Notes on Fair Division

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The concept of fairness is deeply rooted in human nature - I bet everyone of us has screamed "It's not fair!" many times before the age of six. Unfortunately the concept is elusive: it is not easy to give a precise, universally acceptable definition of fairness. More importantly, what looks fair in particular situation is often inefficient in some straightforward sense, so that one has to somehow compromise between fairness and efficiency. We are going to explore the issue in a few simple contexts, in order to become aware of the main positive and negative results which emerge.

First definitions

Economies and allocations. Economies are specified by goods, agents with their preferences, endowments and possibly production possibilities. Without production an economy is specified by the following elements.

- Goods $i = 1, \dots, n$ so bundles $x \in \mathbb{R}^n_+$.

- Agents $h = 1, \ldots, H$ each with preferences $\succcurlyeq_h$ over bundles, $x \succcurlyeq_h y$ complete transitive, with $\succ_h$ and $\sim_h$ denoting strict preference and indifference. Each relation $\succcurlyeq_h$ is assumed continuous, (strongly) monotonic, convex.

- The total endowment of the economy, $\omega \in \mathbb{R}^n_{++}$. Possibly $\omega = \sum_h \omega_h$ if ω_h is h's initial endowment.

Letting $R = (\succcurlyeq_1, \ldots, \succcurlyeq_H)$ be the vector of preferences, the economy is then

$$E = (R, \omega).$$

An *allocation* is a vector $\xi = (\xi_1, \ldots, \xi_H)$, with $\xi_h \in \mathbb{R}^n_+$ to agent h. The allocation is *feasible* if $\sum_h \xi_h \leq \omega$. Denote by $Z(E)$ the set of allocations of E.

Basic properties of allocations. The following are the two basic properties characterizing efficiency and equity.

- An allocation $\xi \in Z(E)$ is *Pareto efficient* if there is no allocation $\zeta \in Z(E)$ such that $\zeta_h \succcurlyeq_h \xi_h$ for all h and $\zeta_h \succ_h \xi_h$ for some h.
- An allocation ξ is *envy-free* if $\xi_h \succcurlyeq_h \xi_k$ for all $h, k = 1, \ldots, H$.

A related, relevant concept is the following:

- An allocation ξ is *egalitarian-equivalent* if there exists a bundle $\bar{x} \in \mathbb{R}^n_+$ such that $\xi_h \sim_h \bar{x}$ for all h.

More abstract contexts. We shall study different models, of which the economy specified above is one. In fact they are all special cases of *coalitional games without transferable payoffs*, see Osborne and Rubinstein (1994). Such a game consists of: a set of players H; a set X (of consequences); a function V which assigns to every *coalition* $S \subseteq H$ a subset $V(S) \subseteq X$; and for each player $h \in H$ a preference relation $\succcurlyeq_h$ on X.

1 Selecting allocations based on no-envy and efficiency

In some situations no-envy and efficiency go a long way towards a solution. We present two examples from Moulin (2003).

Distributing scarce free goods. This is example 6.4 in Moulin (2003). We have to distribute 40 "tickets" to 100 agents: We order them by willingness to pay so that agent $p = 1, \ldots, 100$ values the object at p. We want to assign the tickets to the highest p's, but also wish to design a compensation scheme that eliminates envy. Thus the top 40 agents will receive the ticket but will pay a fee, while the others will receive a cash payment. First of all no envy implies that the fee should be an equal amount for all the agents receiving the object, say τ, and similarly those receiving a payment should receive the same payment ρ. Budget balance requires $40\tau = 60\rho$. For $p > 60$ not to envy someone with no ticket it must be $p - \tau \geq \rho$; the opposite envy is eliminated by requiring $\rho \geq p - \tau$ for all $p \leq 60$. Thus no-envy implies $\tau + \rho \geq p$ for all $p \leq 60$ and $\tau + \rho \leq p$ for all $p > 60$, or $60 \leq \tau + \rho \leq 61$. If the population is large (think of the 40 and 60 as 40% and 60% and argue as before at the margin) we get $60 = \tau + \rho$. This plus $40\tau = 60\rho$ results in $\tau = 36, \rho = 24$.

To view the solution from a different angle suppose the community decides to sell the tickets and then distribute the revenue. Then the quantity demanded at price p is just $100 - p$ (assuming the indifferent agent does not buy it); so the demand equal supply equation is $100 - p = 40$ or $p^{eq} = 60$; total revenue to be distributed is then $60 * 40 = 2,400$. To distribute it evenly you give 24 each: 24 in cash to those with no object, and a discount of 24 to those who pay $36 = 60 - 24$. This is the so called *competitive equilibrium with equal incomes.*

The not so satisfactory aspect of no envy is that cash payments and fees cannot depend on individual valuation. In the Shapely solution different individuals pay or get different sums. Fees and cash payments can be computed to be $\tau(p) = p - 23 - 35, \rho(p) = 40 * \ln(\frac{100}{100-p})$; ρ starts at zero for $p = 0$ and goes up to 36.65 for $p = 60$; those who get the object pay more the higher their valuation. Of course to implement this solution the exact p's must be known.

A Scheduling problem. This is Moulin (2003) Example 7.7. In a scheduling problem a large number of agents need a service but the server can process one agent at a time, so the problem is to determine the order in which the agents are scheduled, that is how long each has to wait to be served. It is convenient here to think of a continuum of agents in the interval $[0, 1]$. We index agents by their waiting cost c per unit of time, and also measure scheduling time s as a real variable in $[0, 1]$; so if agent c is scheduled at time s her waiting cost is $c \cdot s$. Since high-c types are likely willing to pay not to wait too long and vice versa low-c types are willing to accept a payment in exchange for a longer wait we assume cash transfers are possible, and let $\tau(c)$ be the transfer paid by agent c - which can be positive or negative. So an assignment to agent c consists of a pair $(s(c), \tau(c))$.

We can view this as an abstract economy (see the remark above) where objects are such pairs, preferences of agent c are represented by her cost $c \cdot s(c) + \tau(c)$ and feasibility is given by the budget-balance condition $\int_0^1 \tau(c)dc = 0$, which says that the total cash received must be equal to the total payed out.[1]

As a measure of social cost, treating all agents symmetrically, we take the (unweighted) sum of the costs $\int_0^1 (c \cdot s(c) + \tau(c))\, dc = \int_0^1 c \cdot s(c)dc$. Then Pareto efficiency obtains when this sum is minimized, and it can be shown that this obtains when $s(c) = 1 - c$, that is higher costs types served sooner.[2]

Now let us see what no-envy implies for the compensation scheme $\tau(c)$. No envy means that any c must prefer her pair $(s(c), \tau(c))$ to any other pair $(s(c'), \tau(c'))$; that is it must be $c \cdot s(c) + \tau(c) \leq c \cdot s(c') + \tau(c')$

$$c \cdot (1 - c) + \tau(c) \leq c \cdot (1 - c') + \tau(c')$$

so that

$$c^2 - \tau(c) \geq cc' - \tau(c') \quad \textit{for all } c'$$

which says that the function $cc' - \tau(c')$ has its maximum at c. This implies that its derivative there is zero, that is $\tau'(c) = c$, and again it can be shown that this together with $\int_0^1 \tau(c)dc = 0$ implies $\tau(c) = c^2/2 - 1/6$.[3]

Thus in this case no-envy gives a unique solution to the problem. Since $\tau(c) > 0$ for $c > 0.58$, the proposed solution is to have the top 42% refunding the bottom 58% for their longer waiting time. The higher the type the shorter the waiting time and the higher the

[1] If you don't like this formulation suppose there are a finite number of agents c_i and read the feasibility condition as $\sum_i \tau(c_i) = 0$.

[2] To find a minimizing *function* is in general not easy; for a heuristic argument for this case consider the discrete case of a finite number of agents. Then $s(c)$ must be decreasing, because if there are two agents $c < c'$ such that $s \equiv s(c) < s' \equiv s(c')$ then swapping places, that is changing s to a function $\tilde{s}$ such that $\tilde{s}(c) = s'$ and $\tilde{s}(c') = s$ reduces cost since

$$\begin{aligned} cs' + c's &= cs + c(s' - s) + c's' + c'(s - s') \\ &= cs + c's' - (c' - c)(s' - s) < cs + c's' \end{aligned}$$

which means $c\tilde{s}(c) + c'\tilde{s}(c') < cs(c) + c's(c')$. In the discrete case $s(c_i)$ decreasing - higher costs served first - means exactly $s(c_i) = 1 - c_i$, and the continuous limit gives $s(c) = 1 - c$.

[3] $\tau'(c) = c$ implies $\tau = c^2/2 + k$; inserting this into the integral condition gives $0 = \int_0^1 (c^2/2 + k)\, dc = 1/6 + k$whence the result in the text.

price to pay; and symmetrically the lower the type the longer the waiting time and the higher the cash refund. Everyone is indifferent between her lot and that of the others. One could guess that something like that *had* to emerge. The surprising result is that no-envy gives a precise functional form to implement the idea. To be sure, uniqueness and indifference obtain because of the large number of agents.

Let us see what happens if there are only 2 persons in the queue, with costs 1 and 2. For efficiency the latter has to pass first, that is $s(2) = 0, s(1) = 1$. The no-envy inequalities are

$$1 \cdot s(1) + \tau(1) \leq 1 \cdot s(2) + \tau(2) \quad 2 \cdot s(2) + \tau(2) \leq 2 \cdot s(1) + \tau(1)$$
$$1 + \tau(1) \leq \tau(2) \quad \tau(2) \leq 2 + \tau(1)$$
$$1 \leq \tau(2) - \tau(1) \leq 2$$

while the budget-balance condition is $\tau(1) + \tau(2) = 0$ that is $\tau(1) = -\tau(2)$. We obtain $1/2 \leq \tau(2) \leq 1$. For any scheme in the interior each agent strictly prefers her lot to that of the other one.

2 Social orderings and the efficiency/equity tradeoff

A *social ordering function* $\boldsymbol{R}$ associates to economy E a complete transitive relation $\boldsymbol{R}(E)$ over allocations.[4]

The following example is driven by the strongest possible concern for the poorest: it maximizes the minimum utility. It is based on the *maxmin* ordering of vectors in $\mathbb{R}^m$, denoted by $\succcurlyeq_{lexi}$, which is defined as follows: for $x, y \in \mathbb{R}^m$ it is $x \succcurlyeq_{lexi} y$ if the smallest coordinate in x is not smaller than the smallest in y, and if these two are equal the second smallest in x is not smaller than the second smallest in y, and so on. Observe that if $x \neq y$ then necessarily either $x \succ_{lexi} y$ or $y \succ_{lexi} x$. [5] Given $E = (R, \omega)$ and allocation ξ, for $h = 1, \ldots, H$ define the function $\mathcal{U}_h$ by

$$\mathcal{U}_h(\xi_h) = \lambda \iff \xi_h \sim_h \lambda\omega.$$

Agent h at ξ is worse off the lower is λ. Then let $\mathcal{U}(\xi) = (\mathcal{U}_1(\xi_1), \ldots, \mathcal{U}_H(\xi_H))$ denote the vector of these ω-normalized utilities.

[4] This section is adapted from chapter 2 of Fleurbaey and Maniquet (2011).

[5] Formally, re-write vector x with coordinates in increasing order, resulting in the vector $\tilde{x}$ with $\tilde{x}_1 \leq \tilde{x}_2 \cdots \leq \tilde{x}_m$; for example if $x = (2, 3, 1, 3)$ then $\tilde{x} = (1, 2, 3, 3)$; do the same for y. Then $x \succcurlyeq_{lexi} y$ if for a $k \leq m$ we have $\tilde{x}_h = \tilde{y}_h$ for all $h < k$ (if any) and $\tilde{x}_k \geq \tilde{y}$.

The function $\boldsymbol{R}^{\omega lex}$ at E is defined on allocations ξ, ζ by

$$\xi\, \boldsymbol{R}^{\omega lex}(E)\, \zeta \iff \mathcal{U}(\xi) \succcurlyeq_{lexi} \mathcal{U}(\zeta).$$

If for example $\mathcal{U}(\xi) = (0.02, 0.02, \ldots, 0.02)$ and $\mathcal{U}(\zeta) = (0.01, 0.9, \ldots, 0.9)$ then $\xi\, \boldsymbol{P}^{\omega lex}(E)\, \zeta$ because the poorest is better off in ξ.[6] It is a very extreme way of protecting the poor.

A very weak requirement on a social ordering is given by the following axiom, which says that $\boldsymbol{R}$ reflects individual preferences:

Axiom (Pareto dominance)**.** *For any E and allocations ξ, ζ, if $\xi_h \succ_h \zeta_h$ for all h then $\xi\, \boldsymbol{P}(E)\, \zeta$.*

The simplest redistribution axiom may be formulated as follows:[7]

Axiom (Redistribution)**.** *For any E and allocation ξ in E, suppose $\delta \in \mathbb{R}^n_{++}$ is such that $\xi_h - \delta \gg \xi_k + \delta$ for some h, k; then*

$$(\xi_1, \ldots, \xi_h - \delta, \ldots, \xi_k + \delta, \ldots, \xi_H)\, \boldsymbol{R}^{\omega lex}(E)\, \xi.$$

That is, it is okay to transfer goods from a richer guy to a poorer one as long as the donor remains richer after redistribution.

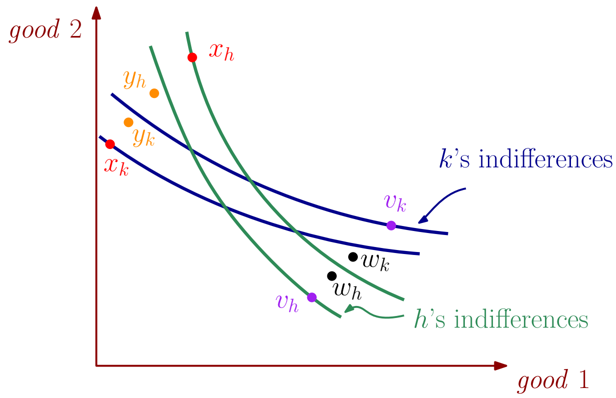
It may come as a surprise that the two above axioms are incompatible with each other:

Proposition. *There exists no social ordering $\boldsymbol{R}$ which satisfies both Pareto Dominance and Redistribution.*

Proof. We only have to exhibit an economy where the two axioms together imply a contradiction. In this economy there are only two agents, h and k, and two goods. In the figure the flatter [resp. steeper] indifference curves represent h's [resp k's] preferences, and four allocations x, y, v, w are depicted.

[6] $\boldsymbol{P}^{\omega lex}$ denotes the strict preference corresponding to $\boldsymbol{R}^{\omega lex}$.

[7] For vectors $x, y \in \mathbb{R}^n$ we write $x \gg y$ if $x_i > y_i$ for all $i = 1, \ldots, n$. For example $\mathbb{R}^n_{++}$ is the set of vectors $x \gg 0$.



Suppose $\boldsymbol{R}$ satisfies both axioms. Then $x\boldsymbol{P}w$ (Pareto), $w\boldsymbol{P}v$ (redistribution), $v\boldsymbol{P}y$ (Pareto) and $y\boldsymbol{P}x$ (redistribution). By transitivity $x\boldsymbol{P}x$, a contradiction. □

This impossibility result is cooked up through a wild combination of preferences. For note that at y Mrs. h - definitely better off than k - would pay to pass to v where she is definitely worse off than k. Similarly at w Mrs. k would pay to pass to x where the wealth rankings between her and h are swapped. The problem is due to indifference curves crossing, and can be overcome. One has to qualify redistribution somehow, and the following alternative gives a positive result.

Axiom (Redistribution-2)**.** *In the previous version add to the hypothesis the requirement that* $z \succcurlyeq_h \xi_h - \delta \Rightarrow z \succ_k \xi_k + \delta$ *for any* $z \in \mathbb{R}^n_+$.

Note that this is violated in the previous configuration: for example $z \equiv v_h \succcurlyeq_h y_h = x_h - \delta$ but $z \prec_k y_k = x_k + \delta$. The following positive result holds:

Proposition. *The maxmin order* $\boldsymbol{R}^{\omega lex}$ *satisfies both Pareto and Redistribution-2.*

Proof. Recall that $\boldsymbol{R}^{\omega lex}$ ranks lexicographically the vectors $\mathcal{U}$. The Pareto axiom is satisfied because if everyone is better off at ξ than at ζ then all λ's are higher, so the smallest is higher too (easy). As to Redistribution-2, let $\zeta = (\xi_1, \ldots, \xi_h - \delta, \ldots, \xi_k + \delta, \ldots, \xi_H)$ be the allocation resulting from the redistribution; we must show that $\zeta\ \boldsymbol{R}^{\omega lex}(E)\, \xi$. Now $\xi_h - \delta \sim_h \lambda_h \omega$, so by the additional requirement it follows that $\lambda_h \omega \succ_k \xi_k + \delta \sim_k \lambda_k \omega$; so the relative rankings of $\mathcal{U}_h$ and $\mathcal{U}_k$ in ζ are the same as in ξ, and the lower one has gone up. Therefore, since the other elements of $\mathcal{U}$ are unchanged, $\mathcal{U}(\zeta) \succcurlyeq_{lexi} \mathcal{U}(\xi)$. □

In general an ordering on allocations has multiple maximal elements, so it does not enable to select a specific allocation. A positive result in this direction is the following:

Proposition. *If the egalitarian-equivalent allocation ξ with $\mathcal{U}(\xi) = (\lambda, \ldots, \lambda)$ is Pareto efficient then it is the unique maximal of the maxmin order $\boldsymbol{R}^{\omega lex}$.*

Proof. Observe that by definition $\mathcal{U}(\xi) = (\lambda, \ldots, \lambda)$ means that each agent is indifferent to $\lambda\omega$; so the allocation is indeed egalitarian-equivalent. If it is Pareto efficient then raising any coordinate to some $\lambda' > \lambda$ implies lowering some other to a $\lambda'' < \lambda$; in other words the smallest element of $\mathcal{U}$ cannot be increased. □

3 Selecting allocations: more examples and difficulties

A sharing capacity simple problem. We start with a simple example, taken from Moulin (2003) and Wikipedia[8], where an allocation is selected directly on the basis of efficiency and fairness principles.

Briefly, two goods, quantities $x, y \leq 100$ (so $\omega = (100, 100)$); three agents A, B, C with utilities $u_A(x, y) = x$, $u_B(x, y) = y$ and $u_C(x, y) = \min\{x, y\}$. There are 40 agents of type A, 30 of type B and 30 of type C. Problem is to divide ω among the three agents.

Let us apply no-envy and Pareto efficiency. By no envy all agents of the same type gets the same bundle; by efficiency A types [resp. B types] get some x units of the first good [resp. some y unit of the second good], and C types get some amount z of each of the two goods. So an allocation is of the form $\xi = ((x, 0), (0, y), (z, z))$; we can ease notation and write it as $\xi = (x, y, z)$. By efficiency we must allocate all of ω so

$$40x + 30z = 100, \quad 30y + 30z = 100.$$

Note that this implies $y = (4/3)x$. By no-envy $x \geq z, y \geq z$ and this implies $z \leq 10/7 = 30/21$ (A vs A and B vs C, there are no other possible envies).

To proceed let us look for an egalitarian equivalent allocation ξ; this should make all agents indifferent between ξ_h and a fixed bundle. Note that if $x > z$ there is no bundle to which A and C types are indifferent, so it must be $x = z$. This implies that it must be $\xi = (30/21, 40/21, 30/21)$; and at this allocation each type is indifferent between ξ_h and the bundle $\bar{x} = (30/21, 40/21)$.

The conclusion is that in this example there is a unique Pareto efficient, envy-free and egalitarian equivalent allocation: $\xi = (30/21, 40/21, 30/21)$.

[8] https://en.wikipedia.org/wiki/Egalitarian_equivalence

Note that this ξ also maximizes the maxmin order $\boldsymbol{R}^{\omega lex}$ among efficient envy-free allocations. Indeed since efficiency and envy-freeness imply $y > x \geq z$ we have to maximize z; this gives $z = 30/21$, and ξ results as before.

As we shall see in the sequel the fair division problem is not always so nicely behaved. A sharp result in this negative direction is presented next section, but before leaving the example we consider another viewpoint.

TO ADD: CEEI, Moulin (2003) p.241

Sharing capacity, more problematic situation. Here we change the sharing capacity example a little, and... problems emerge. Again see Moulin (2003) and Wikipedia[9]. The only change is that the utility of the C type is now $u_C(x, y) = (x + y)/2$. She can use both links. The normalization makes the utility of each player equal to 1 at the bundle $\omega = (100, 100)$.

Efficiency and equal treatment of equals together mean that A types gets some amount x of good 1, B types some amount y good 2, and C types get the complement to 100 of each good, denote these by z_1, z_2; so we now have

$$40x + 30z_1 = 100, \quad 30y + 30z_2 = 100.$$

Let us look for an egalitarian equivalent allocation. Here allocations are of the form $\xi = ((x, 0), (0, y), (z_1, z_2))$. The reference bundle is a $\bar{x} = (\bar{x}_1, \bar{x}_2)$; now $\xi_A \sim_A \bar{x}$ means $x = \bar{x}_1$, and $\xi_B \sim_B \bar{x}$ means $y = \bar{x}_2$; thus $\bar{x} = (x, y)$. Then $\xi_C \sim_C \bar{x}$ means $(z_1 + z_2)/2 = (x + y)/2$. The displayed equalities above then imply $70x + 60y = 200$; the constraints are

$$70x + 60y = 200, \quad 30z_1 = 100 - 40x, \quad 30z_2 = 100 - 30y$$

which imply $x + y = z_1 + z_2$; and they do not determine an allocation uniquely. In the (x, y) plane the line $70x + 60y = 200$ is steeper than $x + y = const.$ so higher x means lower y and lower $z_1 + z_2$: there are conflicting interests among the agents.

One way to resolve them is to look at utility values and use the maxmin order. Observe that $\min\{x, y\} \leq (x+y)/2 \leq \max\{x, y\}$ with inequalities strict if $x \neq y$, so for $x < y$ type A is the one with smallest utility, and for $y < x$ it is type B; it then follows that the maxmin utility vector has $x = y$. This determines the allocation uniquely: $x = y = 200/130 \approx 1.54$, and $z_1 \approx 1.28, z_2 \approx 1.79$. This is the allocation determined by efficiency, egalitarian equivalence

[9] https://en.wikipedia.org/wiki/Egalitarian_equivalence

and maxmin. The (somewhat surprising) problem is that this allocation is not envy-free. Indeed B envies C - and pretty strongly, for B gets $(0, 1.54)$ while C gets $(1.28, 1.79)$.

Let us then see what envy-freeness implies (on top of efficiency and egalitarian equivalence). Envy-freeness here means $x \geq z_1, y \geq z_2$, which together with $x + y = z_1 + z_2$ implies $x = z_1, y = z_2$. We obtain $x = 100/70 \approx 1.43, y = 100/60 \approx 1.67$. These are also the agents' utilities, so type B utility is about 20% higher than A. Type C is in the middle with $(z_1 + z_2)/2 \approx 1.55$.

Let us finally look at Nash solution (introduced below). This maximizes the product of the utilities of all agents, that is

$$u_A^{40} \cdot u_B^{30} \cdot u_C^{30} = x^{40} \cdot y^{30} \cdot \frac{1}{2}\left(\frac{100 - 40x}{30} + \frac{100 - 30y}{30}\right)^{30}$$

where we have substituted the efficiency constraints $30z_1 = 100 - 40x$ and $30z_2 = 100 - 30y$; the non-negativity constraints are $0 \leq x \leq 100/40$ and $0 \leq y \leq 100/30$. To solve it assume the constraints are not binding and check ex-post that the unconstrained solution satisfies it. The FOC's give the following solution: $x = y = 2$ and $z_1 = 2/3, z_2 = 4/3$ so $(z_1 + z_2)/2 = 1$. This is remarkably different than each of the previous two. It is definitely not egalitarian equivalent, since $z_1 + z_2 < x + y$, and obviously not maxmin. It is envy-free, but with an inequality in utilities - types A and B get utility 100% higher than C - considerably stronger than in the envy-free egalitarian equivalent allocation.

A strong impossibility result. Pazner and Schmeidler (1974) exhibit an economy with production where no PO allocation is envy-free. This is a very strong negative result on the possibility of reconciling efficiency and equity. The result is:

Proposition. *There are economies where no PO allocation is envy-free.*

We go over the example to see how it goes.

This kind of difficulties with properties of allocations is the main motivation for the axiomatic approach, which looks for properties of allocation *rules*. To this we turn next.

4 The axiomatic approach

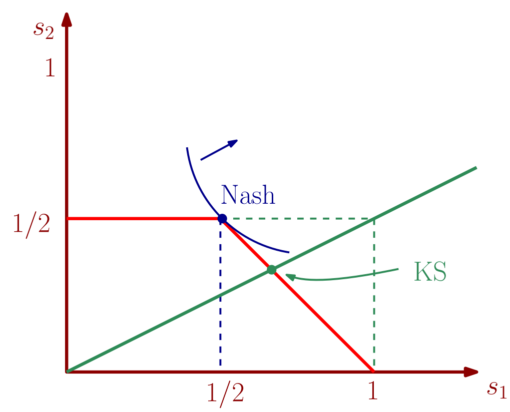
We present here the main axiomatic derivation of selection rules, which are due to Nash (1953) and Kalai and Smorodinsky (1975). In their setup there are two players who can select an element from a set A - which represents an *agreement* - or disagree, in which case the element D obtains. Both have vNM utilities $u_i, i = 1, 2$ over $A \cup \{D\}$, which as we know are invariant over affine transformations. The set of utility pairs which the two players can reach is the set S of pairs $(s_1, s_2) = (u_1(a), u_2(a)), a \in A$ and the point $d = (u_1(D), u_2(D))$. It always is assumed that there is $s \in S$ with $s_i > d_i, i = 1, 2$. The primitive of the problem is the pair (S, d). We are interested in functions f which for each (S, d) select a "fair" selection of an utility pair $f(S, d) \in S$ arising from an agreement. Such functions are called *bargaining solutions*, the pair (S, d) being viewed as a bargaining problem.

Nash Solution. blah blah, axiom etc

Kalai-Smorodinsky. blah blah, axioms and discussion

Nash vs Kalai-Smorodinsky

Example 1. This is from Osborne and Rubinstein (1990) section 2.5. The problem is described in the figure below, where the set S is delimited in red and $d = (0, 0)$. The Nash solution maximizes $s_1 \cdot s_2$ so gives the point $(1/2, 1/2)$. KS is at the intersection of the green line and the boundary of S, that is $(2/3, 1/3)$. Player 2 gets more because her highest utility is twice as high as that of player 1.



Example 2. This is a wage negotiation example, from Osborne and Rubinstein (1990) section 2.4. The agents are a firm and a union; the union represents L workers who can get a wage w_0 outside the firm. The firm can produce $f(\ell)$ units of output if it employs ℓ workers;

the output price is normalized at 1. The production function is concave, $f(0) = 0$ and we assume that at wage w_0 profit $f(\ell) - w_0\ell$ has a maximum $0 < \ell^* \leq L$.

An agreement is a wage-employment pair (w, ℓ). We assume that at (w, ℓ) the firm's payoff is profit, $s_1 = f(\ell) - w\ell$, and the union gets $s_2 = w\ell + w_0(L - \ell)$. Since participation is voluntary agreements must yield non-negative gains to both participants, so the agreement set S contains the pairs (s_1, s_2) such that $f(\ell) \geq w\ell$ $(s_1 \geq 0)$ and $w \geq w_0$ $(s_2 \geq Lw_0)$, with $0 \leq \ell \leq L$. The disagreement point is at $\ell = 0$, where the firm gets zero and the union gets Lw_0; so $d = (0, Lw_0) \in S$. Convexity of S follows by assuming that s_1, s_2 are vNM utilities.[10] Also, $(s_1, s_2) \in S$ imply $s_1 + s_2 = f(\ell) + w_0(L - \ell) \leq f(\ell^*) + w_0(L - \ell^*) \equiv \bar{s}$. So S is the compact convex set

$$S = \{(s_1, s_2) : s_1 + s_2 \leq \bar{s}, s_1 \geq 0, s_2 \geq Lw_0\}.$$

It is easy to see that Pareto optimality obtains if $s_1 + s_2$ is maximal; but $s_1 + s_2$ does not depend on w, and it is maximized by $\ell = \ell^*$; so this is the Nash employment level. It remains to determine the Nash wage w^*. A prominent candidate for a "fair" agreement in this example is given by the wage which equalizes the net gains $s_1, s_2 - Lw_0$, that is making $f(\ell^*) - w\ell^* = (w - w_0)\ell^*$. And this is indeed the Nash solution; for the Nash wage w^* maximizes the product of the net gains $(f(\ell^*) - w\ell^*) \cdot (w - w_0)\ell^*$ over $w_0 \leq w \leq f(\ell^*)/\ell^*$, and the FOC gives precisely $w^* = (w_0 + f(\ell^*)/\ell^*)/2$ - the midpoint between the wage making $s_1 = 0$ and that making $s_2 = Lw_0$.

It is easy to check that KS gives the same answer. Do it yourself.

Exercise. In this example also Shapley gives the same answer.

Example 3. This example is in Moulin (2003) and Wikipedia[11]. There are three options A, B, C available to Alice and George; their vNM utilities are in the following table:

	A	B	C
Alice	60	50	30
George	80	110	150

The disagreement point is assumed to be the worst outcome for each; so the gains from agreement are the following:

[10] This is a technical point you can ignore: a point in the segment between two pairs in S the expected utility of the two players from the corresponding lottery (by linearity of utilities in probabilities).

[11] https://en.wikipedia.org/wiki/Kalai%E2%80%93Smorodinsky_bargaining_solution

	A	B	C
Alice	30	20	0
George	0	30	70

Note that Alice and George have opposite preferences over outcomes: $A \succ_{Alice} B \succ_{Alice} C$ while $C \succ_{George} B \succ_{George} A$. Letting p, q, r be the probabilities of A, B, C, expected utility is $30p + 20q$ for Alice and $30q + 70r$ for George.

The Nah solution maximizes the product of the utilities, equivalently the sum of their logarithms, that is

$$\ln(30p + 20q) + \ln(30q + 70r) = \ln(30p + 20q) + \ln(30q + 70(1 - p - q))$$

over non-negative p, q with $p + q \leq 1$. Assume the constraint $p + q \leq 1$ is not binding, and observe that for any $p + q < 1$ we have $7 - 7p - 4q \geq 7 - 7(p + q) > 0$. Then

$$\begin{aligned} \frac{\partial}{\partial p} &= \frac{3}{3p + 2q} - \frac{7}{7 - 7p - 4q} \\ &= 3\left(\frac{1}{3p + 2q} - \frac{7/3}{7 - 7p - 4q}\right) \leq 0 \\ \frac{\partial}{\partial q} &= \frac{2}{3p + 2q} - \frac{4}{7 - 7p - 4q} \\ &= 2\left(\frac{1}{3p + 2q} - \frac{2}{7 - 7p - 4q}\right) \leq 0 \end{aligned}$$

If $\partial/\partial p = 0$ then $\partial/\partial q > 0$, contradiction; therefore $p = 0$, and then

$$\frac{\partial}{\partial q} = 2\left(\frac{1}{2q} - \frac{2}{7 - 4q}\right) = 0 \iff \frac{1}{2q} = \frac{2}{7 - 4q} \iff q = \frac{7}{8}$$

so this is the optimum (and indeed $p + q \leq 1$ is not binding). So the Nash solution is

$$p = 0 \quad q = \frac{7}{8} \quad r = \frac{1}{8}$$

so the middle "compromise" outcome is used most of the time, and a positive probability is attached to C which is the outcome that George prefers (he gets 70 at C against 30 which Alice gets at A).

The KS solution obtains by equalizing the relative gains:

$$\frac{30p + 20q}{30} = \frac{30q + 70r}{70}$$

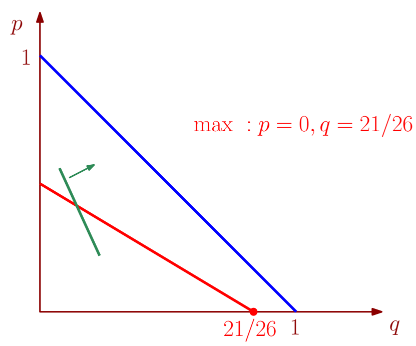
and maximizing this equal value. So the problem is

$$\begin{aligned} &\max \ \frac{30p+20q}{30} \\ s.t. \quad &\frac{30p+20q}{30} = \frac{30q+70r}{70} \\ &p+q+r=1 \end{aligned}$$

equivalently

$$\begin{aligned} &\max \ p + \tfrac{2}{3}q \\ s.t. \quad &p = r - \tfrac{5}{21}q \\ &p+q+r=1 \end{aligned} \qquad or \qquad \begin{aligned} &\max \ p + \tfrac{2}{3}q \\ s.t. \quad &p = \tfrac{1}{2} - \tfrac{26}{42}q \\ &p+q \le 1 \end{aligned}$$

whose solution is in the figure below:



Thus KS solution is

$$p = 0 \quad q = \frac{21}{26} \quad r = \frac{5}{26}$$

Note that the KS r value is $\approx 50\%$ higher than in the Nash solution.

Example 4. This is based on example 3.7 in Moulin (2003). There are two factors capital K and labor L which enter in the production of two firms 1 and 2, firm $i = 1, 2$ producing good i with price p_i. The "utilities" u_i of the firms are given by the values: $u_i(K_i, L_i) = p_i q_i(K_i, L_i)$ where quantities are given by the Leontief production functions

$$q_1(K_1, L_1) = \min\{2K_1, L_1\}, \quad q_2(K_2, L_2) = \min\{K_2, 2L_2\}.$$

So efficient production of firm 1 is with $(K_1, L_1) = (K_1, 2K_1)$ while that of firm 2 is with $(K_2, L_2) = (K_2, K_2/2)$; in other words firm 2's technology is more capital intensive than 1. The economy's total endowment is $\bar{K} = 12, \bar{L} = 24$, and is collective property. This is a standard economy $E = (R, \omega)$ as described at the beginning of this file. Let us assume for now that $p_1 = p_2 = 1$ so that $u_i = q_i$. The problem is to allocate resources to the two production processes.

Since capital is the scarce resource a simple intuition is that firm 1 should be somehow favored as it uses capital relatively less then firm 2. Let us see what our procedures say.

Observe that the only way to use all the resources is to allocate all of them to firm 1: whenever $K_2 > 0$ firm 2 uses $L_2 = K_2/2$ to produce $q_2 = K_2$; if all capital is used up then firm 1 needs $L_1 = 2K_1 = 2(\bar{K} - K_2)$ to produce $q_1 = 2K_1$; then $L_1 + L_2 = 2(\bar{K} - K_2) + K_2/2 = 2\bar{K} - (3/2)K_2 < 2\bar{K} = \bar{L}$. Allocating all resources to firm 1 gives $u_1 = 24, u_2 = 0$, and we can guess this is not going to be "fair" according to Nash or KS rules. Let us see what they prescribe.

The utility Pareto frontier is determined by allocations of the form $(K_1, L_1) = (12-x, 24-2x)$, $(K_2, L_2) = (x, 2x)$ with $0 \leq x \leq 12$ (tough firm 2 only uses $x/2$ units of L_2). Indeed we know that for any $K_2 > 0$ even if $L_1 = 2K_1$ firm 2 will still not use all labour assigned to it, so it would be Pareto inferior to give firm 1 $L_1 < K_1$. These allocations produce utilities $(u_1, u_2) = (24 - 2x, x)$; so the frontier is given by $u_1 = 24 - 2u_2$ for $0 \leq u_2 \leq 12$ that is $u_2 = 12 - u_1/2$ for $0 \leq u_1 \leq 24$ (draw it, it is a line with slope $-1/2$).

The Nash solution maximizes the product of the utilities $u_1 \cdot u_2$ under the constraint $u_2 = 12 - u_1/2$ (the disagreement point is $(0, 0)$; substituting and taking derivative gives

$$0 = \frac{d}{du_1} u_1 \cdot (12 - u_1/2) = 12 - u_1$$

so the Nash solution is $u_1 = 12, u_2 = 6$, obtained by allocating half of each resource to each firm, that is $(K_1, L_1) = (K_2, L_2) = (6, 12)$. Nine units of labour (37.5%) are unemployed.

For the KS solution observe that $\max u_1 = 24, \max u_2 = 12$ so we have to take the intersection between the lines $u_2 = u_1/2$ and $u_2 = 12 - u_1/2$. This gives the same solution as Nash.

This allocation is also egalitarian equivalent since both agents get the same bundle $\bar{x} = (6, 12) = \omega/2$ so $\mathcal{U}_h(\xi_h) = 1/2$; and it is also maximal with respect to the maxmin order $\boldsymbol{R}^{\omega lex}$ because with two agents maxmin reduces to equal (normalized) utilities.

Of course the 37.5% unused labour is something to worry about. This is unavoidable if the two firms cannot communicate. On the other hand suppose they can communicate and trade, starting from the equal individual endowments $(6, 12)$. What happens then? We know that firm 2 is using only 3 units of labour. Observe that if firm 2 transfers the bundle $(K, L) = (1, 2)$ to firm 1 then q_1 goes down by 1 and q_2 goes up by 2; since quantities are values there should then be room for mutually advantageous trade. More generally if firm 2 transfers $(\delta, 2\delta)$ to 1 then $\Delta q_2 = -\delta$ and $\Delta q_2 = 2\delta$; firm 2 ends up with $(6-\delta, 12-2\delta) = (6-\delta, 2(6-\delta))$ so for all $\delta < 6$ firm it has unused labour when firm 1 is producing along the efficient production line $L_1 = 2K_1$ (hence does not need more). This implies that a competitive price for labour in this economy is *zero.* Given this we can concentrate on capital: for each unit transferred to firm 1 (together with 2 units of labor transferred for free), firm 2 loses 1 and firm 1 gains 2; so the trade is Pareto improving for any price between 1 and 2; and the two firms trade until all capital and labour goes to firm 1. Firm 1 will produce $q_1 = 24$ (up from 12) and 2 will go out of production (down from 6). But total utility now includes the cash transfer. If the bargaining power is all on the side of firm 2 the price of capital is 2 so firm 1 ends up with total utility (from cash only) of 12 (up from 6) and firm 1 remains at 12 (24 from production minus 12 payed to 1); if firm 1 has all bargaining power the price is 1, and firm 2's final utility remains at 6 while firm 1's payoff goes up to $24 - 6 = 18$.

The conclusion is that in this example when trade is possible the efficient use of resources is reached (no unused labor or capital). The payoff distribution depends on the relative bargaining power of the two agents. The resulting allocation is known as *competitive equilibrium with equal incomes*, since it is a competitive equilibrium reached from an equal distribution of the total endowment.

Exercise. Analyze the economy with $p_1 \neq p_2$.

5 The cooperative games setup

From textbooks.

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