Uninformed Traders in European Stock Markets

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Abstract. A fully informed agent bets with an uninformed over the capital gains of an asset. A divide–and–choose idea is adapted to induce both trade and revelation of information, but in equilibrium the uninformed buys high and sells low if he is *downside* risk averse. The result may be seen as an informed–price–maker counterpart of some findings of Glosten–Milgrom [2] and Kyle [4] on uninformed agents trading in financial markets.

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Issue

In the models by Glosten–Milgrom [2] and Kyle [4] of a stock market with heterogeneous information, informed and uninformed agents trade with uninformed price-making dealers. The latter face an adverse selection problem arising from insiders, and to protect themselves they quote larger bid-ask spreads than they would otherwise (in [2]), or make the market less liquid (in [4]); thus the uninformed traders bear a negative externality exerted by the informed.

The above traders–dealers model does not apply to european stock markets, because there informed and uninformed agents trade among themselves, with no dealers' intermediation (cfr. [7]): in practice, the single trader posts his buying or selling order after actually observing all the oustanding ones on a computer screen. For the blue chips —the only ones the uninformed really consider trading— there is virtually no price difference between the lowest sellers and the highest buyers, in other words there is a single buy–or–sell price. The present paper is concerned with equilibria of these markets. The idea is that the uninformed enter them because although they realize that the price makers are effectively the informed, they deduce from the presence, at the going price, of informed buyers and informed sellers, that the price correctly reflects available information about the asset (which effectively eliminates information asymmetry). Incidentally, in equilibrium the uninformed will be indifferent between buying and selling at the going price. [1] The structural difference between the present setting and that of Glosten–Milgrom and Kyle is thus that the price maker here is the informed trader. As in their models we shall presume (unmodeled) external competition among informed price-makers, which in our setting forces the informed to make zero bid–ask–spread proposals, as in the story just told. The paper analyses existence and structure of revealing equilibria of a (signalling) game which models the interaction between an informed price–making agent and an uninformed one, which may be informally described as follows:

> Some exogenous non-negative benefits (the dividends net of opportunity costs of cash) accrue to two individuals I and II if they engage in a bet on an event (i.e. if they trade), and I knows the probability of the event occurring. Since this information is worthless if there is no trade, he would want to propose terms of trade which Mr. II can accept; the latter on the other hand must be sure he is treated fairly, otherwise he would refuse to trade; and given that he accepts I's proposal, the latter should have no incentive to be unfair.

[1] Dually, the uninformed do not enter markets with large bid–ask spreads and relatively few traders because there they are not sure which side to take and refrain from trading.

Given the external competitive pressure on the informed side, it seems reasonable to consider the adaptation of the simple cake–splitting rule of fair division theory under which 'One divides, the other chooses' (cfr. e.g. Brams–Taylor [1]), which in our setting becomes 'The informed proposes a trading price, the uninformed chooses whether to be a buyer or a seller'. We study conditions under which this kind of arrangement is an equilibrium, and identify the price distortions and utility losses that the lack of information may produce.

In the game the informed is assumed risk neutral, and the trade–off in the problem faced by the uninformed is between losing smaller amounts with higher probability and losing larger amounts with lower probability. This balance is governed by *downside* risk aversion (positive third derivative of vNM utility function). Asymmetric information and risk aversion notwithstanding, the downside–risk neutral uninformed (quadratic vNM utility) always gets a fair price on the bet he enters in the revealing equilibrium, corresponding to the expected value of the asset conditional on full information. On the other hand, the downside–risk averse uninformed is worse off: he always enters the 'cheaper' bet, but paying more than the above expected value.

This conclusion is our counterpart of that reached by Glosten–Milgrom [2] and Kyle [4] (in the relevant institutional setting): there the uninformed traders bear a negative externality exerted by the informed traders; here they enter unfair bets because of their own downside risk aversion.

We next describe the economic environment where the agents operate, and isolate the strategic problem involved. The resulting two–person game is then analysed, with particular attention to existence and structure of revealing equilibria, and an economic interpretation of the equilibrium is given. Finally an alternative, perhaps more natural game is examined where the informed agent faces a population of uninformed, and it is found that the same type of equilibria exist.

Underlying Economy

Uncertainty. There are two periods: in the first agents trade; in the second uncertainty, represented by a two–state space $S = \{s_1, s_2\}$, is resolved, s_1 occurring with probability θ. There are two (types of) agents, the informed who is risk neutral and knows θ, and the uninformed who is risk averse and does not know θ. θ is drawn in a pre–trade stage according to a given probability distribution, which is common knowledge.

Markets. To justify the two person trading game analyzed in the following section, we make the following two 'special' assumptions on the underlying market structure: (i) only the informed have access to risk–free interest on cash; and (ii) there are two negatively correlated assets, which yield a sure dividend (equal to the risk–free interest on investment) plus a capital risk of a smaller order of magnitude. In the intended interpretation the two assets are stock and bond, and the capital risk on the bond reflects the fact that the uninformed can never remove all uncertainty from their portfolios. The higher interest earned on cash by the informed can be interpreted as modelling better access to credit for professional investors than for the general public.

So let us assume that the informed has access to a risk–free interest r on cash; the uninformed gets zero interest on cash. In addition to cash there are two assets in zero net supply, each being the sum of a risk–free bond plus an Arrow–security; precisely, asset $i = 1, 2$ yields $(1 + r)k\,\mathbf{1}_S + \mathbf{1}_{\{s_i\}}$, where $\mathbf{1}_A$ is the indicator of $A \subseteq S$. It is assumed that $rk > 1$. Also, in any asset trade the quantity is fixed at 1 unit. To simplify notation we employ the following timing convention on payments: if an asset is bought at price $k + p$, then k is paid at the time of trade, while p is paid (if due) when uncertainty resolves.

Remark. We may notice the following implications. *(i)* The uninformed would never sell any asset, for any $k + p$ with $p \leqslant rk$: indeed, since he has no interest on money the sale would result in a net loss of $rk + (1 - p) > 0$ in one state and of $rk - p \geqslant 0$ in the other; analogously, he would be ready to buy any asset at any such price, for it would result in a non-negative net–gain vector equal to the above loss vector. *(ii)* The informed, on the other hand, is indifferent between k today and $(1+r)k$ tomorrow, and therefore his willingness to trade depends only on the terms concerning the risky part of the asset; pressed by competition, he will sign any contract ensuring him nonnegative expected net gains.

Equilibrium. As anticipated, the informed is the price maker. He will quote *selling* prices for the two assets, and the uninformed may buy one unit of *one* asset. [2] Equilibrium obtains if agents agreee to trade at those prices, or not to trade. Of course the trading equilibria Pareto–dominate the no–trade outcome, because in the former case the informed gets non-negative expected profits (otherwise he would refuse to trade), and the uninformed earns interest on cash, albeit at a cost (remember that for him the only way to get interest on cash is to buy risky assets from the informed).

Competition. With no competition on the informed side, the only trading equilibrium would have the informed offering, at all θ, both assets at the highest prices the uninformed can accept, and the uninformed agreeing to buy. But as already said competition forces the informed to make zero bid–ask–spread offers. To see what this means in the present setting, note that for any given θ, selling the Arrow–security $\mathbf{1}_{\{s_1\}}$ at price p is the same as buying $\mathbf{1}_{\{s_2\}}$ at price $1 - p$: both trades

[2] Since in equilibrium the uninformed will be indifferent between buying and selling, the unit–trade restriction is not relevant.

result in the random variable $(p-1, p)$ on S; [3] symmetrically, buying $\mathbf{1}_{\{s_1\}}$ at price p is the same as selling $\mathbf{1}_{\{s_2\}}$ at price $1-p$. Therefore, a buy–or–sell offer in the present context is a pair of selling prices for the two assets of the form $(k+p, k+1-p)$. These must then be the informed agent's feasible offers (besides the no–trade option), for any given θ.

Strategic Interaction. We claim that attention may be focused on the p–part of the $(k+p, k+1-p)$–offer. In fact, it has already been noticed that the informed will be content with making non-negative expected gains on the risky part of the asset, so for him the only strategic choice is p ($p(\theta)$ in fact, one for each θ). Incidentally, it is also clear that he will always quote a price pair with $p \in [0,1]$, for otherwise the uninformed would clearly choose to buy the asset with price lower than k, and the informed would make a sure loss. The uninformed facing a $(k+p, k+1-p)$–offer, on the other hand, will only have to choose which asset to buy, if any (but given $(k+p, k+1-p)$, trading for him is always better than refusing to trade, because of the interest $rk > 1$); and this also depends only on p.

Game Reduction. On the risk–free interest part of the assets, we know that the informed makes zero profit (he gets k to pay $(1+r)k$ the period after, which is exactly what he makes out of the initial k), while the uninformed makes a sure profit of rk (he pays k to get $(1+r)k$, instead of the same k he would end up with if he did not buy any asset). Therefore we can eliminate this aspect altogether from the the payoffs and concentrate the analysis on trade in the Arrow–securities, provided we posit that if the uninformed does not trade he gets a penalty of rk —which we shall do.

At this point we are left with the informed proposing a pair of selling prices $(p, 1-p)$ for the two Arrow–securities. But given the aforementioned byuing–selling symmetry in these two assets, such an offer is clearly equivalent to a buy–or–sell price p for the Arrow–security $\mathbf{1}_{\{s_1\}}$ (by which the informed quotes p and leaves the uninformed the choice whether to buy or sell the asset at that price). Thus we can also eliminate asset two from the picture, and analyse a game with just one asset, where the informed quotes buy–or–sell prices (or makes no offer, if no price gives him non-negative expected profits). To this we now turn.

The Two–person Game

The game studied in this section (a signalling game, cfr. Kreps–Sobel [3]) is the following. There are two players, I and II, and nature, and the game is played in three stages. In the first one nature selects a

[3] We are denoting a function $f\colon S \to \mathbb{R}$ by $(f(s_1), f(s_2))$; given θ, the resulting r.v. will also be called a lottery.

number θ according to a given common–knowledge probability distribution on $\Theta \subseteq [0,1]$ and communicates the selected θ to player I only (player II knowing this). In stage three, a state of the world from $S = \{s_1, s_2\}$ will be selected, with probability $(\theta, 1-\theta)$, and payoffs collected. In stage two player I moves first, and either refuses to trade, in which case the game enters stage three; or quotes a price $p \in [0,1]$; in the latter case player II either buys from player I, or sells to him, the lottery $\mathbf{1}_{\{s_1\}}$ on S at price p.

As to payoffs, let u_i be player i's weakly concave, smooth vNM utility, $i = I, II$. If in stage two player I refuses to trade, then independently of $\theta \in \Theta$ and $s \in S$, he gets $u_{_I}(0)$ and II gets $u_{_{II}}(-rk)$. Otherwise, if θ is selected and players trade at price p, the expected utility that player i gets from buying, resp. selling is

$$B^i(\theta,p) = \theta u_i(1-p) + (1-\theta)u_i(-p),$$
$$S^i(\theta,p) = \theta u_i(p-1) + (1-\theta)u_i(p).$$

<u>*Equilibrium.*</u> Motivated by the underlying economy described in the previous section, we are interested in Nash revealing equilibria where player I plays a pure strategy and never refuses to trade, i.e. plays a one–to–one function $\theta \mapsto \boldsymbol{p}(\theta)$ where $\boldsymbol{p}(\theta)$ is the proposed price (this requires that for any θ he should get more than $u_{_I}(0)$). Player II will play a behavioural strategy, $\boldsymbol{\beta}(p)$ denoting the probability that he buys. So: an ***equilibrium*** is a pair of functions $\theta \mapsto \boldsymbol{p}(\theta)$ from Θ to $[0,1]$, and $p \mapsto \boldsymbol{\beta}(p)$ from $[0,1]$ to $[0,1]$, such that

- for all p such that $\boldsymbol{p}^{-1}(p) \neq \emptyset$, $\boldsymbol{\beta}(p)$ solves

$$\max_\beta \ \mathbb{E}\big[\beta\, B^{II}(\theta,p) + (1-\beta)\, S^{II}(\theta,p) \mid \boldsymbol{p}^{-1}(p)\big]\,,$$

where $\mathbb{E}$ is with respect to the common prior on Θ; and

- for all $\theta \in \Theta$, $\boldsymbol{p}(\theta)$ solves

$$\max_p \ (1-\boldsymbol{\beta}(p))\, B^I(\theta,p) + \boldsymbol{\beta}(p)\, S^I(\theta,p), \tag{1}$$

the maximum being larger than $u_{_I}(0)$ for all θ. An equilibrium is revealing if $\boldsymbol{p}$ is one–to–one.

A simple candidate agreement in the game described above would be the following: the informed proposes to trade at $p = \theta$, the true probability, leaving the uninformed the choice as to which side of the market to take; and the latter tosses a fair coin to decide whether to buy or sell at that price. This is not viable except in special cases (propositions 1 and 3), but variations of it, with consequent price distortions and utility losses, will constitute the central equilibria of the paper (propositions 2 and 4).

<u>*Assumption and Notation.*</u> As anticipated, we will assume throughout that $u_{_I}(x) = x$. We then also let $u := u_{_{II}}$ in the sequel.

Quadratic Case. Here we assume that player II, the uninformed, is downside–risk neutral, i.e. has quadratic vNM u. This is the one case where the simple arrangement mentioned in the introduction is indeed an equilibrium. The economic content of the result is that although the uninformed is risk averse and the informed is the price maker, competition on the informed side is enough to guarantee that the uninformed enter fair bets.

***Proposition* 1.** Assume $\Theta = [0, 1]$, and that u is quadratic. Then the following "fair–bet agreement" is an equilibrium:

$$\forall\theta\ \boldsymbol{p}(\theta) = \theta, \quad \text{and} \quad \forall p\ \boldsymbol{\beta}(p) = 1/2.$$

Proof. Given $\boldsymbol{\beta} \equiv 1/2$, for any θ all p–offers give the risk–neutral player I zero expected profits. And given that I makes his proposal at the true θ, player II is indifferent between buying and selling. Indeed he gets $\theta\, u(1-\theta)+(1-\theta)\, u(-\theta)$ from buying and $\theta\, u(-(1-\theta))+(1-\theta)\, u(\theta)$ from selling; these expressions are equal to $u(0)$ for $\theta = 0, 1$, and for $\theta \in (0, 1)$ they are equal because being quadratic, u satisfies $[u(\theta) - u(-\theta)]/\theta = [u(1 - \theta) - u(-(1 - \theta))]/[1 - \theta]$ for such θ. □

Given player I's indifference, in this case there is a weak incentive for him to reveal the true θ (indeed, to be in the game altogether). In the case considered next revelation will be strictly beneficial for player I.

Downside–risk Averse Uninformed. We have seen that if θ is the true probability, a player with quadratic utility is indifferent between buying and selling at $p = \theta$. This is not true for more general utility functions, and what happens depends on the third derivative.

Downside–risk Aversion in the Model. Recall that at $p = \theta$, 'buying' is the lottery which takes value $1 - \theta$ with probability θ and $-\theta$ with probability $1-\theta$; 'selling' gives the opposite in each event. Of these two lotteries, as is tedious but elementary to check, neither first– or second–order stochastically dominates the other, but buying third–order dominates selling for $\theta < 1/2$, and the opposite occurs for $\theta > 1/2$. Therefore, by a result of Menezes, Geiss and Tressler [5], an individual whose utility function has positive third derivative would prefer buying for $\theta < 1/2$ and selling for $\theta > 1/2$; in other words such an individual would always prefer to put down (and stand to lose) the minimum between θ and $1-\theta$; and the opposite occurs if he has $u''' < 0$; the $u''' = 0$ quadratic utility is indifferent between buying and selling at $p = \theta$ for all θ. Empirical evidence is in favour of this $u''' > 0$ kind of behaviour, called 'downside–risk aversion' (cfr. [5]; axioms for a measure of this kind of risk aversion are e.g. in Modica–Scarsini [6]).

We note here the following immediate consequence of assuming that player II is downside–risk averse: define $p_{II}(\theta)$ as the price which makes

II indifferent between buying and selling given θ —that is, define $p_{II}(\theta)$ by

$$B^{II}(\theta, p_{II}(\theta)) - S^{II}(\theta, p_{II}(\theta)) = 0; \tag{2}$$

then $p_{II}(\theta) - \theta$ has the sign of $1/2 - \theta$.

The magnitude of $p_{II}(\theta) - \theta$ reflects the strength of player II 's downside risk effect; we may imagine it goes up for small values of θ (it is zero at $0, 1/2, 1$), then decreases to zero as $\theta \to 1/2$; its behavior on the right of $1/2$ is symmetric, because $p_{II}(\theta) = 1 - p_{II}(1-\theta)$ —from the equality $B^{II}(1-\theta, 1-p) = S^{II}(\theta, p)$, valid for all (θ, p).

Note also that $p_{II}(\cdot)$ is increasing. To see this differentiate (2) with respect to θ to get $p'_{II}(\theta) = \left(B^{II}_\theta - S^{II}_\theta\right) / \left(S^{II}_p - B^{II}_p\right)$, where subscripts denote partial derivatives ($B^{II}_\theta = \partial B^{II}/\partial\theta$, etc.), or more explicitly (recall $u = u_{II}$)

$$p'_{II}(\theta) = \frac{u(1-p) - u(p-1) + u(p) - u(-p)}{\theta u'(p-1) + (1-\theta)u'(p) + \theta u'(1-p) + (1-\theta)u'(-p)}\,, \tag{3}$$

where all derivatives are taken at $(\theta, p_i(\theta))$; it is immediate to check that numerator and denominator are positive.

Equilibrium. Coming back to our game, note that since $u_I(x) = x$, player I's problem (1) reads

$$\max_p\ 2(p-\theta)(\boldsymbol{\beta}(p) - 1/2)\,. \tag{4}$$

We shall look for an equilibrium with a $\boldsymbol{\beta}(\cdot)$ such that for each θ player I's problem is solved by $p_{II}(\theta)$ and the solution ensures him positive expected profits for almost all θ.

The last condition is relevant, because an equilibrium with player I making zero profits is easily found:

$$\boldsymbol{p} = p_{II}\,, \quad \boldsymbol{\beta} \equiv 1/2. \tag{5}$$

This is a Nash equilibrium, because given I's strategy all $\boldsymbol{\beta}$–functions give II the same payoff, and given $\boldsymbol{\beta} \equiv 1/2$ all $\boldsymbol{p}$–functions give I utility zero; and it is revealing because we have seen that p_{II} is one–to–one. Of course this equilibrium is unattractive for player I, and in fact for him it is payoff–equivalent to the following one, where he does not bother to fine–tune his proposal function $\boldsymbol{p}$:

$$\boldsymbol{p} \equiv \overline{p}\,, \quad \boldsymbol{\beta} \equiv 1/2,$$

where $\overline{p}$ makes II a priori indifferent between buying and selling. Are there better equilibria than the one in (5)? Yes: in fact, with player II downside–risk averse, any equilibrium with $\boldsymbol{p} = p_{II}$ and $\boldsymbol{\beta} \not\equiv 1/2$ Pareto dominates the one in (5). [4]

[4] *Proof*: Player II's position is unchanged; and I has strictly positive utility at any θ such that $\boldsymbol{\beta}(p_{II}(\theta)) \neq 1/2$, because: he could surely propose θ and get zero utility, so in equilibrium his utility must be non–negative; but then if his optimum is at $p_{II}(\theta) \neq \theta$ and $\boldsymbol{\beta}(p_{II}(\theta)) \neq 1/2$, his utility $(p_{II}(\theta) - \theta)(\boldsymbol{\beta}(p_{II}(\theta)) - 1/2) \neq 0$ must be positive.

We now show that there is an equilibrium of the kind just described, provided the support of θ is strictly contained in $[0,1]$. In this equilibrium, player I has positive utility for all $\theta \neq 1/2$; and player II, although he always 'buys cheap', in the sense that $\boldsymbol{\beta}(p) > 1/2$ iff $p < 1/2$, he still buys high and sells low at all $(\theta, \boldsymbol{p}(\theta))$ with $\theta \neq 1/2$, in the sense that $\boldsymbol{\beta}(\boldsymbol{p}(\theta)) > 1/2$ iff $\boldsymbol{p}(\theta) > \theta$.

***Proposition* 2.** If player II is downside–risk averse ($u''' > 0$) and $\Theta = [\underline{\theta}, \overline{\theta}] \subseteq (0,1)$, then there is an equilibrium with $\boldsymbol{p} = p_{II}$ and $\boldsymbol{\beta}(p)$ decreasing, with $\boldsymbol{\beta}(p) \neq 1/2$ for all $p \neq 1/2$.

Proof. It has to be shown that a function $\boldsymbol{\beta}$ as in the statement can be constructed so that at each $\theta \in \Theta$ player I's problem is solved by $p_{II}(\theta)$. We do it in two steps.

First: find a (smooth) $\boldsymbol{\beta}$–function with the property that for each $\theta \neq 1/2$ the first and second order conditions for the maximum of player I's problem (4) are satisfied at $p_{II}(\theta)$, i.e. such that

$$\boldsymbol{\beta}(p_{II}(\theta)) - 1/2 + (p_{II}(\theta) - \theta)\boldsymbol{\beta}'(p_{II}(\theta)) = 0 \tag{6}$$

$$2\boldsymbol{\beta}'(p_{II}(\theta)) + (p_{II}(\theta) - \theta)\,\boldsymbol{\beta}''(p_{II}(\theta)) < 0, \quad \forall\, \theta \in \Theta \setminus \{1/2\}.\ ^{5} \tag{7}$$

To find such a function: letting

$$z(\theta) = \boldsymbol{\beta} \circ p_{II}(\theta) - \tfrac{1}{2} \quad \text{and} \quad \pi(\theta) = \frac{p'_{II}(\theta)}{p_{II}(\theta) - \theta}, \quad \theta \in (0,1) \setminus \{1/2\}$$

and multiplying equation (6) by $\pi(\theta)$, we get the following first-order, linear homogeneous differential equation in θ, well defined for $\theta \in (0,1) \setminus \{1/2\}$:

$$z' + \pi(\theta)\, z = 0, \quad \theta \in (0,1) \setminus \{1/2\}. \tag{8}$$

On each of the subintervals $(0,1/2)$ and $(1/2,1)$, the solution to this equation satisfying $z(\theta_0) = z_0$, with θ_0 an interior point, is

$$z(\theta) = z_0\, \mathbb{E}\big(-\textstyle\int_{\theta_0}^{\theta} \pi(t)dt\big). \tag{9}$$

Let z_1 be the solution to this on $(0,1/2)$ such that $z_1(\underline{\theta}) = 1/2$, and z_2 the solution on $(1/2,1)$ such that $z_2(\overline{\theta}) = -1/2$; and define the function z on Θ by: z_1 on $[\underline{\theta}, 1/2)$, z_2 on $(1/2, \overline{\theta}]$, and $z(1/2) = 0$. Given that $\pi(\theta)$ has the sign of $1/2 - \theta$, it is readily verified that this function z has itself the sign of $1/2 - \theta$ and is decreasing (hence in particular with range in $[-1/2, 1/2]$); incidentally, we shall verify that it is also continuous on Θ.

Then the function $p \mapsto \big(z + \frac{1}{2}\big) \circ p_{II}^{-1}(p)$, decreasing with range $[0,1]$, satisfies (6) on $\Theta \setminus \{1/2\}$ (elementary check). It is still defined only on

[5] That these conditions are sufficient for a *global* maximum will follow from uniqueness of the p satisfying the first order condition at θ.

$p_{II}(\Theta)$, but we extend it to all of $[0,1]$ by setting

$$\boldsymbol{\beta}(p) = \begin{cases} \left(z+\frac{1}{2}\right)\circ p_{II}^{-1}(p) & p \in p_{II}(\Theta) \\ 1/2 & p \in [0,1] \setminus p_{II}(\Theta). \end{cases} \tag{10}$$

This function satisfies (6) on $\Theta \setminus \{1/2\}$ by construction. And for (7): the condition is $(p_{II} - \theta)\beta'' < -2\beta'$ (at $p_{II}(\theta)$); but (6) implies $\beta' \cdot (2p'_{II} - 1) + p'_{II} \cdot (p_{II} - \theta)\,\beta'' = 0$, so that $\beta'' \cdot (p_{II} - \theta) = -\beta' \cdot (2 - 1/p'_{II})$; and the latter is $< -2\beta'$ iff $p'_{II} > 0$, which we know to be true.

Second step: the function defined by (10) satisfies our requirement. Indeed, observe that player I's utility at $(\theta, p_{II}(\theta))$ is positive for $\Theta \setminus \{1/2\}$, because $\boldsymbol{\beta} - 1/2$ has the same sign as $1/2 - \theta$ whence also as $p_{II}(\theta) - \theta$. But then for such θ's $p_{II}(\theta)$ solves I's problem, for by choosing $p = 1/2$ or $p \notin p_{II}(\Theta)$ he gets zero. And for $\theta = 1/2$: he cannot get more than zero, which he obtains by setting $p = 1/2$.

The assertion about the players' payoffs is true because $\boldsymbol{\beta}(\boldsymbol{p}(\theta)) - 1/2$ and $\boldsymbol{p}(\theta) - \theta$ both have the sign of $1/2 - \theta$. □

Economic Interpretation. Remember that 'buying' means buying asset 1 in the two–asset underlying economy we started with, while 'selling' means buying asset 2. Thus the interpretation of the equilibrium just found is that the uninformed buys with higher probability the asset with less downside risk —which is asset 1 iff $\theta < 1/2$. The 'real' economy this model intended to capture, we said, is one with an equity and a bond market. In the interpretation of this economy, a sluggish or falling stock market index is associated with *high* downside risk on the equities, while a rising trend goes with low downside risk on equity. For in the latter case, stock prices reflect the optimistic expectations on future appreciation, and little unexpected further rise is left out (which in our terms, if equity is asset 1, means θ, hence also $p/(k+p)$, small); in the former case on the contrary, a more substantial part of the price paid goes to cover hopes of future gains (θ and $p/(k+p)$ higher). The empirical prediction of the model is therefore that the uninformed tend to enter the stock market in bullish periods, while resort to the bond market in more quite times.

Continuity of $\boldsymbol{\beta}$ at $p = 1/2$. This follows from the fact that any solution to (8) on either half of the unit interval tends to zero as $\theta \to 1/2$. We verify this on $(0, 1/2)$ (other half same story): it is clear from (9) that our claim is equivalent to $\lim_{\theta \uparrow 1/2} \int_{\theta_0}^{\theta} \pi(t)\,dt = \infty$. Now (cfr. (3)) $p'_{II}(1/2) = [u(1/2) - u(-1/2)]/[u'(1/2) + u'(-1/2)] > 0$, so for some $\theta_1 > \theta_0$, $\delta > 0$ it will be $p'_{II}(\theta) > \delta$ on $(\theta_1, 1/2)$; also, $p_{II}(\theta) - \theta < 1/2 - \theta$; therefore

$$\int_{\theta_0}^{\theta} \pi(t)dt > \int_{\theta_1}^{\theta} \pi(t)dt \equiv \int_{\theta_1}^{\theta} \frac{p'_{II}(t)}{p_{II}(t) - t} dt > \delta \int_{\theta_1}^{\theta} \frac{dt}{\frac{1}{2} - t} \to \infty \text{ as } \theta \uparrow \tfrac{1}{2}.$$

<u>*Non–existence with* $\Theta = [0,1]$.</u> The above equilibrium does not seem to survive with $\Theta = (0,1)$; the source of the problem is that any solution to (8) is unbounded. To see this look again at (9) on $(0,1/2)$ (the situation is analogous on the other half): we will prove that $\int_{0+}^{\theta_0} \pi(t)dt = \infty$, which implies the claim. To this end observe that from (3) we have $\lim_{\theta\to 0} p'_{II}(\theta) = [u(1) - u(-1)]/2u'(0) < \infty$; hence for θ sufficiently small: p'_{II} is bounded, and there exists a $k > 1$ such that $p_{II}(\theta) < k\theta$, i.e. $p_{II}(\theta) - \theta < \theta(k-1)$; so for θ' sufficiently small we have $\int_\theta^{\theta'} (p_{II}(t) - t)^{-1}dt > (k-1)^{-1}\int_\theta^{\theta'} t^{-1}dt$, which goes to infinity as $\theta \downarrow 0$. This and boundedness of p'_{II} easily imply $\int_{0+}^{\theta_0} \pi(t)dt = \infty$. Unboundedness of all solutions to (8) clashes of course with the requirement that $\boldsymbol{\beta}$ be in $[0,1]$. Concentrating once again on the left half of the unit interval, the only remedy to the fact that $\lim_{p\to 0} \boldsymbol{\beta}(p) = \infty$ seems to consist in fixing an arbitrarily small p_0, taking the solution to (8) which generates the $\boldsymbol{\beta}$–function with $\boldsymbol{\beta}(p_0) = 1$, and trying to define $\boldsymbol{\beta}$ for $p < p_0$ appropriately. But there is no way to do this, the argument being that, letting θ_0 such that $p_0 = p_{II}(\theta_0)$, for any $\theta < \theta_0$ player I would propose at least p_0 for any specification of $\boldsymbol{\beta}$ for $p < p_0$, and this would destroy informativeness of the proposed p for $p \geqslant p_0$, i.e. upset equilibrium for $\theta > \theta_0$.

A Different Formulation: Uncertainty Over u

In the setup used so far player II's utility $u \equiv u_{II}$ was fixed. We now assume that u is selected according to some probability distribution ν, and that this is all player I knows about II. Our Bayesian game (e.g. Osborne–Rubinstein [8], section 2.6) is in this case the tuple

$$\big\langle N,\, \Omega,\, (A_i)_{i\in N},\, (T_i)_{i\in N},\, (\tau_i)_{i\in N},\, (\pi_i)_{i\in N},\, (\succcurlyeq_i)_{i\in N}\big\rangle,$$

where $N = \{I, II\}$, the type sets are $T_I = (0,1)$ (the domain of θ) and $T_{II} = \mathcal{U}$ (a set of possible u's with a suitable measurable structure); $\Omega = T_I \times T_{II}$; the signaling functions τ_i are the relative projections (uninformative about the other's type); $A_I = A_{II}$ are as in the previous section (p-BOS proposals to make or accept/reject); the prior on Ω will be a common product $\pi_i = \mu \times \nu$, where μ and ν are measures on T_I and T_{II} respectively; and finally, $\succcurlyeq_i$ is i's vNM preference over $A_I \times A_{II} \times \Omega$; more precisely: each element of this set, call it z for a moment, gives rise (independently of the T_{II} coordinate actually) to a monetary lottery, and player i's payoff at z is the expected utility of this lottery according to his vNM u_i on money.

<u>*Equilibrium.*</u> Again we are interested in revealing equilibria. This is then a one–to–one function $\theta \mapsto \boldsymbol{p}(\theta)$ such that

- for any p, each type u solves, with $\theta = \boldsymbol{p}^{-1}(p)$,

$$\max_{\beta} \left[\beta B^{II}(\theta, p) + (1-\beta) S^{II}(\theta, p)\right], \tag{11}$$

where for definiteness we may assume that the ones indifferent between buying and selling toss a fair coin to decide; and

- for all θ, $\boldsymbol{p}(\theta)$ solves the problem in equation (4) page 7, that is

$$\max_p \ 2(p-\theta)(\boldsymbol{\beta}(p) - 1/2)\,,$$

where now $\boldsymbol{\beta}(p)$ is defined as the fraction of types of player II for whom the solution to the problem in (11) above is to buy.

At any given (θ, p) pair (where θ is the true value and p is the proposed price), the fraction of players II who buy, which will be denoted by $\hat{\beta}(\theta, p)$, depends on ν. We start as in the previous section with the simple case where the 'fair–bet agreement' is an equilibrium.

1. Another 'Quadratic' Case. Here we assume that for any θ it is $\hat{\beta}(\theta, \theta) = 1/2$; that is, half of the type-II agents are downside-risk averse, half are downside-risk prone, *or* they all have quadratic utility. In this case again there is an equilibrium with $\boldsymbol{p}(\theta) = \theta$, analogous to that of proposition 1:

Proposition 3. Assume that for any θ it is $\hat{\beta}(\theta, \theta) = 1/2$. Then the function $\boldsymbol{p}(\theta) = \theta$ is an equilibrium.

Proof. Given the assumed function $\boldsymbol{p}$, whatever p is proposed is believed to be equal to the true θ, and this and our assumtpion about $\hat{\beta}$ then imply that the $\boldsymbol{\beta}$ in the definition of equilibrium is just $\boldsymbol{\beta}(p) \equiv 1/2$. On the other hand since player I is risk neutral, given this $\boldsymbol{\beta}$ he has no better proposal at θ than $p = \theta$. □

The Distribution $\boldsymbol{\nu}$. To go beyond the previous case one needs a plausible form of the function $p \mapsto \hat{\beta}(\theta, p)$ for each fixed θ. This depends on ν, and to be in accordance with the empirical evidence mentioned before we *assume in the sequel* that the support of ν is made of u's with positive third derivative.

This implies that for $\theta < 1/2$ it is $\hat{\beta}(\theta, p) = 1$ for all $p \leqslant \theta$, and for $\theta > 1/2$ $\hat{\beta}(\theta, p) = 0$ for all $p \geqslant \theta$. Let us also *assume* that there are types with u''' arbitrarily small. Then, considering first the case of $\theta < 1/2$, at any $p > \theta$ there are some types of player II who would sell (and there will be more as p goes up), i.e. $\hat{\beta}(\theta, \cdot)$ will start decreasing at θ; also, at $p = 1/2$ selling first–order dominates buying, so $\hat{\beta}(\theta, \cdot)$ will reach zero somewhere between θ and $1/2$ (and remain zero up to $p = 1$). Analogously, for $\theta > 1/2$ $\hat{\beta}(\theta, \cdot)$ must be 1 up to some point between $1/2$ and θ, then decrease to reach zero at $p = \theta$.

For $\theta \to 1/2$, both from right and left the families of functions $\hat{\beta}(\theta, \cdot)$ should approach a jump function equal to 1 for all $p < 1/2$ and zero for $p > 1/2$, which is the shape of $\hat{\beta}(1/2, \cdot)$ implied by *any* increasing u; indeed it is immediate to check that for $\theta = 1/2$, for any increasing u buying is better than selling if $p < 1/2$ and worse if $p > 1/2$. For the sake of definiteness we may set $\hat{\beta}(1/2, 1/2) = 1/2$.

Downside–risk Averse Player** II.** If player *II* types assume player *I* plays the invertible strategy $\boldsymbol{p}$, the latter's problem at θ becomes $\max_p[\hat{\beta}(\boldsymbol{p}^{-1}(p), p) - 1/2][p - \theta]$. The strategy $\boldsymbol{p}$ is an equilibrium if this problem is solved by $\boldsymbol{p}(\theta)$ for each θ.

An equilibrium where player I's utility is identically zero is easily found, but it is not satisfactory for the same reasons as in the previous section. [6]

We show next that an equilibrium where player I exploits player IIs' downside risk aversion and gets positive payoff at each θ (except perhaps in a neighborhood of $\theta = 1/2$) also exists.

Proposition 4. Assume $\Theta = [\underline{\theta}, \overline{\theta}] \subseteq (0, 1)$. Then exists an equilibrium with $\hat{\beta} \neq 1/2$ (and player I's utility strictly positive), except possibly in a neighborhood of $\theta = 1/2$. A sufficient condition for $\hat{\beta} \neq 1/2$ for all $\theta \neq 1/2$ is the following:

$$\forall(\tilde{\theta}, \tilde{p}) \text{ s.t. } \hat{\beta}(\tilde{\theta}, \tilde{p}) = \tfrac{1}{2} \;\&\; \tilde{\theta} \neq \tfrac{1}{2} \qquad \lim_{(\theta,p)\to(\tilde{\theta},\tilde{p})} \frac{\hat{\beta}_\theta(\theta, p)}{\hat{\beta}_p(\theta, p)} = 0. \tag{12}$$

<u>*Remarks.*</u> (i) Analogously to proposition 2, in this equilibrium the majority of uninformed buy high —for $\theta < 1/2$— and sell low —for $\theta > 1/2$.

(ii) Assumption (12) says that when $(\tilde{\theta}, \tilde{p})$ splits evenly the player *II* population between buyers and sellers, improving things (θ going up) has an effect of smaller order of magnitude than worsening (p increasing).

Proof. We find it easier to work with the inverse function $\boldsymbol{\theta}(p)$; then player I's problem at θ is $\max_p[\hat{\beta}(\boldsymbol{\theta}(p), p) - 1/2][p - \theta]$, and $\boldsymbol{\theta}$ is the inverse of the equilibrium price function if the solution p is solution for $\theta = \boldsymbol{\theta}(p)$; thus we require that the first and second order conditions at $\boldsymbol{\theta}(p)$ are satisfied by p. FOC is $\hat{\beta}(\boldsymbol{\theta}(p), p) - \frac{1}{2} = (\theta - p)\left[\hat{\beta}_\theta(\boldsymbol{\theta}(p), p) \cdot \boldsymbol{\theta}'(p) + \hat{\beta}_p(\boldsymbol{\theta}(p), p)\right]$, and requiring that this be satisfied with $\theta = \boldsymbol{\theta}(p)$ means requiring that the function $\boldsymbol{\theta}$ solve the following differential equation:

$$\hat{\beta}(\boldsymbol{\theta}(p), p) - \tfrac{1}{2} = (\boldsymbol{\theta}(p) - p)\left[\hat{\beta}_\theta(\boldsymbol{\theta}(p), p) \cdot \boldsymbol{\theta}'(p) + \hat{\beta}_p(\boldsymbol{\theta}(p), p)\right]. \tag{13}$$

[6]For each θ let $\boldsymbol{p}(\theta)$ be such that $\hat{\beta}(\theta, \boldsymbol{p}(\theta)) = 1/2$. Given this (increasing) $\boldsymbol{p}$, for each fixed θ player I gets zero payoff for all p's because by construction $\hat{\beta}(\boldsymbol{p}^{-1}(p), p) = 1/2$, and because of this he can do no better than proposing the given $\boldsymbol{p}(\theta)$.

We shall as always work in the half-interval $\theta < 1/2$; on the other half the construction is entirely analogous. Let p_0 be such that $1/2 < \hat{\beta}(\underline{\theta}, p_0) < 1$, and consider the solution to (13) such that $\boldsymbol{\theta}(p_0) = \underline{\theta}$. Then LHS(13) > 0 at p_0 and $\boldsymbol{\theta}(p_0) < p_0$, so $[\hat{\beta}_\theta(\boldsymbol{\theta}(p), p) \cdot \boldsymbol{\theta}'(p) + \hat{\beta}_p(\boldsymbol{\theta}(p), p)] \equiv d\hat{\beta}(\boldsymbol{\theta}(p), p)/dp < 0$ at p_0. We want to show —Step 1— that as $p \to 1/2$ the function $\hat{\beta}(\boldsymbol{\theta}(p), p)$ decreases to $1/2$, which it may reach at some $p < 1/2$ if (12) is not satisfied. This will imply —Step 2— that this solution $\boldsymbol{\theta}(\cdot)$ also satisfies the second order conditions for I's problem. From this the conclusion will follow that its inverse is the equilibrium we are after.

Step 1. By construction $d\hat{\beta}/dp < 0$ in a (right) neighborhood of p_0; and $\hat{\beta}$ decreasing from $\hat{\beta}(\boldsymbol{\theta}(p_0), p_0) < 1$ implies that $\boldsymbol{\theta}(p) - p < 0$ and —via (13)— $\hat{\beta} > 1/2$.

If $\hat{\beta}$ keeps decreasing for all $p < 1/2$, then as $p \to 1/2$ $\boldsymbol{\theta}(p) \to p$ (otherwise $\hat{\beta} \to 1$, which we are assuming not being the case); hence in this case $\hat{\beta} \to 1/2$ as $p \to 1/2$, and $\boldsymbol{\theta}(p) \to 1/2$.

Otherwise, there is $\tilde{p} < 1/2$ such that $\hat{\beta}(\boldsymbol{\theta}(\tilde{p}), \tilde{p}) = 1/2$; in this case $\hat{\beta}$ must remain constant as $p \to 1/2$. Proof of this: Suppose $\hat{\beta}$ crossed the $1/2$ value downwards at $\tilde{p}$; then in a right neighborhood of $\tilde{p}$ there would be some p with: $\hat{\beta} < 1/2$, $\boldsymbol{\theta}(p) < p$, and $d\hat{\beta}/dp < 0$; but this contradicts (13). Suppose on the other hand that $\hat{\beta}$ came back above $1/2$; then on the right of $\tilde{p}$ there would be a p with $\hat{\beta} > 1/2$, $\boldsymbol{\theta}(p) < p$, and $d\hat{\beta}/dp > 0$, again contradicting (13).

To finish step 1 we must show that if (12) holds, then $\hat{\beta} > 1/2$ for all $p < 1/2$; so assume (12). If $\hat{\beta}$ reached $1/2$ at some $\tilde{p} < 1/2$ (to remain constant after $\tilde{p}$), then $\lim_{p \to \tilde{p}} \hat{\beta} = 1/2$ and $\lim_{p \to \tilde{p}} d\hat{\beta}/dp = 0$; but this is impossible under (12), because: in any compact $[p_0, p_1]$, $\boldsymbol{\theta}'$ is bounded, so (12) implies that for $\hat{\beta}$ close to $1/2$, $d\hat{\beta}/dp$ is bounded away from 0.

Step 2. First notice that $\hat{\beta}$ non–decreasing implies that $\boldsymbol{\theta}' > 0$. This, incidentally, implies that the equilibrium whose existence we are about to show is revealing. Now: the second order condition for player I's problem is that for all p, at $(\boldsymbol{\theta}(p), p)$

$$2(\hat{\beta}_\theta \cdot \boldsymbol{\theta}' + \hat{\beta}_p) + (p - \boldsymbol{\theta})\big[\boldsymbol{\theta}'(\hat{\beta}_{\theta\theta}\boldsymbol{\theta}' + \hat{\beta}_{\theta p}) + \hat{\beta}_\theta \cdot \boldsymbol{\theta}'' + \hat{\beta}_{\theta p} \cdot \boldsymbol{\theta}' + \hat{\beta}_{pp}\big] < 0.$$

But by differentiating (13) with respect to p we see that the left hand side of the last displayed relation is just equal to $\boldsymbol{\theta}' \cdot (\hat{\beta}_\theta \cdot \boldsymbol{\theta}' + \hat{\beta}_p)$; and we know that $\boldsymbol{\theta}' > 0$, so the second order condition holds since we also know that $\hat{\beta}(\boldsymbol{\theta}(p), p)$ is decreasing. To be precise, it is decreasing and then perhaps constant in a right neighborood of $\theta = 1/2$, in which case the second order inequality becomes an equality on that neighborhood;

but that part of strategy which makes $\hat{\beta}$ constant at 1/2 we know is (weakly) optimal from the previous proposition. $\square$

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