Examples of Bayesian Games[1]

Contents

[1] Salvatore Modica 2023. Based on Rubinstein-Osborne *Game Theory,* Osborne *Introduction to Game Theory,* Maschler et al *Game Theory,* Eichberger *Game Theory for Economists,* Tadelis *Game Theory,* notes by Debraj Ray and possibly others. I collect here the examples I use in class for convenience. If they were songs this would be just a collection of covers.

1 Definition of Bayesian Game and Bayesian Equilibrium

In Bayesian games - like Battle of Sexes with uncertainty - payoffs depend on action profiles $a = (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n = A$ but also on type profiles $t = (t_1, \ldots, t_n) \in T_1 \times \cdots \times T_n = T$. So $u_i = u_i(a, t)$; and for reasons clear in a moment u_i is assumed to be a vNM utility. Note that each t determines a game on A (with payoffs $u_i(a, t)$). Type profiles are also referred to as *states*. The sets A and T are assumed finite; and we write $a = (a_i, a_{-i}), t = (t_i, t_{-i})$ with obvious meaning. On T there is a commonly known prior distribution P, which assigns positive probability to each t_i, formally $P(t_i) := P(\{t_i\} \times T_{-i}) > 0$ for all t_i. Player i is informed of her type t_i and updates probabilities on T_{-i} by conditioning on t_i; thus she may use probabilities $P(t_{-i} \mid t_i) = P(t)/P(t_i)$. A pure strategy of player i is a function $s_i \colon T_i \to A_i$ from types to actions, that is $t_i \in T_i \mapsto s_i(t_i) \in A_i$. A profile of pure strategies is denoted by $s \colon T \to A$ where $t \mapsto s(t) = (s_1(t_1), \ldots, s_n(t_n))$. The payoff of i under s is taken to be expected utility:

$$u_i(s) := \sum\nolimits_t u_i(s(t), t)P(t). \tag{ex-ante}$$

So the game is defined by strategies $s_i \colon T_i \to A_i$ and payoffs as above. A pure-strategy Bayesian equilibrium is a Nash equilibrium of this game, that is a profile s^* such that $u_i(s^*) \geq u_i(s_i, s^*_{-i})$ for all i, s_i. The idea is always the same: *mutual best responses*.

Write $s(t) = (s_i(t_i), s_{-i}(t_{-i}))$. Then

$$u_i(s) = \sum\nolimits_{t_i} P(t_i) \sum\nolimits_{t_{-i}} u_i(s_i(t_i), s_{-i}(t_{-i}), t_i, t_{-i})P(t_{-i} \mid t_i).$$

Since $P(t_i) > 0$ for all i this expression shows that, given s_{-i}, the strategy s_i which maximizes u_i from the ex-ante perspective (as in (ex-ante) above) is such that for each t_i the chosen $s_i(t_i)$ maximizes, with respect to a_i, the conditional expectation $\sum_{t_{-i}} u_i(a_i, s_{-i}(t_{-i}), t_i, t_{-i})P(t_{-i} \mid t_i)$ relevant after observing t_i. This is the usual way to look for and compute equilibria: s^* is an equilibrium if for all i for each t_i the action $s_i^*(t_i)$ solves

$$\max_{a_i} \sum\nolimits_{t_{-i}} u_i(a_i, s_{-i}^*(t_{-i}), t_i, t_{-i}) P(t_{-i} \mid t_i).$$

Notice that we can view the game as being played by the $T_1 + \cdots + T_n$ players (i, t_i), where (i, t_i) has action set A_i and payoff $\sum_{t_{-i}} u_i(a_i, s_{-i}^*(t_{-i}), t_i, t_{-i}) P(t_{-i} \mid t_i)$.

Remark. In some examples further down the line we are going to have a continuum of types. In those cases expectations are integrals, but they will mostly be elementary.

2 First examples

2.1 Simplest Battle of Sexes with uncertainty

The game is described in Osborne: player 1, the boy, is uninformed of the type of player 2, the girl, who has two types, say l and r for left and right. The relevant payoff matrices are the following:

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

and each has, from the point of view of player 1, probability 1/2. Let us look for pure equilibria. If the boy plays S then l plays S and r plays B (best responses of course), that is the girl plays SB; but then the payoff of 1 under S is $0.5 * 1 + 0.5 * 0$, while by playing B he would get $0.5 * 0 + 0.5 * 2$; so there is no equilibrium where 1 plays S. If he plays B then she would play BS and he would get 1, while S would yield him 0.5. Thus the profile (B, BS) is an equilibrium.

In terms of the game played by $(1, (2, l), (2, r))$ - with obvious notation for the girl's strategies, we get the following payoffs - in the order $(1, (2, l), (2, r))$ - where asterisks mark best responses:

	BB	BS	SB	SS
B	$2^*, 1^*, 0$	$1^*, 1^*, 2^*$	$1^*, 0, 0$	$0, 0, 2^*$
S	$0, 0, 1^*$	$0.5, 0, 0$	$0.5, 2^*, 1^*$	$1^*, 2^*, 0$

which confirms that (B, BS) is the only pure equilibrium of the game.

2.2 Opponent of unknown strength

Again player 1 has one type, player 2 has two types t and t', and player 1 is uninformed about 2's type. The interpretation here is that player 1 is uninformed of the strength of player 2. Type t has probability p. For both actions are F for fight and Y for yield. The respective payoff matrices are the following (the strong t type of 2 is on the left):

$$\begin{array}{ccc} & F & Y \\ F & \boldsymbol{-k,1} & 1,0 \\ Y & 0,1 & 0,0 \end{array} \qquad \begin{array}{ccc} & F & Y \\ F & \boldsymbol{1,-1} & 1,0 \\ Y & 0,1 & 0,0 \end{array}$$

You are player 1. Look at the payoffs. The strong opponent will fight for sure; the weak one only if you yield. So if you fight you get $-kp+1-p=1-(1+k)p$; if you don't you get zero. So you fight if $1-(1+k)p>0$ that is $p<1/(1+k)$. Thus the equilibrium is (F, FY) if $p<1/(1+k)$, and (Y, FF) otherwise. Notice how the range of π where you fight shrinks as k gets larger (of course!).

2.3 Entrant/Incumbent

Player 1 is a potential entrant and has a single type; player 2 is the incumbent and is of two possible types t, t'. He is of type t with probability p, assumed $\neq 1/2$ to avoid unnecessary complications. The actions for 1 are E, N - for entry or not - and for 2 are A, F - for accommodate or fight. The two possible payoff matrices are the following (only difference marked as bold), where again we can interpret the left type of the incumbent as "strong":

$$\begin{array}{ccc} & A & F \\ E & 1,1 & -1,\mathbf{2} \\ N & 0,3 & 0,3 \end{array} \qquad \begin{array}{ccc} & A & F \\ E & 1,1 & -1,\mathbf{-1} \\ N & 0,3 & 0,3 \end{array}$$

We may look for equilibria where 1 enters, then for equilibria when 1 does not. Note that if does not enter he gets zero for sure. If 1 enters then player 2 plays FA, in which case E yields $1-2p$, and this is positive iff $p<1/2$. Therefor there is an equilibrium where 1 enters only for $p<1/2$, in which case the equilibrium is (E, FA).

Next look for equilibria where 1 plays N. Every strategy of 2 is a best reply to N; we must select those against which N is best reply. Best reply to AA is E, so AA is out. From the above computation we deduce that N is best reply to FA for $p>1/2$ and to AF for

$p < 1/2$, so (N, FA) is an equilibrium for $p > 1/2$ and (N, AF) is an equilibrium for $p < 1/2$ (notice that in the right game NF is an equilibrium along with EA). Finally, (N, FF) is an equilibrium for all p.

To sum up: for $p < 1/2$ equilibria are (E, FA), (N, FF) and (N, AF); for $p > 1/2$ they are (N, FF) and (N, FA).

2.4 More information may hurt

This is an interesting phenomenon, which cannot occur in single person decision problems: more information may hurt both the informed and the uninformed. Consider the following two two-player games:

	L	M	R
T	$1,2$	$1,0$	$1,3$
B	$4,4$	$0,0$	$0,5$

	L	M	R
T	$1,2$	$1,3$	$1,0$
B	$4,4$	$0,5$	$0,0$

Suppose first that both players assign equal probability to each of them (formally, both have two types and both are uninformed about the other player's type). Then L is a dominant action for 2 (if 2 plays T it gives 2 while the others yield 1.5; if 1 plays B it gives 4 while the others give 2.5); so in any equilibrium 2 plays L; then 1 plays B, and both get 4 for sure. Now suppose player 2 becomes informed of her own type. Then her best reply is RM against both T and B; but player 1's best reply to RM is T, so the equilibrium payoff is 1 for player 1 and 3 for (both types of) player 2. Both are worse off. Here L is a "compromise" action, which 2 can only play if uninformed; once 1 knows that 2 is informed then L is out, and 1's then dominant action T can make both worse off.

2.5 Quantity competition: the Cournot oligopoly model

We modify the model we have already studied which we recall for convenience. We had n firms, with demand price $p(q) = a - Q$ with $Q = \sum q_i$, and costs $c_i(q_i) = cq_i$, where $a > c$. Firms maximized profits $\pi_i(q) = q_i p(q) - cq_i = q_i(a - Q) - cq_i$, and we found that the only equilibrium was symmetric with (letting $\sigma = a - c$)

$$q_i^{eq} = \frac{a-c}{1+n} = \frac{\sigma}{1+n} \ \forall i$$

We now assume there are only two firms and that firm 1 has cost $c_1 = c$, but suppose that 2's cost c_2 may be with equal probabilities $c^H = c + \delta$ or $c^L = c - \delta$ so $Ec_2 = c$.[2] Firm 2 only knows its value, so its cost is its type (and the two types have equal probability). An equilibrium is a triple of choice (q_1, q_2^H, q_2^L) with the mutual best response property (in expected value terms). Expected payoffs of the three effective players are, letting $Eq_2 = (q_2^H + q_2^L)/2$,

$$
\begin{aligned}
u_1(q_1, q_2^H, q_2^L) &= \sum\nolimits_{t=H,L} \frac{1}{2} q_1(a - q_1 - q_2^t) - cq_1 = q_1(a - c - Eq_2 - q_1) \\
u_2^t(q_1, q_2^H, q_2^L) &= q_2^t(a - q_1 - q_2^t) - c_2^t q_2^t = q_2^t(a - c_2^t - q_1 - q_2^t), \quad t = H, L.
\end{aligned}
$$

The optimal choices are easily seen to be (all functions are parabolas..)

$$
q_1 = \frac{a - c - Eq_2}{2} \qquad q_2^t = \frac{a - c_2^t - q_1}{2} \quad t = H, L.
$$

From 2's best responses we get $Eq_2 = (a - c - q_1)/2$ (since $Ec_2 = c$) and plugging this into the first we obtain $q_1 = (a - c)/3$. Plugging this in gives the equilibrium, where we use $\sigma = a - c$:

$$
q_1 = \frac{\sigma}{3} \qquad q_2^H = \frac{\sigma}{3} - \frac{\delta}{2} \qquad q_2^L = \frac{\sigma}{3} + \frac{\delta}{2}.
$$

So firm 1 produces as much as in the full information case, the high cost firm produces a little less and the low cost one a little more.

Here too more information is worse for all

It is interesting to compare equilibrium payoffs to those which would arise if 2's costs were observed by both firms. Of course we are heading to a question we already encountered: does more information hurts? We know that in the just computed equilibrium $Eq_2 = q_1$ so firm 1's payoff is $q_1(\sigma - 2q_1) = (\sigma/3)^2$; the expected payoff of firm 2 is $(\sigma/3)^2 + (\delta/2)^2$, see footnote.[3]

[2] δ is assumed small enough that all the quantities computed below are positive.

[3] We have

$$
\begin{aligned}
&\frac{1}{2} q_2^H(a - c_2^H - q_1 - q_2^H) + \frac{1}{2} q_2^L(a - c_2^L - q_1 - q_2^L) \\
=&\frac{1}{2}\left[\frac{a-c}{3} - \frac{\delta}{2}\right]\left[a - c - \delta - q_1 - \frac{a-c}{3} + \frac{\delta}{2}\right] + \frac{1}{2}\left[\frac{a-c}{3} + \frac{\delta}{2}\right]\left[a - c + \delta - q_1 - \frac{a-c}{3} - \frac{\delta}{2}\right] \\
=&\frac{1}{2}\left(\frac{a-c}{3} - \frac{\delta}{2}\right)^2 + \frac{1}{2}\left(\frac{a-c}{3} + \frac{\delta}{2}\right)^2 = \left(\frac{\sigma}{3}\right)^2 + \left(\frac{\delta}{2}\right)^2
\end{aligned}
$$

Now suppose 2's costs become known to both; then we have two full information games each occurring with probability 1/2. The first has $c_1 = c, c_2 = c + \delta$; the second has $c_1 = c, c_2 = c - \delta$. In these games, respectively: $u_1 = q_1(\sigma - q_2 - q_1)$, maximized at $q_1 = (\sigma - q_2)/2$; $u_2 = q_2(\sigma \mp \delta - q_1 - q_2)$, maximized at $q_2 = (\sigma \mp \delta - q_1)/2$. It is elementary to check that the systems have the following solutions: in the high cost game $q_1(H) = (\sigma + \delta)/3, q_2(H) = (\sigma - 2\delta)/3 = q_1(H) - \delta$; in the low cost game $q_1(L) = (\sigma - \delta)/3, q_2(L) = (\sigma + 2\delta)/3 = q_1(L) + \delta$. And that payoffs are: in the high cost game $u_1(H) = q_1(H)^2$ and $u_2(H) = q_2(H)^2$; in the low cost game $u_1(L) = q_1(L)^2$ and $u_2(L) = q_2(L)^2$.[4] Therefore from an ex-ante point of view the two firms' expected payoffs are the following:

$$\text{Firm 1: } \frac{1}{2}\left(\frac{(\sigma+\delta)^2}{9} + \frac{(\sigma-\delta)^2}{9}\right) = \left(\frac{\sigma}{3}\right)^2 + \left(\frac{\delta}{3}\right)^2 > \left(\frac{\sigma}{3}\right)^2$$

$$\text{Firm 2: } \frac{1}{2}\left(\frac{(\sigma-2\delta)^2}{9} + \frac{(\sigma+2\delta)^2}{9}\right) = \left(\frac{\sigma}{3}\right)^2 + \left(\frac{2}{3}\delta\right)^2 > \left(\frac{\sigma}{3}\right)^2 + \left(\frac{1}{2}\delta\right)^2.$$

So *both* firms are better off if firm 2 foregoes its private information *in the entire range of parameters values.*

2.6 Price Competition: the Bertrand oligopoly model

We consider the model without uncertainty first. There are two price-setting firms with demand functions

$$d_1(p_1, p_2) = a - p_1 + \frac{1}{2}\frac{p_2}{p_1} \qquad d_2(p_1, p_2) = b - p_2 + \frac{1}{2}p_1$$

where d_i, p_i are quantities and prices, and a, b are constants (parameters).[5] Assume for simplicity that production costs are zero; then payoffs $u_i(p_{1,}p_2)$ are just gross revenues $p_i d_i$.

(i) Find the Nash equilibrium prices (best responses are found via the zeros of the deriva-

[4] To compute payoffs note that in the high cost game $q_1 + q_2 = 2q_1 - \delta = 2q_2 + \delta$; in the low cost game $q_1 + q_2 = 2q_1 + \delta = 2q_2 - \delta$; so payoffs are: in the high cost game $u_1(H) = q_1(\sigma - q_2 - q_1) = [(\sigma + \delta)/3]^2 = q_1(H)^2$ and $u_2(H) = q_2(\sigma - 2\delta - 2q_2) = [(\sigma - 2\delta)/3]^2 = q_2(H)^2$; in the low cost game $u_1(L) = [(\sigma - \delta)/3]^2 = q_1(L)^2$ and $u_2(L) = [(\sigma + 2\delta)/3]^2 = q_2(L)^2$.

[5] To be pedantic quantities demanded are $\max\{d_i, 0\}$ but we can neglect this.

tives). The answer is (details in footnote[6])

$$p_1 = \frac{a}{2}, \quad p_2 = \frac{b}{2} + \frac{1}{4}p_1 = \frac{b}{2} + \frac{1}{8}a.$$

Now suppose a and b can take two values $\{a_H, a_L\}$ and $\{b_H, b_L\}$ respectively, where the value of a is known to firm 1 but not 2 and similarly b is known to firm 2 only. So each player has two types and we have to specify $p_1(a_H), p_1(a_L)$ for player 1 and $p_2(b_H), p_2(b_L)$ for player 2. The players' conditional probabilities π_i are as follows:

$$\pi_1(b_H \mid a_H) = \pi_2(a_H \mid b_H) = 4/5$$
$$\pi_1(b_L \mid a_L) = \pi_2(a_L \mid b_L) = 2/3$$

where of course $\pi_1(b_L \mid a_H) = 1/5$ etc. since conditional probabilities must sum up to 1. Thus each firm believes that it is likely that its competitor faces the same demand condition as itself. In the obvious notation $(i,H),(i,L)$ for $i = 1,2$ the payoff of $(1,H)$ is

$$\begin{aligned} Eu_{1,H} &= \pi_1(b_H \mid a_H)\left[a_H p_1 - p_1^2 + \frac{1}{2}p_2(b_H)\right] + \pi_1(b_L \mid a_H)\left[a_H p_1 - p_1^2 + \frac{1}{2}p_2(b_L)\right] \\ &= a_H p_1(a_H) - p_1(a_H)^2 + \frac{1}{2}Ep_2(b \mid a_H), \end{aligned}$$

and the others are defined similarly.

(ii) Find the Bayesian equilibrium of the game. (*Hint:* you should find e.g. $p_2(b_L) = \frac{b_L}{2} + \left[\frac{1}{24}a_H + \frac{1}{12}a_L\right]$).[7]

(iii) Suppose demand for our two firms has been constant for a while and then at some

[6]Solution. From $u_1(p_1,p_2) = ap_1 - p_1^2 + \frac{1}{2}p_2$ and $u_2(p_1,p_2) = bp_2 - p_2^2 + \frac{1}{2}p_1p_2$ by equating derivatives to zero we get

$$0 = a - 2p_1, \qquad 0 = b - 2p_2 + \frac{1}{2}p_1$$

from which the result is direct.

[7]The payoffs are the same as in the certainty case except that the opponent's price is replaced by its expectation. For $t = H, L$ we have $Eu_{2,t} = \left[b + \frac{1}{2}E(p_1(a) \mid b_t)\right]p_2 - p_2^2$. Thus in equilibrium

$$\begin{aligned} p_1(a_H) &= \frac{a_H}{2} \quad p_1(a_L) = \frac{a_L}{2} \\ p_2(b_H) &= \frac{b_H}{2} + \frac{1}{4}E(p_1(a) \mid b_H) = \frac{b_H}{2} + \frac{1}{8}E_2(a \mid b_H) \\ p_2(b_L) &= \frac{b_L}{2} + \frac{1}{4}E(p_1(a) \mid b_L) = \frac{b_L}{2} + \frac{1}{8}E_2(a \mid b_L). \end{aligned}$$

point firm 2 believes the demand conditions of its competitor may deteriorate, in the sense that $E_2(a \mid b_H)$ and $E_2(a \mid b_L)$ decrease. On the basis of the result above can we predict its price reaction? [8]

2.7 Infection

2.8 Providing a Public good

2.8.1 Study Group

In this example types are not finite, but as you will see this does not create problems. Two students $i = 1, 2$ work on a problem in two different rooms. Student i can solve the problem - choice $s_i = 1$ - at utility cost $0 < c < 1$, or do nothing - choice $s_i = 0$ - at zero cost. If the problem is solved - by one of them or both - student 1 gets t_1^2 and student 2 gets t_2^2, where $0 \leq t_i \leq 1$ for $i = 1, 2$ (higher types care more about the result). So student i of type t_i has payoff

$$u_i(s_i, s_j, t_i) = \max\{s_1, s_2\}t_i^2 - s_i c.$$

The two types have independent uniform distributions on $[0, 1]$, that is cumulative distribution function $F(t_i) = t_i$.[9] Player i knows her type and that the other has cdf F. So this is a Bayesian game, where a strategy of player i is a function $s_i(t_i)$ from $[0, 1]$ to $\{0, 1\}$ (one of the two possible choices for each type). We want to find equilibrium strategies.

To analyze the game let $z_i = \text{Prob}(s_i = 1)$. If player 1 chooses $s_1 = 1$ she gets $t_1^2 - c$; under $s_1 = 0$ she gets t_1^2 if 2 solves the problem and zero if 2 does not, so her expected utility is $t_1^2 \cdot z_2 + 0 \cdot (1 - z_2)$. Therefore she chooses $s_1 = 1$ if and only if $t_1^2 - c \geq t_1^2 z_2$ or $t_1^2(1 - z_2) \geq c$. Analogously $s_2 = 1$ if and only if $t_2^2(1 - z_1) \geq c$. Notice that the best response of i depends on $s_j(\cdot)$ only through z_j.

Let us first see if it is possible to have $z_2 = 1$ in equilibrium (player 2 always contributing). If $z_2 = 1$ then 1 never contributes ($t_1^2 - c < t_1^2 z_2$ for all t_1) so that $z_1 = 0$; but then $s_2 = 1$ if $t_2^2 - c \geq 0$ or $t_2 \geq \sqrt{c}$ so $z_2 = 1 - \sqrt{c} < 1$, contradiction. Analogously it cannot be $z_1 = 1$. A similar argument shows that $z_1, z_2 > 0$,[10] and we can conclude that $0 < z_1, z_2 < 1$.

[8] Solution: The result says it will lower the price.

[9] Recall that this means that $\text{Prob}(t_i \leq x) = x$ for all $x \in [0, 1]$.

[10] Suppose $z_2 = 0$ then $z_1 = 1 - c^{1/2}$; but then

$$z_2 = 1 - \sqrt{c/(1 - z_1)} = 1 - \sqrt{c/c^{1/2}} = 1 - c^{1/4} > 0$$

Therefore the two threshold conditions can be written as

$$t_1 \geq \sqrt{\frac{c}{1-z_2}}, \quad t_2 \geq \sqrt{\frac{c}{1-z_1}}$$

which say

$$z_1 = 1 - \sqrt{\frac{c}{1-z_2}}, \quad z_2 = 1 - \sqrt{\frac{c}{1-z_1}}$$
$$c = (1-z_1)^2 (1-z_2) = (1-z_1)(1-z_2)^2$$
$$z_1 = z_2 = 1 - c^{1/3}.$$

Therefore the equilibrium strategies are the same, given by

$$s_i(t_i) = \begin{cases} 0 & t_i < c^{1/3} \\ 1 & t_i \geq c^{1/3}. \end{cases}$$

Remark. Consider player i in isolation. If she chooses $s_1 = 1$ she gets $t_1^2 - c$; under $s_1 = 0$ she gets zero for sure - remember that in the group her expected utility in that case is $t_i^2 \cdot z_j + 0 \cdot (1 - z_j) > 0$. So in isolation she solves the problem if $t_1 \geq \sqrt{c}$. For $\sqrt{c} < t_i < c^{1/3}$ she works if she is alone but not if in the group. This is not surprising: in the group you can hope that somebody else does the work. In the group only the most motivated students will work hard.

2.8.2 Same idea, different model

Consider the following variant of the "study group" example above. Benefit is 1 for both players but effort costs are different: for player i it is c_i; and both $c_1$and c_2 are distributed uniformly on $[0, 2]$ that is with cdf P given by $P(c) = c/2$ for $c \in [0, 2]$. Show that the unique Bayes Nash equilibrium is that each player contributes in the interval $[0, c^*]$ where $c^* = 2/3$. Note that in isolation each player would contribute if $c_i \leq 1$.

Solution. In this case a strategy for player i is a map $s_i(c_i)$ from $[0, 2]$ to $\{0, 1\}$ (contribute or not), and the payoff - which does not depend on c_j - is

$$u_i(s_i, s_j, c_i) = \max\{s_1, s_2\} - c_i s_i.$$

Now for the equilibrium. Let $z_j = \Pr(s_j = 1)$. If i does not contribute she gets 1 with probability z_j (zero with probability $1 - z_j$); if she does she gets $1 - c_i$; therefore she will contribute if $z_j \leq 1 - c_i$ or $c_i \leq 1 - z_j \equiv c_i^*$. In other words we have $z_i = P(c_i^*)$ with $c_i^* = 1 - z_j$. Therefore $c_i^* = 1 - z_j = 1 - P(c_j^*) = 1 - P(1 - z_i) = 1 - P(1 - P(c_i^*))$; that is both c_1^*, c_2^* must satisfy the equation $x = 1 - P(1 - P(x))$. In our case this gives directly $x = c_1^* = c_2^* = 2/3$.

2.8.3 A slight generalization

Suppose c_1 and c_2 are distributed in an interval $[\underline{c}, \overline{c}]$ with $\underline{c} < 1 < \overline{c}$ according to the cdf P (in the previous case it was $[\underline{c}, \overline{c}] = [0, 2]$ and $P(x) = x/2$ for $0 \leq x \leq 2$). As before $z_i = P(c_i^*)$ with $c_i^* = 1 - z_j$.

Now if $z_j = 0$ then $c_i^* = 1$ that is $z_i = P(1)$; on the other hand in this case $c_j^* = 1 - P(1)$ so $z_j = 0$ if $1 - P(1) \leq \underline{c}$. The conclusion is that if $1 - P(1) \leq \underline{c}$ there are two equilibria where one player contributes for $c_i \leq 1$ and the other never contributes. And these are the only equilibria since if $c_i^* < 1$ then a fortiori $z_j = 0$ so it must be $c_i^* = 1$.

If on the other hand $1 - P(1) > \underline{c}$ then we can proceed as in the base model and deduce that c_1^*, c_2^* must satisfy the equation $x = 1 - P(1 - P(x))$ that is $1 - x = P(1 - P(x))$. For $x = 1$ we have $1 - x = 0 = P(\underline{c}) < P(1 - P(1)) = P(1 - P(x))$ while for $x = \underline{c}$ we have $P(1 - P(x)) = P(1 - P(\underline{c})) = P(1) < 1 - \underline{c} = 1 - x$ so there are solutions in the open interval $(\underline{c}, 1)$, which are symmetric equilibria.

2.9 Committee voting under unanimity

We frame this example in terms of a group of jurors who have to decide whether to convict or acquit a defendant who may be guilty or not guilty.[11] The defendant is convicted only if *all* the jurors agree on conviction. The defendant is guilty with probability π, and each juror's payoff is 1 if a correct decision is taken and zero otherwise. Each juror has two actions, A and C for acquit and convict. For a juror in isolation C yields 1 with probability π and zero otherwise, and similarly A gives $1 - \pi$, so he prefers C if $\pi > 1 - \pi$ that is $\pi > 1/2$. We will assume $\pi > 1/2$.

Each juror receives a signal on the defendant. The latter may be G or NG (guilty or not guilty), and the signal is g or ng; we assume that the probability that the signal is correct is p: $P(g \mid G) = P(ng \mid NG) = p$, and we assume $p > 1/2$. The signal is the juror's type. So each

[11] General remarks on the relevance of committee voting and a more complete analysis of the problem are in Osborne's book, chapter 9.

one has two possible types, and therefore a strategy is a pair of actions, one for each signal; we order them so that strategy XY with $X, Y \in \{A, C\}$ means X under g and Y under ng.

Again let us look at the case of a single juror. When is it optimal to "follow the signal", that is to play CA? By Bayes rule we have $P(G \mid g) = \pi p/[\pi p + (1-\pi)(1-p)] > \pi$ so under g you choose C.[12] On the other hand $P(G \mid ng) = \pi(1-p)/[\pi(1-p) + (1-\pi)p] < \pi$; thus under ng you choose A if $\pi(1-p)/[\pi(1-p)+(1-\pi)p] < 1/2$ which is equivalent to $p > \pi$, that is the quality of the signal is high enough.[13]

Now suppose there are two jurors. Is the profile (CA, CA) an equilibrium? Recall that for conviction both jurors have to choose C. Suppose juror 2 chooses CA. Will juror 1 choose A under ng? The crucial point is that his decision is relevant *only* if the juror 2 chooses C (which occurs if his type is g);[14] therefore juror 1 computes

$$P(G \mid ng, g) = \frac{\pi p(1-p)}{\pi p(1-p) + (1-\pi)p(1-p)} = \pi.$$

Since $\pi > 1/2$ the answer is no; therefore "following signals" is not an equilibrium.

Note that with n jurors the situation is even worse: juror i's decision is again relevant only if all the others choose C; therefore assuming all the others choose CA, upon receiving the not guilty signal ng he computes

$$P(G \mid ng, g, \ldots, g) = \frac{\pi(1-p)p^{n-1}}{\pi(1-p)p^{n-1} + (1-\pi)p(1-p)^{n-1}}$$

and since $(1-p)p^{n-1} > p(1-p)^{n-1}$ this is even greater than π. His own ng evidence loses weight given that all others are taken to have the g signal. Indeed

$$P(G \mid ng, g, \ldots, g) = \frac{\pi(1-p)}{\pi(1-p) + (1-\pi)p[(1-p)/p]^{n-1}} \to 1 \text{ as } n \to \infty$$

since $p > 1/2$. What are equilibria of this game? All choosing AA is an obvious equilibrium. Another is all choosing CC under the same condition $p > \pi$ of the single juror case, since when all $j \neq i$ choose CC then i is in the same position as when he is alone. These equilibria are not "satisfactory", but the problem is not the equilibrium concept - it is unanimity.

We will explore existence of a symmetric mixed equilibrium in the next section.

[12] the denominator is smaller than p since $p > 1-p$: $\pi p + (1-\pi)(1-p) < \pi p + (1-\pi)p = p$

[13] $2\pi(1-p) < \pi(1-p) + (1-\pi)p \iff \pi(1-p) < (1-\pi)p \iff p > \pi$

[14] We shall make a similar point in the next example and in the context of common value auctions.

2.10 Adverse selection: a market for lemons

There are two players, a seller of a car and a potential buyer. The quality of the car - which is the seller's type, known to the seller but not to the buyer - is t, uniformly distributed on $[0,1]$. Here there is a continuum of types, but again all we need to know for now is that the uniform distribution on $[0,1]$ has density $f(t)=1$, and that for any function g we want to integrate it is $Eg=\int_0^1 g(t)dt$. In particular the expected value of the car is $Et=\int_0^1 tdt=1/2$. The buyer's strategy is a bid $b\in[0,1]$; the seller has two actions, A for accept and R for reject, so a strategy of the seller is a function $s:[0,1]\to\{A,R\}$. The value of the car to the seller is t, therefore obviously the seller's best response to b is A if and only if $b\geq t$. There is nothing more to say about him. The value of the car to the buyer is assumed to be $a+t$ with $0\leq a<1$, so that since $a+t\geq t$ it would be efficient (better for both) to trade for all t. The payoff to the buyer if she bids b is then $a+t-b$ if the offer is accepted and zero otherwise. Given the above strategy of the seller we then have $u(b)=I(t\leq b)(a+t-b)$ where $I(F)$ denotes the indicator of the event $F\subseteq[0,1]$, and therefore

$$\begin{aligned}Eu(b) &= \int_0^1 I(t\leq b)(a+t-b)dt = \int_0^b (a+t-b)dt \\ &= b(a-b)+b^2/2 = b(2a-b)/2\end{aligned}$$

For equilibrium the buyer must maximize this with respect to b, which gives $b^{eq}=a$. Thus the car is sold if $t\leq a$, which implies that the expected value of the traded car is

$$E(t\mid t\leq a)=\frac{\int_0^1 tI(t\leq a)dt}{P(t\leq a)}=\frac{\int_0^a tdt}{a}=a/2.$$

This is *lower* than the overall average $Et=1/2$. Only "lemons" are traded. The idea is simple: you can't bid $1/2$, for in that case the expected value of the car you get is $1/4$. You have to bid (possibly much) less.

2.11 BoS with uncertainty on both sides

This is based on OR exercise 27.2. Here the two players like to go out together but neither knows whether the other prefers B or S. Each player's type consists of his/her preferred alternative, and it is then convenient to use b and s to denote types; so each player can be of type $t_i\in\{b,s\}$, and $T=\{bb,bs,sb,ss\}$. The corresponding four games (in the same order) are then:

	B	S
B	$2,2$	$0,0$
S	$0,0$	$1,1$

	B	S
B	$2,1$	$0,0$
S	$0,0$	$1,2$

	B	S
B	$1,2$	$0,0$
S	$0,0$	$2,1$

	B	S
B	$1,1$	$0,0$
S	$0,0$	$2,2$

Player i is informed of her/his type, so for example player 2 knows whether $t_2 = b$ (that is $t \in \{bb, sb\}$) or $t_2 = s$ (that is $t \in \{bs, ss\}$). For each $i = 1, 2$ the pure strategies are XY with $X, Y \in \{B, S\}$ - as in the previous examples, player 1 playing SB means that $(1, b)$ plays S and $(1, s)$ plays B, etc. To find the equilibria of this game first observe that in each state the unique best reply of each player is to match the choice of the other. This implies that (BB, BB) and (SS, SS) are equilibria for any probability P on T, and that any other pure-strategy equilibrium has both players play type-dependent strategies (in the sense that different types play different strategies).

To see what these other equilibria may be we impose some symmetry on the probability P on T. Specifically, we assume that for any (i, t_i) the conditional probability that j has the same preference as i is a constant p; formally, that

$$P(t_j = x \mid t_i = x) = p \quad \text{for all } i \neq j \text{ and } x = b, s.$$

The first of the candidate equilibria just described is (BS, BS). So suppose 2 plays BS; if $(1, b)$ plays B he gets $2 * p + 0 * (1 - p)$, while playing S results in the payoff $0 * p + 1 * (1 - p)$; so $(1, b)$'s best reply is B if $p \geq 1/3$; analogously, $(1, s)$'s best reply is S if $0 * (1 - p) + 2 * p \geq 1 * (1 - p) + 0 * p$ that is again if $p \geq 1/3$. Hence player 1's best reply to BS is BS if $p \geq 1/3$. The symmetric argument shows that the same holds for player 2. Hence if $p \geq 1/3$ we have the equilibrium (BS, BS). Analogous calculations show that for $p \geq 2/3$ the profile (SB, SB) is an equilibrium;[15] this makes sense, for if p is close to 1 then she shares your preferences so if she plays according to the opposite preferences you should too. Note that in these *pure strategy* equilibria, with probability $P\{t_i \neq t_j\}$ the two players make different choices and get zero. Lastly, the same kind of computations show that for $p \leq 1/3$ the profiles (BS, SB) and (SB, BS) are equilibria.[16] In these two cases the two players make different choices with

[15]Suppose that 2 plays SB; then $(1, b)$'s best reply is S if $p * 1 + (1 - p) * 0 \geq p * 0 + (1 - p) * 2$ that is $p \geq 2/3$; by the same token $(1, s)$'s best reply is B if $p \geq 2/3$.

[16]Check (BS, SB). If 2 plays SB then 1's best reply is BS if $p \leq 2/3$; if 1 plays BS then by the first calculation 2's best reply is SB if $p \leq 1/3$; so (BS, SB) is an equilibrium if $p \leq 1/3$. The case (SB, BS) is analogous.

probability $P\{t_i = t_j\}$.

3 Mixed equilibria

3.1 Premise: mixed vs behavior strategies

In the Battle of Sexes with uncertainty we are going to find that in some range of the parameters there is no pure-strategy equilibrium, so we look for mixed equilibria.

In Bayesian games a pure strategy of i is a function mapping $t_i \in T_i \mapsto s_i(t_i) \in S_i$. A mixed strategy is a probability distribution on pure strategies. For the games we analyze it is equivalent - and convenient - to consider a different type of mixing, whereby each type t_i of i chooses a distribution on S_i. These are called *behavior strategies*.[17] Of course each t_i choosing a degenerate distribution is a pure strategy of i. In the following examples we work with behavior strategies.

There is no need for general notation. To see what is involved, player 2 in the example below has two types t and t', and two actions B and S (hence 4 pure strategies). A behavior strategy is specified by a distribution $(x, 1-x)$ on $\{B, S\}$ used by t and another one $(y, 1-y)$ used by t'.

3.2 A first example

This is from Maschler et al. Example 9.54. There are two players, and for a change *the uninformed player is player 2*. Player 1 has two types say again l, r. The games in the two states are the following, and each has probability 1/2:

	L	R
T	$1,0$	$0,2$
B	$0,3$	$1,0$

	L	R
T	$0,2$	$1,1$
B	$1,0$	$0,2$

So the three players here are $(1,l), (1,r)$ and 2. Player 2 chooses L or R, a pure strategy of 1 is of the form XY with $X, Y \in \{T, B\}$ (where for example BT means $(1,l)$ plays B and $(1,r)$ plays T). As to mixed strategies, we let q the probability 2 assigns to L and x [resp. y] the probability with which $(1,l)$ [resp. $(1,r)$] plays T.

[17] This is a consequence of Kuhn's Theorem, to discuss which would take us too far aside.

To analyze the game first observe that player 2 must mix, that is in equilibrium it must be $0 < q < 1$. Indeed if 2 plays L for sure then 1 plays TB but then L gives zero for sure while R gives 2 for sure. If on the other hand 2 plays R for sure then 1 will play BT and then R gives expected payoff 0.5 and L gives 2.5.

Then 2 must be indifferent between L and R, which implies

$$\frac{1}{2} \cdot 3(1-x) + \frac{1}{2} \cdot 2y = \frac{1}{2} \cdot 2x + \frac{1}{2} \cdot [y + 2(1-y)]$$

that is $x = (1+3y)/5$. Since $1/5 < x < 4/5$, $(1, l)$ must be indifferent between T and B, so it must be $q = 1/2$. For this value $(1, r)$ is also indifferent between her two strategies. We conclude that the equilibria have

$$q = 1/2, x = (1+3y)/5 \text{ and } 0 \le y \le 1.$$

Equilibrium payoffs: player 1 of any type gets 1/2, while player 2 gets (e.g. from L)

$$\frac{1}{2} \cdot 3(1-x) + \frac{1}{2} \cdot 2y = \frac{12+y}{10} \in \{12/10, 13/10\}.$$

Does more information hurt once again? Would player 1 pay to keep for her private information? We compare the Bayesian equilibrium to two possible alternatives.

1. When 2's types are realized they become common knowledge. In this case the players play the left and the right game with equal probability. In the left game the unique equilibrium is mixed, with $q = 1/2, p = 3/5$; player 1 gets 1/2, player 2 gets 6/5. In the right game $q = 1/2, p = 2/3$; player 1 gets 1/2, player 2 gets 4/3. So 1's expected payoff is 1/2 while 2 gets $0.5(6/5+4/3) = 6/5+0.67/10$. 1 is indifferent to the original game. As to 2, if she think that the equilibria on $0 \le y \le 1$ are played with uniform probability she is better off without private information.

2. Types remain unknown to both types. Hence 1 and 2 play ex ante, assigning probability 1/2 to each of the two games. The matrix below contains the expected payoffs:

	L	R
T	$\frac{1}{2}, 1$	$\frac{1}{2}, \frac{3}{2}$
B	$\frac{1}{2}, \frac{3}{2}$	$\frac{1}{2}, 1$

Pure equilibria are off diagonals, where 1 gets $1/2$ and 2 gets $3/2 > 6/5 + 0.67/10$. In the mixed equilibria $p = 1/2$ and $0 \leq q \leq 1$ payoffs are the same. Therefore in this case 1 is again indifferent and 2 strictly prefers not to become informed of her type.

3.3 Symmetric mixed equilibrium in the committee voting example

We look for a symmetric mixed equilibrium in which g chooses C and ng chooses C with probability β, for each player. We assume $p > \pi$ which is the condition guaranteeing that the single juror would follow the signal, and also that p is close to π.

To find the equilibrium β again assume all $j \neq i$ play according to the candidate strategy, and consider type ng of i. Again his decision is relevant only if all others play C so he computes $P(G \mid ng \text{ and all } j \neq i \text{ choose } C)$, and this must be equal to $1/2$ for him to be indifferent between C and A. Now if the defendant is guilty j play C with probability $p + (1-p)\beta$, while if he is not guilty that probability is $1 - p + p\beta$; therefore (always by Bayes rule)

$$P(G \mid ng \text{ and all } j \neq i \text{ choose } C) = \frac{\pi(1-p)[p+(1-p)\beta]^{n-1}}{\pi(1-p)[p+(1-p)\beta]^{n-1} + (1-\pi)p[1-p+p\beta]^{n-1}}$$

and this is $= 1/2$ iff

$$\begin{aligned} \pi(1-p)[p+(1-p)\beta]^{n-1} &= (1-\pi)p[1-p+p\beta]^{n-1} \qquad (\star) \\ \Pi \equiv \left[\frac{\pi/(1-\pi)}{p/(1-p)}\right]^{1/(n-1)} &= \frac{1-p+p\beta}{p+(1-p)\beta} \\ \Pi\,[p+(1-p)\beta] &= 1-p+p\beta \\ \beta &= \frac{\Pi p - (1-p)}{p - \Pi(1-p)} \end{aligned}$$

Given $p > \pi$ but close to π we have $\Pi < 1$ but close to 1. It is then easy to check that both numerator and denominator above are positive,[18] and that $\beta < 1$.

We still have to check that type g wants to play C; for this it must be $P(G \mid g \text{ and all } j \neq i \text{ choose } C) \geq 1/2$; this probability is

$$\frac{\pi p[p+(1-p)\beta]^{n-1}}{\pi p[p+(1-p)\beta]^{n-1} + (1-\pi)(1-p)[1-p+p\beta]^{n-1}}$$

[18] The numerator is $(\Pi+1)p - 1$; denominator is $(\Pi+1)p - \Pi$; both are positive if p is close to π.

and it is $\geq 1/2$ if $\pi p[p + (1-p)\beta]^{n-1} \geq (1-\pi)(1-p)[1-p+p\beta]^{n-1}$; but this follows from $(\star)$ and $p > 1/2$.

3.4 The Battle of Sexes with uncertainty on one side, $0 < \pi < 1$.

Recall that the game (from Osborne section 9.1) is an incomplete information version of the battle of sexes. For concreteness we think of player 1 as a boy and player 2 as a girl. The situation is that she may or may not like to be with him and he does not know it. So the two possibilities are the ones we know:

$$\begin{array}{ccc} & B & S \\ B & 2,1 & 0,0 \\ S & 0,0 & 1,2 \end{array} \qquad \begin{array}{ccc} & B & S \\ B & 2,0 & 0,2 \\ S & 0,1 & 1,0 \end{array}$$

where the left game has probability $0 < \pi < 1$. We denote by p the probability of B for player 1, and by x and y the probabilities of B for $(2,l)$ and $(2,r)$.

The pure equilibria To study the game we start looking for pure strategy equilibria.

If 1 plays B then 2's best response is BS. And against this $B \succ_1 S$ iff $2\pi > 1 - \pi$ i.e. $\pi > 1/3$. So for $\pi > 1/3$ we have the pure equilibrium (B, BS).

If 1 plays S then 2's best response is SB. Against this $S \succ_1 B$ iff $\pi > 2(1-\pi)$ i.e. $\pi > 2/3$. So for $\pi > 2/3$ we have the equilibrium (S, SB).

Therefore:

a) for $1/3 < \pi < 2/3$ equilibrium is (B, BS). Here π is large enough so that 1 grabs the occasion to meet 2 on the left if 2 plays BS, and $1 - \pi$ is large enough that if 2 plays SB then enjoying B in good company in the right game has higher expected utility than enjoying S in the left game.

b) for $\pi > 2/3$ there are two equilibria, (B, BS) and (S, SB) - those of the left game.

For $\pi < 1/3$ there is no equilibrium in pure strategies. In this case we are with high probability in the right game, where there is no pure equilibrium.

The mixed equilibrium Let us study the case $\pi < 1/3$. We know that 1 must mix, $0 < p < 1$. So 1 must be indifferent between B and S. This gives

$$2x\pi + 2(1-\pi)y = \pi(1-x) + (1-\pi)(1-y)$$
$$3\pi x + 3(1-\pi)y = 1$$
$$\pi x + (1-\pi)y = 1/3.$$

Equilibrium x, y must satisfy this condition for 1 to mix. But they must also be best replies to the equilibrium p. So we look at player 2. $(2, l)$ prefers B iff $p > 2/3$; $(2, r)$ prefers S iff $p > 1/3$. So:

For $p > 2/3$ 2's best response is BS to which (given $\pi < 1/3$) 1's best response is S, that is $p = 0$. This is out.

For $p < 1/3$ 2's best response is SB to which 1's best reply is B (again because $\pi < 1/3$), that is $p = 1$. This is out too.

For $1/3 < p < 2/3$ 2's best response is SS to which 1's best reply is obviously S that is $p = 0$. So no equilibria there. The only candidates are $p = 1/3$ and $p = 2/3$.

If $p = 1/3$ then $(2, l)$ plays S that is $x = 0$ and $(2, r)$ is indifferent; so we have equilibrium with y given by $\pi \cdot 0 + (1-\pi)y = 1/3$ that is $y = 1/3(1-\pi)$.

If $p = 2/3$ then $(2, r)$ then plays S that is $y = 0$; but then given $\pi < 1/3$ there is no $x \in [0, 1]$ which solves $\pi x + (1-\pi) \cdot 0 = 1/3$.

The conclusion is that for $\pi < 1/3$ there is a unique (mixed) equilibrium, where 1 plays $p = 1/3$, $(2, l)$ plays S for sure, and $(2, r)$ plays B with probability $y = 1/3(1-\pi) < 1/2$.

Moral of the story

Lets us look at the equilibrium payoff of the $(2, l)$ (the girl who likes him) in the mixed equilibrium. She plays S and he plays B with probability $1/3$ so she gets $(2/3) \cdot 2 \approx 1.33$. In the original BoS without uncertainty the mixed equilibrium is $p = 2/3, q = 1/3$ so that her payoff is $pq + 2(1-p)(1-q) = 2/9 + 2 \cdot 2/9 = 2/3 \approx 0.67$.

In other words: given that the boy lets the girl know that he wants to go out with her, if the girl who actually wants to go out with him too makes him believe she quite possibly (precisely with probability higher than $2/3$) does not, then she ends up *much* better off. Since - as we all know too well - this behavior is the rule, the moral of this story is: sorry guys, but girls are so much smarter than us!

4 Simple Auctions

The basic setup is the same as in the standard case, with each of the n bidders having a valuation of the object which is now her type $v_i \in T_i$, private information to her (we use the letter v for types in this context). Here again types are a continuum. For each i the action set is $A_i = \mathbb{R}_+$ containing her bid b_i. A strategy of player i is then a function $s_i \colon T_i \to \mathbb{R}_+$, and i's payoff depends on the bids profile $b = (b_1, \ldots, b_n)$ and her type: $u_i = u_i(b, v_i)$. The fact that u_i does not depend on v_{-i} characterizes the so called *private value* auctions. We will introduce common value auctions later. The payoff is different in the first- and second-price auctions as in the standard case. We deal with the two cases next. We start with private value auctions, then turn to common value ones.

4.1 Private value auctions

4.1.1 Second-price

In the second-price auction the highest bidder pays the second highest bid, and if there is a tie at the top the highest bidders share the object. Precisely,

$$u_i(b, t_i) = \begin{cases} 0 & \text{if } \max_{j \neq i} b_j > b_i \\ v_i - \max_{j \neq i} b_j & \text{if } \max_{j \neq i} b_j < b_i \\ (v_i - b_i)/m & \text{if } \max_{j \neq i} b_j = b_i \text{ and } b_j = b_i \text{ for } m \text{ players} \end{cases}$$

For this auction we only want to show that - as in the standard case - bidding your value, that is the strategy $s_i(v_i) = v_i$, is weakly dominant.

To see this observe that since v_{-i} is unknown to i so is j's bid $s_j(v_j)$. Therefore the other players' maximum bid $\max_{j \neq i} s_j(v_j) \equiv X$ is a random variable from the point of view of i. Let $P(F)$ be the probability that i assigns to the event $X \subseteq F$ (conditional on v_i). Consider bidding $b_i < v_i$. Then (always conditional on v_i), denoting as usual by $I\{F\}$ the indicator of the event F and applying the auction's rule we get (considering possible m-way ties)

$$\begin{aligned}
& Eu_i(b_i, b_{-i}, v_i) \\
&= \int_{\mathbb{R}_+} [I\{X < b_i\} * (v_i - x) + I\{X = b_i\} * (v_i - x)/m + I\{b_i < X < v_i\} * 0 + I\{X \geq v_i\} * 0] \, dP(x) \\
&\leq \int_{\mathbb{R}_+} [I\{X < b_i\} * (v_i - x) + I\{X = b_i\} * (v_i - x) + I\{b_i < X < v_i\} * (v_i - x) + I\{X \geq v_i\} * 0] \, dP(x) \\
&= Eu_i(v_i, b_{-i}, v_i).
\end{aligned}$$

Here a way to read the integral is to think that X is discrete and identify the integral with the sum of values times probability as in the discrete case.

For bids $b_i > v_i$ the argument is similar:

$$\begin{aligned}
& Eu_i(b_i, b_{-i}, v_i) \\
&= \int_{\mathbb{R}_+} [I\{X \leq v_i\} * (v_i - x) + I\{v_i < X < b_i\} * (v_i - x) + I\{X = b_i\} * (v_i - x)/m + I\{X > b_i\} * 0] \, dP(x) \\
&\leq \int_{\mathbb{R}_+} [I\{X \leq v_i\} * (v_i - x) + I\{v_i < X < b_i\} * 0 + I\{X = b_i\} * 0 + I\{X > b_i\} * 0] \, dP(x) \\
&= Eu_i(v_i, b_{-i}, v_i)
\end{aligned}$$

since now in $\{v_i < X < b_i\}$, and $\{X = b_i\}$ we have $t_i - x < 0$. This completes the argument.

4.1.2 First-price

Here we assume that types v_i are independent and each has distribution P uniform on $[0, 1]$.[19] In particular any single type has probability zero, and for all $x \in [0, 1]$ we have $P(v_i < x) = x$. Independence implies that on $[0, 1]^k$, $k \leq n$ the v_i's induce the product probability P^k which on rectangles $R = [x_1, y_1) \times \cdots \times [x_k, y_k)$ takes values $P^k(R) = \Pi_{i \leq k}(y_i - x_i)$; in particular,

[19] Underlying there is a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ on which the v_i are $[0, 1]$-valued random variables. The assumption is that these rv's are independent, and that on $[0, 1]$ they all induce the uniform distribution $P = \text{Leb}$.

The Lebesgue measure on $[0, 1]$ is most easily defined as the distribution F - where $F(x) = P(\omega \leq x)$ - given by $F(x) = x$. Its derivative is the constant density $f(x) = 1$, and by the fundamental theorem of calculus we have, for $0 \leq a \leq b \leq 1$,

$$P(\omega \in [a, b]) = F(b) - F(a) = \int_a^b f = b - a$$

so that Leb assigns to each interval a probability equal to its length.

for given i the probability that $v_j < x\ \forall j \neq i$ is $P^{n-1}([0,x)^{n-1}) = x^{n-1}$.

Action sets are again $A_i = \mathbb{R}_+$ for all i, containing bids b_i, and we use b for elements of $A = A_1 \times \cdots \times A_n$ and v for types profiles. The utility $u_i(b,v)$ is defined by the first-price rule as follows. If there is some $b_j > b_i$ then i gets zero; if b_i is the sole highest bid she gets $v_i - b_i$; if b_i is one of $m \leq n$ highest bids then i gets $(v_i - b_i)/m$. A strategy for player i is again a function $s_i\colon [0,1] \to \mathbb{R}_+$ which specifies the bid $s_i(v_i)$ of type v_i.

We look for equilibria where all the strategies s_i are strictly increasing and differentiable. Since the v_i's are iid uniform ties have probability zero, and since the strategies are increasing $P(s_i(v_i) < x) = P(v_i < s_i^{-1}(x)) = s_i^{-1}(x)$. We also restrict attention to existence of a *symmetric* equilibrium, where $s_i = s_j \equiv \varsigma$ for all i,j. We are going to show that such an equilibrium exists and is given by $\varsigma(v_i) = (1-1/n)v_i$. Thus in this equilibrium player of type v_i bids a little less than v_i.

Given strategy profile s_{-i}, the conditional expected utility which type v_i of player i gets if she bids b_i is simply the payoff $v_i - b_i$ multiplied by the probability that hers is the winning bid,[20] and given symmetry this is $(v_i - b_i) \cdot P^{n-1}([0,\varsigma^{-1}(b_i))^{n-1}) = (t_i - b_i) \cdot \left(\varsigma^{-1}(b_i)\right)^{n-1}$. The symmetric equilibrium best response $\varsigma(v_i)$ is given by the maximum of this expression with respect to b_i. To find it set the derivative equal to zero, obtaining that for $b_i = \varsigma(v_i)$ it must be

$$0 = -\left(\varsigma^{-1}(b_i)\right)^{n-1} + (n-1)(v_i - b_i)\left(\varsigma^{-1}(b_i)\right)^{n-2} \cdot \frac{d}{db_i}\varsigma^{-1}(b_i).$$

From this, using $b_i = \varsigma(v_i) \iff v_i = \varsigma^{-1}(b_i)$ and $d\varsigma^{-1}(b_i)/db_i = 1/\varsigma'\left(\varsigma^{-1}(b_i)\right) = 1/\varsigma'(v_i)$, we get that the equilibrium strategy must satisfy

$$\begin{aligned}
v_i^{n-1}\varsigma'(v_i) &= (n-1)(v_i - \varsigma(v_i))v_i^{n-2} \\
v_i^{n-1}\varsigma'(v_i) + (n-1)\varsigma(v_i)\cdot v_i^{n-2} &= (n-1)v_i^{n-1} \\
\frac{d}{dv_i}v_i^{n-1}\varsigma(v_i) &= (n-1)v_i^{n-1} \\
v_i^{n-1}\varsigma(v_i) &= \int_0^{v_i}(n-1)s^{n-1}ds = \frac{n-1}{n}v_i^n \\
\varsigma(v_i) &= \left(1 - \frac{1}{n}\right)v_i.
\end{aligned}$$

This proves our claim.

[20] In this case type spaces are not finite so the expectation is an integral. But since its value is so simple to compute there is no point in getting into formal details.

4.1.3 Revenue equivalence

In both first price and second price auctions the expected price paid by the winner, which is the expected revenue of the owner of the auctioned object, is the same (in the second price auction we consider the equilibrium where each player bids his valuation). We show this in the two-player case. In the first price auction each player i with valuation v_i bids $v_i/2$, so the expected price paid by the winner is half the highest realized valuation. In the second price auction each bidder bids his value v_i, and the winner - again the highest realized valuation - pays the second highest bid. *Given that the latter is less than the winner's valuation*, its expected value is again half of that (recall that valuations are distributed uniformly).

4.2 Common value auctions

This is taken from the book by Osborne (ch.9), to which the reader is referred for more details and for an account of the auctions of the radio spectrum in the USA. In the private values case discussed so far a player's valuation depends only on her type. In the common value case her valuation depends also on the other players' valuations. The possibility arises when players valuations depend on some (partial) information they have collected on the auctioned object - say an oil field, or a jar containing some unknown quantity of gold coins. We shall consider a simple case with two players. If my opponent has a low valuation it means she had bad news on the value of the object, and this pushes down my own valuation. The crucial point in this context is that if I win I must take into account the fact that my opponent's valuation is lower than mine.

In this context we must distinguish between players' values and the *signals* they received, representing their information on the value of the object. So a player's type is her signal t_i (unknown to the others); and her value depends on all signals, say $v_i = g(t_1, \ldots, t_n)$. This g function is assumed increasing in all arguments. Again actions are non-negative bids b_i, and the price $\rho(b)$ paid by the winner at the bid profile $b = (b_1, \ldots, b_n)$ is as before equal to the highest bid in first-price auctions and to the second highest bid in second-price auctions. For example if $b_i > \max_{j \neq i} b_j$ player i wins and gets $v_i - \rho(b) = g(t_1, \ldots, t_n) - \rho(b)$; etc.

We will study the case of two players with valuations $v_i = at_i + \gamma t_j$ with $a \geq \gamma > 0$, $i = 1, 2 \neq j$. Note that $\gamma = 0$ would mean private values. We assume the two signals are iid uniform on $[0, 1]$. Recall that this implies $P([0, x)) = x \;\forall x \in [0, 1]$.

4.2.1 Second Price

In such an auction an equilibrium is given by $b_i = (a+\gamma)t_i$, $i = 1, 2$. To prove this suppose player 2 is bids $b_2 = (a+\gamma)t_2$; we must show that then player 1's best response is $b_1 = (a+\gamma)t_1$. Keep in mind that ties have zero probability under uniform distribution so we use strict inequalities. Player 1 must maximize her expected payoff with respect to b_1. If she loses she gets zero, so her expected payoff is the probability of winning times the expected payoff if she wins.

Now $0 \leq b_2 < b_1 \iff 0 \leq t_2 < b_1/(a+\gamma)$ which has probability $b_1/(a+\gamma)$. So this is the winning probability. Payoff is expected value minus expected price paid, the latter being 2's bid. For both terms *we must condition on the fact that 1 wins*, that is on $b_2 \in [0, b_1)$; therefore proceeding as in section 2.10 we find that the conditional expected value of t_2 is $\frac{1}{2}b_1/(a+\gamma)$. Thus 2's expected bid is $(a+\gamma) * \frac{1}{2}b_1/(a+\gamma) = b_1/2$, and 1's expected valuation is $at_1 + \gamma * \frac{1}{2}b_1/(a+\gamma)$. In conclusion 1's expected payoff from b_1 is

$$\frac{b_1}{a+\gamma}\left[at_1 + \frac{\gamma b_1}{2(a+\gamma)} - \frac{b_1}{2}\right] = \frac{a}{2(a+\gamma)^2} b_1 \left[2(a+\gamma)t_1 - b_1\right]$$

which is a parabola with maximum at $b_1 = (a+\gamma)t_1$. By symmetry the same holds for player 2, and the result follows.

4.2.2 First price

The only change from the previous case here is that the price paid by the winner is now the highest bid. We next show that in this case the symmetric equilibrium has $b_i = (a+\gamma)t_i/2$. So suppose player 2's bid is $b_2 = (a+\gamma)t_2/2$. Since now $0 \leq b_2 < b_1 \iff 0 \leq t_2 < 2b_1/(a+\gamma)$, proceeding as before you should be able to show that 1 wins with probability $2b_1/(a+\gamma)$ and that the conditional expected value of t_2 is $b_1/(a+\gamma)$. At this point plug in t_2's expected value into $at_1 + \gamma t_2$, and write down 1's expected payoff, recalling that the price paid is now simply b_1. You should find that it is a parabola maximized at $b_1 = (a+\gamma)t_1/2$. Details in footnote.[21] Again symmetry completes the argument.

[21] From $0 \leq b_2 < b_1 \iff 0 \leq t_2 < 2b_1/(a+\gamma)$ we deduce directly that 1 wins with probability $2b_1/(a+\gamma)$ (twice as before of course). And proceeding as the previous case we find that the conditional expected value of t_2 is $b_1/(a+\gamma)$. Thus 1's expected value if she wins is $at_1 + \gamma b_1/(a+\gamma)$, and since she pays b_1 her expected payoff if she wins is

$$\frac{2b_1}{a+\gamma}\left[at_1 + \frac{\gamma b_1}{(a+\gamma)} - b_1\right] = \frac{2ab_1}{(a+\gamma)^2}\left[(a+\gamma)t_1 - b_1\right],$$

4.2.3 Revenue equivalence

That the expected price paid by the winner in the two types of auctions is the same holds for common value auctions as well (again considering the truthful equilibrium in the second price auction). In both cases we deduce (proceeding as before) that it is $(a+\gamma)\bar{t}/2$, where $\bar{t}$ is the highest realized type.

4.3 Double auction: trade under incomplete information

This is from Fudenberg-Tirole, to which the reader is referred for a more detailed discussion. Here player 1 is a seller of an object she can produce at cost c and player 2 is a potential buyer whose valuation of the object is v. These are the players' types, as usual privately known, and they are independent and uniformly distributed in $[0,1]$. *Both simultaneously place a bid*: the seller asks $b_1(c)$ and the buyer offers $b_2(v)$. If $b_1 > b_2$ no trade occurs; if $b_1 \leq b_2$ trade occurs at price $(b_1+b_2)/2$. If no trade occurs both get zero; if it does the seller gets $u_1 = (b_1+b_2)/2 - c$ and the buyer $u_2 = v - (b_1+b_2)/2$.

We show existence of an equilibrium with linear strategies, $b_1(c) = \alpha_1 + \beta_1 c \in [\alpha_1, \alpha_1+\beta_1]$ and $b_2(v) = \alpha_2 + \beta_2 v \in [\alpha_2, \alpha_2+\beta_2]$. The uniform distributions give constant densities $1/\beta_1$ and $1/\beta_2$ respectively.

Given the buyer's strategy, $b_2 \geq b_1 \iff v \geq (b_1-\alpha_2)/\beta_2$ so the seller's expected utility as a function of b_1 is

$$Eu_1(b_1) = \frac{1}{\beta_1}\int_{(b_1-\alpha_2)/\beta_2}^{1} [(b_1+\alpha_2+\beta_2 v)/2 - c]\, dv$$

which she maximizes with respect to b_1. This integral is elementary, and it is found to equal a constant times

$$(\alpha_2+\beta_2-b_1)\left[b_1 - \frac{4c-(\alpha_2+\beta_2)}{3}\right]$$

which is a parabola, maximized at the midpoint of the roots $b_1(c) = \frac{2}{3}c + \frac{\alpha_2+\beta_2}{3}$. Thus if the buyer's strategy is linear so is the seller's, with $\beta_1 = 2/3$ and $\alpha_1 = (\alpha_2+\beta_2)/3$. Similarly for the buyer, given the seller's strategy $b_1 \leq b_2 \iff c \leq (b_2-\alpha_1)/\beta_1$; therefore she maximizes

again a parabola, maximized at $b_1 = (a+\gamma)t_1/2$.

with respect to b_2 her expected utility

$$Eu_2(b_2) = \frac{1}{\beta_2} \int_0^{(b_2-\alpha_1)/\beta_1} [v - (\alpha_1 + \beta_1 c + b_2)/2]\, dc$$

which is another elementary integral, which can be checked to equal a constant times

$$(b_2 - \alpha_1) \left[\frac{4v - \alpha_1}{3} - b_2 \right]$$

and this is another parabola, maximized at $b_2(v) = \frac{2}{3}v + \frac{\alpha_1}{3}$. Thus if the seller's strategy is linear so is the buyer's, with $\beta_2 = 2/3$ and $\alpha_2 = \alpha_1/3$. From this and $\alpha_1 = (\alpha_2 + \beta_2)/3$ (found above) we get $\alpha_1 = 1/4$ and $\alpha_2 = 1/12$. Thus the equilibrium is

$$b_1(c) = \frac{1}{4} + \frac{2}{3}c \qquad b_2(v) = \frac{1}{12} + \frac{2}{3}v.$$

The relevant point is that trade occurs if $\frac{1}{4} + \frac{2}{3}c \leq \frac{1}{12} + \frac{2}{3}v$ that is if $v \geq c + 1/4$, while there would be gains from trade for both payers whenever $v \geq c$. Assuming that with complete information gains from trade are always realized, incomplete information considerably hampers trade opportunities. Indeed the probability of trade shrinks from $1/2$ to $9/32 \approx 0.28$.[22]

Little exercise Consider the following pair of strategies. The seller asks $1/2$ if $c \leq 1/2$ and 1 otherwise, and the buyer offers $1/2$ if $v \geq 1/2$ and zero otherwise. Is this an equilibrium? What is the probability of trade under this profile?

[22] Given uniform distributions of c and v the probability that $v \geq c$ is $\int_0^1 (1-c)dc = 1/2$, while the probability that $v \geq c + 1/4$ is $\int_0^{3/4} (1 - c - 1/4)dc = 9/32$.