Two-person Zero-Sum Games

OR section 2.5 calls them "strictly competitive" games. They are 2-player games where whatever is good for player 1 is bad for 2 and vice versa; formally, for any profiles a, b we have $a \succcurlyeq_1 b \iff b \succcurlyeq_2 a$. Assuming existence of vNM utilities, if u_1 represents $\succcurlyeq_1$ then $u_2 = -u_1$ represents $\succcurlyeq_2$. Since for such a pair $u_1 + u_2 = 0$ such games are usually called "zero-sum". We work with such pairs.

The result on zero-sum games we are going to prove is that in such games equilibria coincide with the profiles of conservative strategies.[1] Recall that in general this is false - think of the chicken game for instance. Intuitively, from the point of view of any player, considering for each possible action "the worst that can happen" is rational because the worst that can happen to you is actually the best for your opponent.

Characterization of Nash Equilibrium

We shall use the fact that if $f(x) \geq g(x)$ for all x then $\max f(x) \geq \max g(x)$. Indeed otherwise, letting x^g the maximum of g, we would have $g(x^g) > \max f(x) \geq f(x^g)$, contradiction. Analogous argument establishes that it must also be $\min f(x) \geq \min g(x)$.[2]

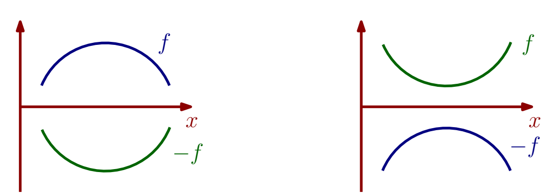
Let us first restate OR (Osborne-Rubinstein) Lemma 22.1.

Lemma 1 (OR Lemma 22.1)**.** *Given $u_2 = -u_1$ we have*

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$$

and the y-solution is the same for both problems.

Proof. We know that for any function f we have $-\max f = \min(-f)$ and $\max(-f) = -\min f$, and that the solutions are the same - see the picture below:



Thus since $u_2 = -u_1$ we have

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = \max_{y \in A_2}[-\max_{x \in A_1} u_1(x, y)] = -\min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$$

[1]Conservative strategies are also known as "maxmin" strategies. We shall use the term conservative following OR.

[2]Maths note that you can ignore. We assume that all the min and max exist. To see that there may be an existence problem observe that for example $f(x) = x$ has neither maximum nor minimum on $(0, 1)$; it is $\sup_{(0,1)} f = 1$ but there is no $x \in (0, 1)$ such that $f(x) = \sup_{(0,1)} f$. Maxima and minima surely exist if the action sets are finite.

as asserted. □

Now we use notation $u_1 = u, u_2 = -u$. We are talking about a two–person zero–sum game with action sets A_1, A_2 and $u_1 = u,\ u_2 = -u$. In this context (x^*, y^*) is Nash iff

$$\forall x, y \quad u(x, y^*) \le u(x^*, y^*) \le u(x^*, y). \tag{1}$$

Conservative strategies maximize over worst outcomes. The formal definitions of conservative strategies and of $\underline{v}$ and $\overline{v}$ are the following:

- $x^* \in A_1$ is conservative if $\min_{y \in A_2} u(x^*, y) = \max_{x \in A_1} \min_{y \in A_2} u(x, y)$;
- $y^* \in A_2$ is conservative if $\max_{x \in A_1} u(x, y^*) = \min_{y \in A_2} \max_{x \in A_1} u(x, y)$;
- $\underline{v} \equiv \max_{x \in A_1} \min_{y \in A_2} u(x, y)$ - called the lower value of the game
- $\overline{v} \equiv \min_{y \in A_2} \max_{x \in A_1} u(x, y)$ - the upper value.

Remark. $\underline{v} \le \overline{v}$. Proof of this: for any x, y it is $\max_x u(x, y) \ge u(x, y)$, whence for any *fixed* x (taking min with respect to y on both sides) it is $\overline{v} \ge \min_y u(x, y)$, and from this (take max with respect to x) $\overline{v} \ge \underline{v}$.

In case $\underline{v} = \overline{v}$ this common number is denoted by v, *or* $v(G)$ *if needed*, and is called the *value* of the game.

Lemma 2. *In any Nash equilibrium* (x^*, y^*) *we must have* $\underline{v} \le u(x^*, y^*) \le \overline{v}$.

Proof. In any equilibrium (x^*, y^*) player 1 should get $u(x^*, y^*) \ge \underline{v}$. Indeed in equilibrium it must be $u(x^*, y^*) = \max_{x \in A_1} u(x, y^*)$; but $u(x, y^*) \ge \min_{y \in A_2} u(x, y)$ for all x, so $\max_{x \in A_1} u(x, y^*) \ge \max_{x \in A_1} \min_{y \in A_2} u(x, y) = \underline{v}$. By the same token it must be $u_2(x^*, y^*) = -u(x^*, y^*) \ge \max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = -\overline{v}$ (last equality by Lemma 1). □

The following proposition (Proposition 22.2 in OR) considerably strengthens these inequalities, and also establishes that in two-person zero-sum games equilibria coincide with the profiles of conservative strategies. We rephrase the proposition as follows.

Proposition (OR Proposition 22.2)**.** *In a two-person zero-sum game: (1) An equilibrium exists if and only if* $\underline{v} = \overline{v}$*; (2) In this case* (x^*, y^*) *is Nash if and only if it is conservative; (3) In any equilibrium* (x^*, y^*) *one has* $u(x^*, y^*) = v$.

Proof. Assume that an equilibrium (x^*, y^*) exists. Then

$$\overline{v} \le \max_x u(x, y^*) = u(x^*, y^*) = \min_y u(x^*, y) \le \underline{v}$$

(equalities by definition of equilibrium, inequalities by definition of $\underline{v}, \overline{v}$); hence the two inequalities above are in fact equalities, and $\overline{v} = \underline{v}$. Also, $u(x^*, y^*) = v$, and $\underline{v} = \min_y u(x^*, y)$ and $\overline{v} = \max_x u(x, y^*)$, that is x^* is conservative and y^* is conservative.

Assume now that $\underline{v} = \overline{v}$ and let (x^*, y^*) be a conservative profile. Then

$$\min_y u(x^*, y) = \underline{v} = \overline{v} = \max_x u(x, y^*)$$

the first equality because x^* is conservative, the last one because y^* is conservative. Therefore the inequalities $\min_y u(x^*, y) \leq u(x^*, y^*) \leq \max_x u(x, y^*)$ are in fact equalities, which means that (x^*, y^*) is Nash. All assertions now follow. □

Thus in a zero-sum game the equilibria coincide with the profiles of conservative strategies, and in any of them $u(x^*, y^*) = v$: player 1 gets the value and 2 the opposite.

An obvious but important consequence of the proposition is that if $\underline{v}(G) < \overline{v}(G)$ then G has no Nash equilibria; for a simple example think of matching pennies, where $\underline{v} = -1, \overline{v} = 1$. In such games, that is where the A_i's are finite, we know that there exists a Nash equilibrium in mixed strategies, action sets becoming $\Delta(A_i)$ (this follows from OR Proposition 33.1).

For an example of a game with value (and equilibrium) consider the following:

	L	C	R
T	$2, -2$	$0, 0$	$1, -1$
M	$4, -4$	$-3, 3$	$2, -2$
B	$1, -1$	$-2, 2$	$-2, 2$

Here it is easy to verify that $\underline{v} = \overline{v} = 0$, and that the only (pure strategy) equilibrium coincides with the unique conservative profile, namely TC.

Mixed extension of zero-sum games

Given a zero-sum game

$$G = \langle \{1, 2\}, (A_1, A_2), (u, -u) \rangle$$

where $u_1 = u$ and $u_2 = -u$, consider its mixed extension, which we denote by G^Δ. Since

$$U_2(\alpha) = \sum\nolimits_a \alpha(a) u_2(a) = -\sum\nolimits_a \alpha(a) u_1(a) = -U_1(\alpha)$$

the mixed extension is also zero-sum, and we let $U = U_1 = -U_2$. Thus

$$G^\Delta = \langle \{1, 2\}, (\Delta(A_1), \Delta(A_2)), (U, -U) \rangle.$$

Pure strategies are included, as degenerate mixed strategies. Notice that in a pure profile we have $U(a) = u(a)$.

By OR Proposition 33.1 every game G with finite strategy sets has a mixed Nash equilibrium. Therefore if A_1, A_2 are finite, as we assume, G^Δ has an equilibrium. As we have just seen if G is zero-sum so is G^Δ, so we can apply Proposition 22.2 (restated above); this implies that G^Δ has a value $v(G^\Delta)$ and that all equilibria (α_1^*, α_2^*) are conservative and $U(\alpha_1^*, \alpha_2^*) = v(G^\Delta)$. Here of course α_1^* is conservative if it solves $\max_{\alpha_1} \min_{\alpha_2} U(\alpha_1, \alpha_2)$ and α_2^* is conservative if it solves $\min_{\alpha_2} \max_{\alpha_1} U(\alpha_1, \alpha_2)$.

First we observe that the following holds for pure equilibria:

Proposition 3. *a^* is an equilibrium of G if and only if it is an equilibrium of G^Δ.*

Proof. If a^* is an equilibrium of G^Δ there are no profitable deviations in the sets $\Delta(A_i)$ so a fortiori there aren't in the smaller sets A_i. If on the other hand a^* is an equilibrium of G, that is for each i one has $u_i(a_i, a_{-i}^*) \leq u_i(a_i^*, a_{-i}^*)$ for all a_i, therefore (using equation (32.2)) $U_i(\alpha_i, a_{-i}^*) = \sum_{a_i} \alpha_i(a_i) u_i(a_i, a_{-i}^*) \leq u_i(a^*) = U_i(a^*)$ so a^* is also an equilibrium of G^Δ. $\square$

Next the relation between values of G and G^Δ; we let $\underline{v}(G), \overline{v}(G)$ the lower and upper values of G.

Proposition 4. $\underline{v}(G) \leq v(G^\Delta) \leq \overline{v}(G)$. *If G has a value $v(G)$ then $v(G) = v(G^\Delta)$.*

Proof. We show that $\underline{v}(G) \leq v(G^\Delta) \leq \overline{v}(G)$, which clearly implies the second assertion. Let (a_1^*, a_2^*) be conservative. Then

$$\begin{aligned}\underline{v}(G) &= \max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = \min_{a_2 \in A_2} u(a_1^*, a_2) = \min_{a_2 \in A_2} U(a_1^*, a_2) \\ &= \min_{\alpha_2 \in \Delta(A_2)} U(a_1^*, \alpha_2) \leq \max_{\alpha_1 \in \Delta(A_1)} \min_{\alpha_2 \in \Delta(A_2)} U(\alpha_1, \alpha_2) = v(G^\Delta).\end{aligned}$$

The other half is analogous. $\square$

Example (Matching Pennies). For matching pennies game, call it G, it is immediate that $-1 = \underline{v}(G) < \overline{v}(G) = 1$ (and there is no equilibrium). In the mixed extension on the other hand $\underline{v}(G^\Delta) = \overline{v}(G^\Delta) = 0$, so $v(G^\Delta) = 0$ and this as we know is what players get in equilibrium. Let's check that $\underline{v}(G^\Delta) = \overline{v}(G^\Delta) = 0$. We have seen that $U(p,q) = (2p-1)(2q-1)$ (where p and q are the probabilities with which players 1 and 2 play H). Consider $\min_q U(p,q)$; if $p > 1/2$ then $\min_q U(p,q) = -(2p-1) < 0$ (achieved for $q = 0$); if $p < 1/2$ then $\min_q U(p,q) = (2p-1) < 0$ (achieved for $q = 1$); if $p = 1/2$ clearly $\min_q U(p,q) = 0$. Therefore $\underline{v}(G^\Delta) = \max_p \min_q U(p,q) = 0$, and the only conservative strategy of player 1 is $p = 1/2$. Analogously from player 2's problem we deduce that $\overline{v}(G^\Delta) = 0$ and that her only conservative strategy is $q = 1/2$.