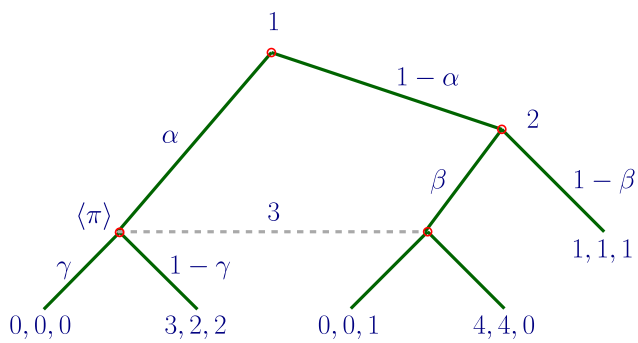
Example of sequential equilibrium

Find the sequential equilibria of the three-player game represented below. Actions are always left or right, we just write behavioral probabilities for simplicity; π is the probability attached to the left node of 3's information set. As usual we also write $\succcurlyeq_i$ for i's preferences. So we look for pairs of strategies (α, β, γ) and belief π which are sequentially rational and consistent. (*Answer:* (α, β, γ) with $\alpha = \beta = 0$ and $\gamma \geq 3/4$ and $\pi \leq 1/3$)



Solution. First we show that it cannot be $0 < \alpha < 1$ (that is the first player cannot mix). By contradiction, suppose it were so. Then 1 must be indifferent between L and R so that $3(1-\gamma)$ must equal $1-\beta+4\beta(1-\gamma)$ that is $2-3\gamma = \beta(3-4\gamma)$. Also, given $1-\alpha > 0$ it is $L \succcurlyeq_2 R$ iff $4(1-\gamma) \geq 1$ that is $\gamma \leq 3/4$. But then as you can easily check for no value of γ can the equality $2-3\gamma = \beta(3-4\gamma)$ be satisfied (for example if $\gamma < 3/4$ then $\beta = 1$ so..). Whence either $\alpha = 0$ or $\alpha = 1$.

Suppose $\alpha = 1$. Then $\gamma = 0$; but then it should be $\beta < 1$ (otherwise 1 would deviate to R and get 4) which implies $R \succcurlyeq_2 L$ that is $\gamma \geq 3/4$ - contradiction.

Therefore in equilibrium it must be $\alpha = 0$. Given this, if $\beta > 0$ then $\gamma = 1$ which implies $\beta = 0$ - another contradiction. Hence also $\beta = 0$. To sustain the play R, R we need $R \succcurlyeq_{1,2} L$ and this is possible if γ is such that $\gamma \geq 3/4$. That is we need $L \succcurlyeq_3 R$ which means $1-\pi \geq 2\pi$ or $\pi \leq 1/3$.

The conclusion so far is that the sequentially rational systems are strategies (α, β, γ) with $\alpha = \beta = 0$ and $\gamma \geq 3/4$ and beliefs $\pi \leq 1/3$. To finish we show that these π's are consistent. We need sequences of fully mixed $(\alpha_n, \beta_n, \gamma_n) \to (\alpha, \beta, \gamma)$ and $\pi_n \to \pi$ with

$$\pi_n = \frac{\alpha}{\alpha + (1-\alpha)\beta}$$

Note that $\pi \leq 1/3$ is equivalent to $(1-\pi)/\pi \geq 2$ so it suffices to have $(1-\pi_n)/\pi_n = (1-\alpha_n)\beta_n/\alpha_n \geq 2$. To this end we may take $\gamma_n = \gamma, 0 < \alpha_n \to 0$ and $\beta_n = c\alpha_n/(1-\alpha_n)$ with any $c \geq 2$ (of course $\beta_n \to 0$).