be $Q \leq \sigma$.

Now observe that $\pi_i = q_i[(\sigma - \sum_{j \neq i} q_j) - q_i]$ is a parabola in q_i with maximum $b_i(q_{-i}) = (\sigma - \sum_{j \neq i} q_j)/2 \geq 0$ (non-negative because $\sum_{j \neq i} q_j \leq Q \leq \sigma$).[7] Equilibrium is given by the

[6] We should write $p(Q) = \max\{0, a - Q\}$ but the results would not change.
[7] Note for later that the best response is unique.

solution of the system $q_i = b_i(q_{-i})\ \forall i$, which in our case is

$$2q_i + \sum_{j\neq i} q_j = \sigma \quad i = 1,\dots,n$$

Writing this as $q_i + Q = \sigma$ and summing over i we get $(1+n)Q = n\sigma$ that is $Q = n\sigma/(1+n)$; then from $q_i + Q = \sigma$ we get

$$q_i^{eq} = \frac{\sigma}{1+n}\ \forall i$$

Note that the equilibrium is unique and symmetric (all firms produce the same quantity).

Economics of the equilibrium

1. First observe that total profit is *not* maximized. Indeed $\sum \pi_i = Q(\sigma - Q)$ (a parabola) has maximum $Q^* = \frac{1}{2}\sigma$ while in equilibrium $Q^{eq} = \frac{n}{n+1}\sigma > Q^*$: firms produce "too much" relative to collusive outcome. For maximum total profit each should produce $q_i^* = Q^*/n = \sigma/2n$, but they produce $\sigma/(1+n) > \sigma/2n$ (for any $n \geq 2$). The maximized total profit is $\sum \pi_i^* = Q^*(\sigma - Q^*) = \sigma^2/4$, so that each firm gets $\pi_i^* = \sigma^2/4n$.

2. Equilibrium price $p(q^{eq}) > c$ but it goes to c for $n \to \infty$:

$$p(q^{eq}) = a - Q^{eq} = a - \frac{n}{n+1}\sigma \downarrow a - \sigma = c$$

3. Not only the profit of each single firm, but also *total* profits tend to zero as $n \to \infty$:

$$\sum \pi_i(q_i^{eq}) = Q^{eq}(\sigma - Q^{eq}) = \frac{n\sigma}{n+1}(\sigma - \frac{n\sigma}{n+1}) = \frac{n\sigma^2}{(1+n)^2} \downarrow 0$$

Points 2 and 3 show that for large n the oligopolistic market becomes competitive.

4. Will the firms collude and play $q_i^*, 1 = 1,\dots,n$, or will they end up playing the near-competitive Nash equilibrium? They are in effect in a prisoners dilemma, assuming the possible actions are q_i^* or q_i^{eq}. To see this observe that $\pi_i(q_i^*, q_{-i}^*) = \pi_i^* = \sigma^2/4n$; on the other hand $\pi_i(q_i^{eq}, q_{-i}^*) = q_i^{eq}\left[\left(\sigma - \sum_{j\neq i} q_j^*\right) - q_i^{eq}\right] = \frac{\sigma^2}{1+n}\left[\frac{n+1}{2n} - \frac{1}{1+n}\right]$, and it can be checked by elementary algebra that the last expression is larger than $\sigma^2/4n$, so that $\pi_i(q_i^{eq}, q_{-i}^*) > \pi_i(q_i^*, q_{-i}^*)$. Since by construction $\pi_i(q_i^{eq}, q_{-i}^{eq}) > \pi_i(q_i^*, q_{-i}^{eq})$ we have a prisoners dilemma where the only equilibrium is $q_i = q_i^{eq}\ \forall i$ which gives every firm a profit smaller than what they get if $q_i = q_i^*\ \forall i$.

The dilemma is more severe the larger is n. Indeed the competitive profit is $\pi_i(q_i^{eq}, q_{-i}^{eq}) = q_i^{eq}[\sigma - Q^{eq}] = \frac{\sigma}{1+n}[\sigma - \frac{n\sigma}{1+n}] = \left(\frac{\sigma}{1+n}\right)^2$ while as we have seen $\pi_i(q_i^*, q_{-i}^*) = \sigma^2/4n$, therefore as $n \to \infty$

$$\frac{\pi_i(q_i^*, q_{-i}^*)}{\pi_i(q_i^{eq}, q_{-i}^{eq})} = \frac{(1+n)^2}{4n} \uparrow \infty.$$

Antitrust policy If the firms collude then $Q^* = \frac{1}{2}\sigma < \frac{n}{n+1}\sigma = Q^{eq}$ and $p(q^*) = a - Q^* > a - Q^{eq} > p(q^{eq})$. So they do it at the consumers' expenses in the sense that consumers surplus goes down. And since as we have discussed collusive agreements may be sustainable among firms who have long term relations, the law forbids them. On the other hand firms usually use informal means to enforce cartels, and there enters the antitrust authority who has the difficult task of contrasting collusive looking for evidence of their existence when it suspects they are in place.

2.3 A Partnership Game

Two partners $i = 1, 2$ share a firm and have to decide how much effort to put into it. Each chooses effort level $x_i \geq 0$ at cost x_i^2; the firm's profit is $\pi = 4(x_1 + x_2 + cx_1x_2)$ where $0 < c < 1$. Profit is shared, so $u_i(x_1, x_2) = \pi/2 - x_i^2$. Compute Nash equilibrium. *Hint.* This is a pair (x_1^*, x_2^*) where both payoffs are maximized, use calculus to find best response of i as a function of $x_j, j \neq i$, call it $b_i(x_j)$ and solve the resulting system $x_i = b_i(x_j), i \neq j = 1, 2$.

Solution ...

2.4 Reporting a Crime (Osborne Chapter 3)

Again we give a succinct description of the game and refer to the reader to the book for a fuller account. $n \geq 2$ people (the players) observe a crime, and each has two possible actions: report to the police or not. Let $0 < c < 1$. The payoff of player i is the following: zero if nobody reports; $1 - c$ if she reports; and 1 if someone else reports (we have normalized utility as we learned from decision theory). Thus c represents the cost of reporting.

Solution ...

2.5 Crime and Police (Watson)

The two players are the police P and a criminal C. The police choose enforcement level $x \geq 0$ and the criminal chooses crime intensity $y \geq 0$. Police payoff is $u_P(x, y) = -(c^4x + y^2/x)$, so that c^4 is enforcement unit cost (assume $c > 0$, the power you will see is there to simplify reading of the solution) and the level of enforcement mitigates the negative y^2 effect of crime intensity. The criminal payoff is $u_C(x, y) = \sqrt{y}/(1 + xy)$, where we may interpret $1 + xy$ as the probability of escaping and $\sqrt{y}$ as the utility of crime, with decreasing marginal utility. Compute Nash equilibrium (the procedure is the same as in the previous exercise) and make sure the dependence of equilibrium on c makes sense.

Solution ...

2.6 A Public Good Game

Consider the two-person game where player A has strategy set $x \geq 0$ and B chooses $y \geq 0$, with payoffs given, for some $\epsilon > 0$, by

$$\pi^A(x, y) = \ln(x + y) - x \quad \pi^B(x, y) = (1 + \epsilon)\ln(x + y) - y$$

(a) Find the Nash equilibrium of the game (*hint:* observe that the derivative π^A_x is decreasing, hence if non positive at $x = 0$ the optimum is zero; analogous consideration applies to π^B).
(b) Compute the efficient allocation, i.e. the one maximizing total payoff $\pi = \pi^A + \pi^B$.

Solution (a) To find best responses we must consider possible corner solutions at zero. For player A we have

$$\pi^A_x = \frac{1}{x + y} - 1.$$

Since $\pi^A_x(0, y) \leq 0 \iff y \geq 1$ we get that the best response of A to y is given by

$$b^A(y) = \begin{cases} 1 - y & y < 1 \\ 0 & y \geq 1. \end{cases}$$

For player 2 analogously

$$\pi^B_y = \frac{1 + \epsilon}{x + y} - 1$$

and $\pi^B_y(x, 0) \leq 0 \iff x \geq 1 + \epsilon$ whence

$$b^B(x) = \begin{cases} 1 + \epsilon - x & x < 1 + \epsilon \\ 0 & x \geq 1 + \epsilon. \end{cases}$$

The Nash system is $x = b^A(y), y = b^B(x)$. If $0 \leq y < 1$ we have $x = b^A(y) \iff x = 1 - y \leq 1$ so $y = b^B(x) \iff y = 1 + \epsilon - x$; then the system becomes $x + y = 1, x + y = 1 + \epsilon$ which is impossible. Thus it must be $y \geq 1$. This implies $x = b^A(y) \iff x = 0$, and $y = b^B(x) \iff y = 1 + \epsilon$. This is the only Nash: $x^* = 0, y^* = 1 + \epsilon$. Player B does all the work.

(b) $\pi = \ln(x + y) - x + (1 + \epsilon)\ln(x + y) - y$ so

$$\pi_x = \pi_y = \frac{2 + \epsilon}{x + y} - 1$$

and $\pi_x = \pi_y = 0 \iff x + y = 2 + \epsilon$. Any pair satisfying this is efficient, and note that

$x^{\text{eff}} + y^{\text{eff}} > x^* + y^*$.

2.7 Electoral Campaign

Two parties 1,2 are campaigning in N regions and they have to allocate their campaign time T among them. So a strategy of 1 can be described by $x = (x_1, \ldots, x_N)$ where x_i is time spent in region i, with $\sum_i x_i = T$; and similarly a strategy for 2 is a $y = (y_1, \ldots, y_N)$ such that $\sum_i y_i = T$. Thus each has a continuum of actions.

In each region the probabilities of winning are determined by the time spent as follows:

$$\Pr(1 \text{ wins in region } i) = \frac{x_i}{x_i + y_i}, \quad \Pr(2 \text{ wins in region } i) = \frac{y_i}{x_i + y_i}$$

(notice that one of them wins for sure). Winning in region i gives v_i seats in parliament. Thus under profile (x, y) party 1's expected number of seats is $\sum_i v_i \cdot \frac{x_i}{x_i+y_i}$, similarly for 2 (note that expected total number of seats is $\sum_i v_i \cdot \frac{x_i}{x_i+y_i} + \sum_i v_i \cdot \frac{y_i}{x_i+y_i} = \sum_i v_i \equiv V$ for any profile).

Parties maximize the difference between its expected seats and those of the adversary; so under profile (x, y) payoff of party 1 is

$$\sum_i v_i \cdot \frac{x_i}{x_i + y_i} - \sum_i v_i \cdot \frac{y_i}{x_i + y_i} = 2\sum_i v_i \cdot \frac{x_i}{x_i + y_i} - V$$

and that of party 2 is the opposite - it is a zero-sum game.

Recalling that when maximizing a sum $\sum_i u(z_i)$ with u concave subject to $\sum_i z_i = T$ the optimality condition is $u'(z_i) = u'(z_j)$ for all i, j, find the Nash equilibrium of this game and verify that the equilibrium expected payoff of the two parties is zero.

Solution Apply the condition $u'(z_i) = u'(z_j)$ for each player to get

$$\frac{v_i x_i}{v_j x_j} = \frac{v_i y_i}{v_j y_j} = \left(\frac{x_i + y_i}{x_j + y_j}\right)^2$$

From the first equality $x_i/x_j = y_i/y_j$. Using the time constraint we get

$$T = x_1(1 + \frac{x_2}{x_1} + \cdots + \frac{x_n}{x_1}) = y_1(1 + \frac{y_2}{y_1} + \cdots + \frac{y_n}{y_1})$$

from which we deduce $x_i = y_i$ for all i. Therefore

$$\frac{v_i x_i}{v_j x_j} = \left(\frac{x_i + y_i}{x_j + y_j}\right)^2 = \left(\frac{x_i}{x_j}\right)^2$$

whence $x_i/x_j = v_i/v_j$; similar argument holds for y, and we arrive at

$$\frac{x_i}{x_j} = \frac{y_i}{y_j} = \frac{v_i}{v_j}.$$

This characterizes equilibrium: time spent in region i is proportional to its value.

For player 1 we then get $x_i = \frac{v_i}{v_1} x_1,\ i = 2, \ldots, n$; from $T = x_1 \left(1 + \frac{v_2 + \cdots + v_N}{v_1}\right) = x_1 \frac{V}{v_1}$ we obtain

$$x_i = \frac{v_i}{V} \cdot T, \quad i = 1, \ldots, N$$

From $y_i = x_i$ we see that in equilibrium each party has probability of winning 1/2 in each region, and each gets expected payoff of zero (which is the value of the game).

2.8 Accident Law (Osborne Ch.3)

Here is a concise statement of the game; for a more extensive description see the Osborne book. There are two players, a *pedestrian* (player 1) and a *driver* (player 2). For both the action set is $A_i = [0, \infty)$, interpreted as the *amount of care* taken to avoid an accident. Profile (a_1, a_2) causes a total (expected) loss of $L(a_1, a_2) > 0$, naturally assumed to be decreasing in both arguments. Player 1 pays a fraction $\rho(a_1, a_2)$. So the payoffs in the game are

$$u_1(a_1, a_2) = -a_1 - \rho(a_1, a_2)L(a_1, a_2) \quad \text{and} \quad u_2(a_1, a_2) = -a_2 - (1 - \rho(a_1, a_2))L(a_1, a_2).$$

Suppose $(\hat{a}_1, \hat{a}_2)$ is the only pair which minimizes the total loss $a_1 + a_2 + L(a_1, a_2)$ the players suffer. This we may assume to represent a "socially desirable" outcome. The question is, can a game be designed which admits $(\hat{a}_1, \hat{a}_2)$ as the *only* Nash equilibrium? This is a prototype of what is known as an *implementation problem.*[8] Since players incentives are determined by the assignment rule ρ one has to look for a rule - a law - which induces $(\hat{a}_1, \hat{a}_2)$ as the only Nash. We are going to show one such rule is given by what is known as "negligence with contributory negligence". This consists of two thresholds C_1, C_2 of care level which give rise to the following zero-one valued ρ:

$$\rho(a_1, a_2) = \begin{cases} 1 & \textit{if } a_1 < C_1 \textit{ and } a_2 \geq C_2 \\ 0 & \textit{if } a_1 \geq C_1 \textit{ or } a_2 < C_2. \end{cases}$$

So the driver pays if the pedestrian is careful enough or if she is not careful enough. The result that we are going to prove is that if $(C_1, C_2) = (\hat{a}_1, \hat{a}_2)$ then the only Nash equilibrium of the game is $(\hat{a}_1, \hat{a}_2)$. Notice that in equilibrium the driver bears the entire loss.

[8] It is the same kind of problem King Solomon faced.

Solution

With $(C_1, C_2) = (\hat{a}_1, \hat{a}_2)$ the rule is

$$\rho(a_1, a_2) = \begin{cases} 1 & \textit{if } a_1 < \hat{a}_1 \textit{ and } a_2 \geq \hat{a}_2 \\ 0 & \textit{if } a_1 \geq \hat{a}_1 \textit{ or } a_2 < \hat{a}_2. \end{cases}$$

First we show that $(\hat{a}_1, \hat{a}_2)$ is Nash. For any a_2, 1's best response $B_1(a_2) \leq \hat{a}_1$ - this is clear. Now find $B_1(\hat{a}_2)$. For any $a_1 < \hat{a}_1$ we have

$$u_1(a_1, \hat{a}_2) = -a_1 - L(a_1, \hat{a}_2) \leq -\hat{a}_1 - L(\hat{a}_1, \hat{a}_2) < -\hat{a}_1 = u_1(\hat{a}_1, \hat{a}_2)$$

the first inequality because $(\hat{a}_1, \hat{a}_2)$ maximizes $-(a_1 + a_2 + L(a_1, a_2))$.[9] So $B_1(\hat{a}_2) = \hat{a}_1$. As to player 2, again clearly $B_2(a_1) \leq \hat{a}_2$ for any a_1; and for $a_2 < \hat{a}_2$

$$u_2(\hat{a}_1, a_2) = -a_2 - L(\hat{a}_1, a_2) < -\hat{a}_2 - L(\hat{a}_1, \hat{a}_2) = u_2(\hat{a}_1, \hat{a}_2)$$

again by definition of max. Hence $B_2(\hat{a}_1) = \hat{a}_2$. Therefore $(\hat{a}_1, \hat{a}_2)$ is Nash.

We must still prove that $(\hat{a}_1, \hat{a}_2)$ is the *unique* Nash. We know that $B_1(a_2) \leq \hat{a}_1$ for any a_2; we now show that for any $a_1 \leq \hat{a}_1$ we have $B_2(a_1) = \hat{a}_2$. Hence in any Nash $a_2 = \hat{a}_2$, and since $B_1(\hat{a}_2) = \hat{a}_1$ we get uniqueness.

Consider $a_1 \leq \hat{a}_1$. For $a_1 = \hat{a}_1$ we know that $B_2(\hat{a}_1) = \hat{a}_2$. Suppose then $a_1 < \hat{a}_1$; we already know that $B_2(a_1) \leq \hat{a}_2$; but for $a_2 < \hat{a}_2$ we have

$$u_2(a_1, a_2) = -a_2 - L(a_1, a_2) < -a_2 - L(\hat{a}_1, a_2) < -\hat{a}_2 - L(\hat{a}_1, \hat{a}_2) < -\hat{a}_2 = u_2(a_1, \hat{a}_2)$$

the first inequality because L is decreasing in a_1, the second by definition of maximum, the third because $L > 0$, and the final equality by application of the rule ρ; hence for $a_1 < \hat{a}_1$ too we have $B_2(a_1) = \hat{a}_2$.

3 Back to Finite Action Spaces

3.1 A 4x2 Game (Watson)

In the following game: (a) Find all the equilibria; (b) Compute player 1's expected payoff in equilibrium; (c) If player 1 is willing to buy your advice, what can you tell her?

[9] Recall from the zero-sum lecture that $\min f = -\max(-f)$ and the solution of the two problems is the same.

	x	y
a	12, 0	0, 6
b	11, 1	1, 5
c	10, 2	4, 2
d	9, 3	6, 0

Solution (a) First observe that player 2 must mix. Let p the probability of strategy x. Next observe that player 1 will never play b: you can easily check that for $p > 1/2$ it is dominated by a and for $p < 5/7$ it is dominated by d, so for any p it is dominated. Next observe (easy to check, obvious notation):

$$a \succsim_1 c \iff a \succsim_1 d \iff c \succsim_1 d \iff p \geq 2/3$$

Now: if $p > 2/3$ action a dominates but if 1 plays a player 2 would play y for sure; and for $p < 2/3$ action d dominates but analogously 2 cannot mix if 1 plays d. Hence $p = 2/3$ in equilibrium, and for this value $a \sim_1 c \sim_1 d$. We are left to impose $x \sim_2 y$, that is

$$2p_c + 3p_d = 6p_a + 2p_c.$$

This yields $p_d = 2p_a$, with $p_c = 1 - p_a - p_d = 1 - 3p_a$. In conclusion, noting that $0 \leq p_c \leq 1$, the equilibria are given by $p = 2/3$ and

$$\begin{cases} 0 \leq p_a \leq 1/3 \\ p_d = 2p_a \\ p_c = 1 - 3p_a \end{cases}$$

Notice that there is an equilibrium where 1 plays only c, and one where she plays only a and d (with $p_a = 1/3, p_d = 2/3$. (b) Player 1's expected payoff is always 8.

(c) First you can recommend her *not* to play b. For the rest you might explain that player 2 will play x two thirds of the time, hence she can either play c, or mix between a, c and d always using d twice as many times as a.

3.2 A Parametric Chicken Game

Consider the following Chicken game. Two players, actions *Stop* and *Go* (from *crossroads* interpretation). Individual payoffs are the following:

	S	G
S	$1\ ,\ 1$	$1-\lambda\ ,\ 1+\gamma$
G	$1+\gamma\ ,\ 1-\lambda$	$0\ ,\ 0$

with $\gamma > 0$ and $0 < \lambda < 1$. The pure strategy equilibria are SG and GS.

(a) Compute the mixed equilibrium of the game (as a function of λ, γ) and the resulting equilibrium payoff of both players (is this greater or smaller than 1?) (b) Consider the case $\lambda \to 1$. What happens in equilibrium? (quite a bad state of affairs). Can you explain why? (consider playing S or G) (c) Derive the outcome if both play maxmin. Is it more reasonable than mixed equilibrium? (d) (if applicable) Consider now correlated equilibria with 3 states, probability q on SS and the rest equally split on the off-diagonal elements. For which values of q is the resulting distribution a correlated equilibrium? (e) (if applicable) Compute the correlated equilibrium payoffs (of any player) at q. Assume $\lambda > \gamma$ so that the average payoff at SS is larger than at SG and GS. What is the maximum correlated equilibrium payoff in this case? Is it larger than the mixed equilibrium payoff?

Solution ...

3.3 A three-player Game

Recall that in the 2×2 case (two players, two actions) we computed mixed equilibria by imposing indifference conditions among the actions with positive probability: indifference of expected payoff of player 1 determined play probabilities of 2 and vice versa.

Here we consider a game with *three* players, each still with two actions. In this case to compute the expected payoff of say player 1 we have to weight the four possibilities corresponding to what the others do (in the 2×2 case there were two possibilities only, what 2 was doing). Consider the game below. In the picture 1 and 2 choose as before but there is also player 3 who chooses *Left* or *Right,* meaning she chooses one of the two matrices; the payoffs are ordered naturally; mixing probabilities are also indicated: 1 plays *Up* with probability p, 2 *Left* with probability q, 3 plays *Left* with probability r. If 1 plays *Up* her expected payoff is $r \cdot q \cdot 0 + r(1-q) \cdot (-4) + (1-r)q \cdot 3 + (1-r)(1-q) \cdot 1$ - etc.

3 Left: r

		2 : q ***Left***	***Right***
1 : p	***Up***	$0, 0, 0$	$-4, 1, 2$
	Down	$1, -4, 2$	$2, 2, -2$

3 Right

		2 : q ***Left***	***Right***
1 : p	***Up***	$3, 3, -2$	$1, -4, 2$
	Down	$-4, 1, 2$	$0, 0, 0$

Now: first show that there is no pure strategy equilibrium; next find the fully mixed equilibrium. Note that 1 and 2 are symmetric in the computation of the expected payoffs

(*Left* corresponding to *Up* and *Right* to *Down*) so in equilibrium it must be $p = q$; this greatly simplifies computation. The answer is $(p, q, r) = (1/2, 1/2, 8/15)$. Finally show that any mixed equilibrium must be fully mixed; this implies that the equilibrium you have found is the only one.

Solution of the latter point Player 3 must mix: if she plays a pure strategy then 1 and 2 have a unique Nash to which player 3 would respond with the other pure strategy. Then at least one of the other two must mix as well, otherwise 3 would have a pure best response. Suppose 1 mixes; if 2 doesn't then 3's best replies are pure (contradicting the fact that she must mix); so 2 must mix as well.

3.4 Another Three-player Game

There are three players. Each player chooses one of two actions, C or D and the payoffs can be written in bi-matrix form. If player 3 plays C the payoff matrix for the actions of players 1 and 2 is the prisoner's dilemma game in the left matrix, where player 3 prefers that 1 and 2 cooperate (play CC); if player 3 plays D players 1 and 2 are in the coordination game in the right matrix, where player 3 prefers that they defect (play DD):

	C	D
C	$6,6,5$	$0,8,0$
D	$8,0,0$	$2,2,0$

	C	D
C	$10,10,0$	$0,8,5$
D	$8,0,5$	$2,2,5$

(a) Compute the Nash equilibria of this game. For the mixed equilibria let α^i denote the probability with which player i plays C. (there are three Nash equilibria: DDD; one where 3 plays D and 1 and 2 mix 50-50 between C and D; and a fully mixed one $\alpha^1 = \alpha^2 = 1 - \alpha^3 = 1/\sqrt{2} \approx 0.7$)

Solution First of all there is no Nash where $\alpha^3 > 1/2$ for if 1 and 2 play DD (as they have to in equilibrium) player 3 prefers D ($\alpha^3 = 0$). Same for $\alpha^3 = 1/2$: if 1 and 2 play CC player 3 strictly prefers C; if they play DD she strictly prefers D.

So for Nash it must be $\alpha^3 < 1/2$. The CC equilibrium for 1 and 2 cannot be part of equilibrium because then 3 prefers C ($\alpha^3 = 1$). Hence 1 and 2 must either play DD or mix. If they play DD then 3's best response is D that is $\alpha^3 = 0$ and therefore DDD is Nash. Suppose then 1 and 2 mix. From $\alpha^1 = \alpha^2 = 1/\left(2(1-\alpha^3)\right)$ we see that $\alpha^1 = \alpha^2 \geq 1/2$. Player 3 prefers D strictly if $5\left(1 - \left(\alpha^1\right)^2\right) > 5\left(\alpha^1\right)^2$ that is if $\alpha^1 = \alpha^2 < 1/\sqrt{2} \approx 0.7$, so the only Nash in this range has $\alpha^1 = \alpha^2 = 1/2, \alpha^3 = 0$. For $\alpha^1 = \alpha^2 = 1/\sqrt{2}$ there is a fully mixed equilibrium with $\alpha^1 = \alpha^2 = 1/\sqrt{2}$ and α^3 given by $1/\left(2(1-\alpha^3)\right) = 1/\sqrt{2}$ that is

$\alpha^3 = 1 - 1/\sqrt{2} \approx 0.3$. Finally there are no equilibria with $\alpha^1 = \alpha^2 > 1/\sqrt{2}$ because for such values 3 strictly prefers C and we have seen that this cannot happen in equilibrium.

(b) Compute the payoff vectors in the Nash equilibria.

Solution In DDD it's $(2,2,5)$. In the partially mixed equilibrium the group members get 5 while player 3 gets $5 * 3/4 = 3.75$. In the fully mixed equilibrium players 1 and 2 get $2 + 3\sqrt{2} \approx 6.24$ and 3 gets 2.5. Algebra for the last case. For 1 and 2

$$\alpha^3 \left(6\left(\alpha^1\right)^2 + 8\alpha^1(1-\alpha^1) + 2(1-\alpha^1)^2\right) + (1-\alpha^3)\left(10\left(\alpha^1\right)^2 + 8\alpha^1(1-\alpha^1) + 2(1-\alpha^1)^2\right)$$
$$8\alpha^1(1-\alpha^1) + 2(1-\alpha^1)^2 + 6\left(\alpha^1\right)^2 + 4(1-\alpha^3)\left(\alpha^1\right)^2$$
$$8\frac{1}{\sqrt{2}}(1-\frac{1}{\sqrt{2}}) + 2(1-\frac{1}{\sqrt{2}})^2 + 6\left(\frac{1}{\sqrt{2}}\right)^2 + 4\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2$$
$$8\frac{1}{\sqrt{2}}\frac{\sqrt{2}-1}{\sqrt{2}} + (\sqrt{2}-1)^2 + 3 + 2\frac{1}{\sqrt{2}}$$
$$4(\sqrt{2}-1) + (\sqrt{2}-1)^2 + 3 + \sqrt{2}$$
$$(\sqrt{2}-1)(4+\sqrt{2}-1) + 3 + \sqrt{2}$$
$$(\sqrt{2}-1)(3+\sqrt{2}) + 3 + \sqrt{2}$$
$$\sqrt{2}(3+\sqrt{2}) = 2 + 3\sqrt{2} \approx 6.24$$

For 3:

$$\alpha^3\frac{5}{2} + (1-\alpha^3) * 5\left(1 - \left(\alpha^1\right)^2\right)$$
$$(1-\frac{1}{\sqrt{2}})\frac{5}{2} + (1-\alpha^3) * \frac{5}{2}$$
$$(1-\frac{1}{\sqrt{2}})\frac{5}{2} + \frac{1}{\sqrt{2}} * \frac{5}{2} = 2.5$$

3.5 Yet another one (Li Calzi)

The game is the following, where 3 chooses the matrix (M or N):

M	L	R
T	$1,1,1$	$0,0,0$
B	$0,0,0$	$2,1,1$

N	L	R
T	$0,0,0$	$1,2,1$
B	$1,1,2$	$0,0,0$

To find all Nash equilibria of the game we search separately for those where $0,1,2$ or 3 players use mixed strategies. We are going to have to solve a system of equations so it is convenient to let x, y, z the probabilities with which player 1 plays T, 2 plays L and 3 plays M. In pure strategies it is elementary to check that there are 4 equilibria: TLM, BRM, BLN and TRN.

One player mixes and the other two use pure strategies: in this case we easily check that if two players use pure strategies the best response of the remaining player is a unique pure strategy, so there can be no equilibrium of this kind.

Now one player uses a pure strategy and the others mix; starting with player 3 suppose she plays M (that is $r = 1$). In the resulting 2×2 game between 1 and 2 the mixed equilibrium is $x = 1/2$ and $y = 2/3$. Then player 3's payoff from M is $U_3(1/2, 2/3, 1) = xy + (1-x)(1-y) = 1/3 + 1/6$; if she plays N she gets $U_3(1/2, 2/3, 0) = 2(1-x)y + x(1-y) = 2 \cdot 1/2 \cdot 2/3 + 1/2 \cdot 1/3 = 2 \cdot 1/3 + 1/6 > U_3(1/2, 2/3, 1)$; so $r = 1$ is not equilibrium. Suppose $r = 0$ (3 plays N); then the mixed equilibrium of the resulting game is $x = 1/3, y = 1/2$; and proceeding as above we see that actually $U_3(1/3, 1/2, 0) > U_3(1/3, 1/2, 1)$ so $(x, y, z) = (1/3, 1/2, 0)$ is an equilibrium. Repeating the analysis with players 1 and 2 we find the equilibria $(x, y, z) = (0, 1/3, 1/2), (1/2, 0, 1/3)$.

Assume finally that all 3 players mix. Player 1's indifferences give

$$
\begin{gathered}
zy + (1-z)(1-y) = 2z(1-y) + (1-z)y \\
2y + 3z - 5zy = 1
\end{gathered}
$$

and analogously, working out the indifferences of the other two we arrive at the following system:

$$
\begin{aligned}
2x + 3y - 5xy &= 1 \\
2y + 3z - 5yz &= 1 \\
2z + 3x - 5zx &= 1
\end{aligned}
$$

The solution to this, details in footnote, is $x = y = z = (5 \pm \sqrt{5})/10$ - two more equilibria.[10] In conclusion there are 9 equilibria.

---

[10] We solve the system in three steps.

1. Establish that it must be $x, y, z \neq 0$

$x = 0 \Rightarrow y = 1/3, z = 1/2 \Rightarrow 2y + 3z - 5yz \neq 1$. Analogously $y, z \neq 0$

2. Show that it must be $x = y = z$

1st equation is $x(2-5y) = 1 - 3y$. Now $2 - 5y = 0$ & $x \neq 0 \Rightarrow x(2-5y) \neq 1 - 3y$, so $2 - 5y \neq 0$. Analogously $2 - 5x \neq 0, 2 - 5z \neq 0$. We then derive

$$
x = \frac{1-3y}{2-5y}, y = \frac{1-3z}{2-5z}, z = \frac{1-3x}{2-5x}
$$

3.6 Matching Pennies with Spy

In this elaboration of the matching pennies game, a spy peeks at player 2's move and reports her observation to player 1, who then plays. The spy's report is correct with probability π.

Player 2 obviously has just two strategies, H and T. Player 1 on the other hand can choose what to play depending on the spy's report. So we can represent the set A_1 of her 4 actions as pairs ij with $i, j \in \{H, T\}$, where the first coordinate indicates what she plays if the spy reports H and the second her play after report T. Thus for example TH means she plays the opposite of what the spy says. Then the (expected) payoffs of player 1 are represented in the following table (for example HT, whereby she plays what the spy reports, gives $\pi \cdot 1 + (1-\pi) \cdot (-1) = 2\pi - 1$; etc). Player 2 always gets the opposite of 1 so the game is still a zero-sum game.

	H	T
HH	1	-1
HT	$2\pi - 1$	$2\pi - 1$
TH	$1 - 2\pi$	$1 - 2\pi$
TT	-1	1

To analyze the equilibria of the mixed extension of this game as π varies it is convenient to derive the conservative strategies of player 2 directly as minmax strategies. Recall that in equilibrium a player uses only the pure actions giving her highest utility against the mixed strategies of the others. Then, letting as usual q denote the probability that she plays H, the value of the mixed extension of the game, which we denote by $v(\pi)$, can be written as follows

so with a little algebra we get

$$x - y = \frac{1-3y}{2-5y} - \frac{1-3z}{2-5z} = \cdots = \frac{z-y}{(2-5y)(2-5z)}$$
$$z - y = \cdots = \frac{z-x}{(2-5x)(2-5z)}$$
$$z - x = \cdots = \frac{y-x}{(2-5x)(2-5y)}$$

whence

$$x - y = \frac{y-x}{(2-5x)^2(2-5y)^2(2-5z)^2}$$

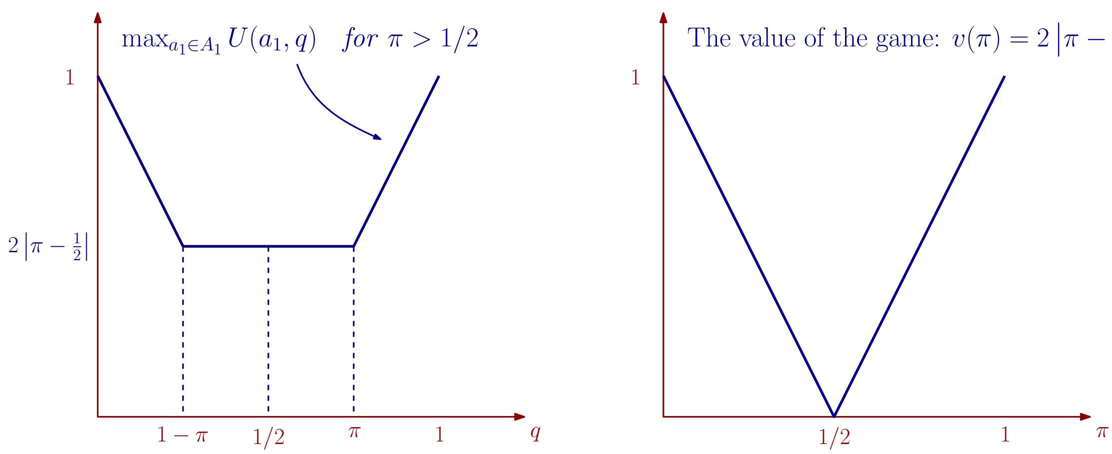
Thus if $x \neq y$ we get a contradiction (-1 equal positive number). Analogously $y = z$.

3. At this point the three equations are all $5x - 5x^2 = 1$ which is solved by $x = (5 \pm \sqrt{5})/10$. So the solution of the system is $x = y = z = (5 \pm \sqrt{5})/10$.

(note for example that if 2 plays q and 1 plays HH the latter gets $q \cdot 1 + (1-q)\cdot(-1) = 2q-1$):

$$\begin{aligned} v(\pi) &= \min_{0\le q\le 1} \max_{a_1\in A_1} U(a_1,q) \\ &= \min_{0\le q\le 1} \max\{2q-1, 2\pi-1, 1-2\pi, 1-2q\} \\ &= \min_{0\le q\le 1} \max\{2\left|\pi-\frac{1}{2}\right|, 2\left|q-\frac{1}{2}\right|\} \\ &= \min_{0\le q\le 1} \begin{cases} 2\left|\pi-\frac{1}{2}\right| & \textit{if} \quad \left|q-\frac{1}{2}\right| \le \left|\pi-\frac{1}{2}\right| \\ 2\left|q-\frac{1}{2}\right| & \textit{if} \quad \left|q-\frac{1}{2}\right| \ge \left|\pi-\frac{1}{2}\right| \end{cases} \\ &= 2\left|\pi-\frac{1}{2}\right|. \end{aligned}$$

The function $\max_{a_1\in A_1} U(a_1,q) = \max\{2\left|\pi-\frac{1}{2}\right|, 2\left|q-\frac{1}{2}\right|\}$ is displayed in the left panel of the figure below for $\pi > 1/2$. In this case its minimum is achieved for any $1-\pi \le q \le \pi$. The resulting value function $2|\pi-1/2|$ is in the right panel.



Consider $\pi \ge 1/2$. Notice first that when truth telling has probability higher than 1/2 it is never a best response to do the opposite of what the spy suggests (indeed TH is worse than HT whatever 2 does so it cannot be played with positive probability in equilibrium). From the left panel of the figure it is clear that in any equilibrium it must be $1-\pi \le q \le \pi$. Given this, the equilibrium play of 1 is not hard to find. Let $\alpha_1 = (p_1, \ldots, p_4)$ where $p_1$is the probability of HH etc. Then by inspecting the payoff matrix we deduce that: if $q = \pi$ then $\alpha_1 = (p, 1-p, 0, 0)$ for any $0 \le p \le 1$; if $q = 1-\pi$ then $\alpha_1 = (0, p, 0, 1-p)$ for any $0 \le p \le 1$; and if $1-\pi < q < \pi$ then $p_2 = 1$.

For $\pi < 1/2$ the argument is analogous: $\pi \le q \le 1-\pi$; for interior q we have $p_3 = 1$; if $q = \pi$ then $\alpha_1 = (0, 0, p, 1-p)$ for any p; and if $q = 1-\pi$ then $\alpha_1 = (p, 0, 1-p, 0)$ for any p.

Another look at the game In this game it is interesting to express the the value in terms of the correlation between 2's play and the spy's announcement. To do this we change interpretation and give "monetary" values to actions: say H has value h and T has value $t > h$. Similarly we give values h and t to the corresponding spy's reports. Let $\Omega_1 = \{h, t\}$ represent the value of 2's play and $\Omega_2 = \{h, t\}$ the value of the spy's announcement. Let $\Omega = \Omega_1 \times \Omega_2$, and P a probability on Ω. Let X and Y the projections (random variables), that is for $\omega = (\omega_1, \omega_2)$ let $X(\omega) = \omega_1$ and $Y(\omega) = \omega_2$. They represent play and announcement. The hypothesis we have made above is that

$$\mathsf{P}(Y = h \mid X = h) = \mathsf{P}(Y = t \mid X = t) = \pi.$$

From this we get $\mathsf{P}(Y = X) = \pi$.[11] Since the value of the game is independent of the equilibrium played we may take the equilibrium where 2 plays H with probability $1/2$ (which is part of an equilibrium for all π); in this context this means assuming $\mathsf{P}(X = h) = \mathsf{P}(X = t) = 1/2$. From this it follows that $EX = (h+t)/2$; and that $E(Y) = \frac{1}{2}\left[E(Y \mid X = h) + E(Y \mid X = t)\right] = \frac{1}{2}\left[\pi h + (1-\pi)t + \pi t + (1-\pi)h\right] = E(X)$. We also have $Var(X) = [(t-h)/2]^2 = Var(Y)$ (since in both cases the square distance from the mean is $[(t-h)/2]^2$ with probability 1). Finally, $Cov(X, Y) = [(t-h)/2]^2 \cdot \mathsf{P}(X = Y) - [(t-h)/2]^2 \cdot \mathsf{P}(X \neq Y) = \frac{1}{2}(t-h)^2 \cdot (\pi - \frac{1}{2})$.

Therefore correlation coefficient $\rho(X, Y)$ satisfies

$$|\rho(X, Y)| = \left| \frac{Cov(X, Y)}{\sqrt{Var(X)\, Var(Y)}} \right| = 2\left|\pi - \frac{1}{2}\right| = v(\pi).$$

That is, the value is *exactly* the absolute value of the correlation coefficient between 2's play and spy's announcement.

3.7 Battleships

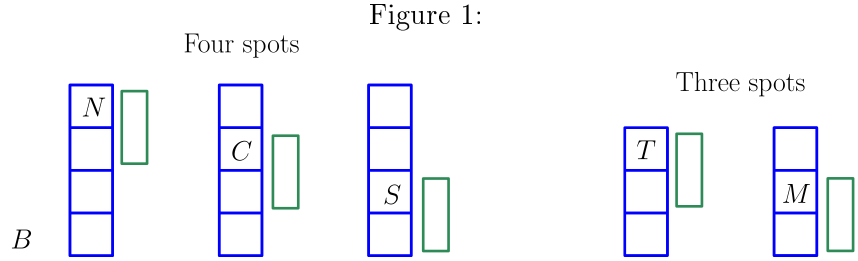
This is an elaboration of a game described in Binmore's *Fun and Games*. As we all know there are two players who both bomb and hide, and the one who sinks all the other's ships first wins. We are going to consider the case where there is only one ship, and will start by looking at the further simplified case where one player bombs and the other one hides. In fact, since for each player the two roles of bomber and hider are totally separate, the analysis of this asymmetric case effectively solves the bombing-and-hiding case. We describe the details next.

[11] Because

$$\begin{aligned}\mathsf{P}(Y = X) &= \mathsf{P}(X = h) \cdot \mathsf{P}(Y = X \mid X = h) + \mathsf{P}(X = t) \cdot \mathsf{P}(Y = X \mid X = t) \\ &= \mathsf{P}(X = h) \cdot \mathsf{P}(Y = h \mid X = h) + \mathsf{P}(X = t) \cdot \mathsf{P}(Y = t \mid X = t) = \pi.\end{aligned}$$

One bombs, the other hides John has a ship of length 2 that he may place in two of four aligned squares of length one, as in the left panel of the figure. Alberta cannot see it but she wants to sink it (two good hits for that), and she can shoot as many times as she wants, John truthfully reporting the outcome after each shot. John wants to live as long as possible, while Alberta wants to win as quickly as she can. So it is a zero-sum game, where we may take the number of shots up to sinking as John's payoff. Obviously John has 3 strategies, which we call N, C, S (the position of the prow); the tricky part is to show that Alberta is basically in the same position, in the sense that she has only three undominated strategies (she has four, but one is redundant because it replicates one of the other three). At each shot she can target N, C, S and the bottom spot B. Start thinking of Alberta's strategies and you will realize that it takes a little effort to describe them conveniently. Once we get to them the game is trivial.

Figure 1:



To start organizing thoughts it is useful to look at an even simpler version of this game, with three spots instead of four, as in the right panel. John 's strategies are T and M; Alberta's targets are here T, M, B. But M is a sure and necessary hit, so she really has two strategies, represented by first-shot choices T or B.

(a) Write down the two-by-two matrix of the 3-spot game with John's payoff as entries, and find equilibrium and value (just to check, the value is $2\frac{1}{2}$, understandably). (b) Guided: turn to the 4-spot game and do the same, that is write the game matrix and find equilibrium and value (value is $2\frac{2}{3}$, John is better off because he has more room to hide). Start by arguing that in any equilibrium John has to fully mix; to prove this a full specification of Alberta's strategies is not needed.[12] This implies that Alberta will not play weakly dominated strategies in equilibrium so we can ignore them. To find a dominated strategy consider starting with target N. Show that however you continue you can do weakly better by starting with C instead. To do this use what you know about the 3-spot game to show that there are two possibilities starting with N, one for example is $N_m C$ meaning "start with N, if you miss target C" (it is obvious what to do if you hit); there are obviously two starting with C, one

[12]Proof: if John mixes between N and C only then Alberta's best response is not to target B in the first three shots; to this John's best reply is S which gives a sure 4. It follows similarly that he cannot mix between any two alternatives.

being $C_h N$ which means "start with C, if you hit target N" (it is obvious what to do if you miss). Write down the corresponding payoffs under John's N, C, S and compare.

Solution of this If you start with N and hit then you win by shooting C. Suppose you miss; then you are left with three spots and as we know your choice is then effectively C or B. So starting with N you have $N_m C$ and $N_m B$. Starting with C, if you miss you win with S, B; if you hit then you can continue with N or S. Consider $C_h N$ and compare John's payoffs:

	$N_m C$	$N_m B$	$C_h N$
N	2	2	2
C	3	4	3
S	4	3	3

From this we see that $C_h N$ weakly dominates both.

Analogously, starting with B must be weakly dominated too. At this point you can get down to 4 strategies for Alberta (two $C_h X$ and two $S_h X$). Write down the 3×4 matrix and observe that two strategies are equivalent. Reduce the matrix to a 3×3, and you are done.

Solution to final step the 3×4 matrix is on the left, and it reduces to the 3×3 on the right.

	$C_h N$	$C_h S$	$S_h C$	$S_h B$
N	2	3	3	3
C	3	2	2	3
S	3	3	3	2

	$C_h N$	$C_h S$	$S_h B$
N	2	3	3
C	3	2	3
S	3	3	2

Now work on the latter. Call q_1, q_2, q_3 Alberta's mixing probabilities; John's indifference between N and C gives $q_1 = q_2$, and that between C and S gives $q_2 = q_3$; so it is a uniform $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. John is symmetric, also uniform. The value is given by John's payoff from any of his strategies, and it is $2\frac{2}{3}$.

Bombing and hiding (c) Who should be expected to win? Assume that both play the Nash equilibrium as bombers and hiders (it can be argued that this will actually be the case in the long run). Solution in footnote. [13]

[13] The answer is the one who starts. Because the expected number of shots to sink is $2\frac{2}{3}$ for each, so in expectation the one who starts has an advantage.