MEASURING DOWNSIDE RISK AVERSION

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ABSTRACT. We provide comparative global conditions for downside risk aversion, which are similar to the ones studied by Ross for risk aversion. We define a coefficient of downside risk aversion, and study its local properties.

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1. INTRODUCTION

Identified by Menezes, Geiss and Tressler [10], an increase in downside risk is defined as a mean–variance preserving transformation —a mean–preserving spread plus a mean–preserving contraction— which shifts dispersion from the right to the left of a distribution (this is to be precise an 'elementary' change, a general one being given by a sequence of such transormations). To visualize we recall their introductory example: on the set $\{0, 1, 2, 3\}$ consider the two lotteries given by the probability vectors $p = (0, 3/4, 0, 1/4)$ and $p' = (1/4, 0, 3/4, 0)$; they have same mean and variance, and most people report preference for the former. In fact $p' = p + s + c$ where $s = (1/4, -1/2, 1/4, 0)$ is a spread and $c = (0, -1/4, 1/2, -1/4)$ is a contraction *occurring on the right of the spread*; thus p' is obtained from p by shifting dispersion from right to left, and the change from p to p' is the prototype increase in downside risk.

It is proved in [10] that all the functions with convex derivative *dislike* a probability change if and only if it is an increase in dowside risk; and accordingly, u is defined to be downside risk averse if it has convex derivative (if smooth: *positive* third derivative). On the other hand there is in the literature no *measure* of downside risk aversion, resulting either from an analysis paralleling the classical Arrow–Pratt development which lead to the risk aversion coefficient $r_u = -u''/u'$ (cfr. Pratt [11]), or following the subsequent approach by Ross (ref. [13]), which resulted in a 'stronger measure' of risk aversion. We show in this note that the latter (global) approach generalizes to the case of downside risk, while the former does not (section 2). However the analogue of r_u, namely $u'''/u' \equiv d_u$, has some natural, and interesting, local properties, which we report in section 3. In section 4 we briefly spell out the connection between d_u and the concept of local risk vulnerability defined by Gollier–Pratt [2]. [1]

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[1] We mention that two measures appear in the literature related to the third derivative of the utility function, but both apply to specific two–period problems. One is Prudence, introduced by Kimball [7] as $-u'''/u''$; he shows that prudence is a measure of the strength of an investor's motives to make precautionary savings in the standard two–period consumption–savings decision under uncertainty. On this see also Menegatti [9]. The other is Cautiousness: Huang [5, 6] defines it as $u''' \cdot u'/(u'')^2$ and uses it to analyse option buyers and sellers in a two–period equilibrium model, contrasting his conclusions with those of Leland [8].

2. Comparative Downside Risk Aversion

We provide here a comparative criterion for downside risk in the spirit of Ross [13]. The proposition which follows is the analogue of his main theorem for the case of downside risk aversion, with third instead of second derivatives and 3-convex instead of 2-convex order between random variables. For random variables X, Y say that Y has more downside risk than X if all functions with convex derivative prefer X to Y; from [10], equivalently Y is obtained from X via a sequence of mean variance preserving transformations. In the following proposition we take smooth functions defined on $\mathbb{R}$. [2]

Proposition 1. *For u, v increasing functions with convex derivative the following three conditions are equivalent:*

(i) $\exists \lambda > 0\ \forall x, y \quad \frac{u'''(x)}{v'''(x)} \geq \lambda \geq \frac{u'(y)}{v'(y)}$

(ii) $\exists \phi \colon \mathbb{R} \to \mathbb{R}$ *with* $\phi' \leq 0, \phi''' \geq 0$ *and* $\lambda > 0$ *such that* $u = \lambda v + \phi$

(iii) If Y has more downside risk than X, $\mathbb{E}u(Y) = \mathbb{E}u(X - \pi_u)$ *and* $\mathbb{E}v(Y) = \mathbb{E}v(X - \pi_v)$, *then* $\pi_u \geq \pi_v$.

Proof. That (i) implies (ii) is evident: defining $\phi = u - \lambda v$ and differentiating one obtains $\phi' = u' - \lambda v'$ and $\phi''' = u''' - \lambda v'''$, both non-positive given the signs of the derivatives of u and v.

(ii) $\Rightarrow$ (iii) is as in Ross, noting that $\mathbb{E}\phi(Y) \leq \mathbb{E}\phi(X) \leq \mathbb{E}\phi(X - \pi_v)$ (first inequality from $\phi''' \geq 0$, second from $\phi' \leq 0$). To spell it out:

$$\begin{aligned}\mathbb{E}u(X - \pi_u) &= \mathbb{E}u(Y) = \lambda\mathbb{E}v(Y) + \mathbb{E}\phi(Y) \\ &\leq \lambda\mathbb{E}v(Y) + \mathbb{E}\phi(X) = \lambda\mathbb{E}v(X - \pi_v) + \mathbb{E}\phi(X) \\ &\leq \lambda\mathbb{E}v(X - \pi_v) + \mathbb{E}\phi(X - \pi_v) = \mathbb{E}u(X - \pi_v).\end{aligned}$$

We turn to (iii) $\Rightarrow$ (i). X, Y will be finite lotteries, and to facilitate comparison we describe their distributions on the union of their supports, which is, with $x, y \in \mathbb{R}$, $v = x + 3\epsilon$ and $\epsilon > 0$, the set $\{x-\epsilon, x, x+\epsilon, v-\epsilon, v, v+\epsilon, y\}$; the probabilities are, for X and Y respectively, $(0, p/2, 0, p/4, 0, p/4, 1-p)$ and $(p/4, 0, p/4, 0, p/2, 0, 1-p)$, with $p \in (0, 1)$. So Y is obtained from X via a mean preserving spread plus a contraction on its right, offsetting the change in variance; it can be checked directly that the change from X to Y is a mean–variance preserving transformation according to the definition of Menezes, Geiss and Tressler [10]; hence, according to their result, Y has more downside risk than X.

By differentiating the equality $\mathbb{E}u(Y) = \mathbb{E}u(X - \pi_u)$ with respect to ϵ one easily finds, with $\pi_u = \pi_u(\epsilon)$: $\pi_u'(0) = \pi_u''(0) = 0$, and

$$\pi_u'''(0) = \frac{\frac{9}{2} p\, u'''(x)}{pu'(x) + (1-p)u'(y)}.$$

Given this, the conclusion is again as in Ross. Indeed given this, from $\pi_u \geq \pi_v$ it follows that for any choice of x, y, p it must be

$$\frac{p\, u'''(x)}{pu'(x) + (1-p)u'(y)} \geq \frac{p\, v'''(x)}{pv'(x) + (1-p)v'(y)},$$

that is

$$\frac{u'''(x)}{v'''(x)} \geq \frac{pu'(x) + (1-p)u'(y)}{pv'(x) + (1-p)v'(y)},$$

[2] A word of caution on the fact that we take functions defined on all of $\mathbb{R}$: the remarks of Pratt [12] on Ross obviously apply to the present setting too. For example, if u and v have constant risk aversions $a > b > 0$, it is easily checked that condition (i) in the proposition which follows cannot be satisfied for all x, y, for it requires $x - y \leq [2\ln(a/b)]/(a - b)$.

which is true for all x, y, p iff for all x, y it is

$$\frac{u'''(x)}{v'''(x)} \geq \frac{u'(y)}{v'(y)},$$

which implies (i). □

This result suggests analyzing the global properties of what seems the most natural candidate for an analogue of the Arrow–Pratt coefficient r_u, namely the 'downside risk aversion coefficient' defined as

$$d_u(x) = \frac{u'''(x)}{u'(x)}. \tag{1}$$

Does this measure have the analogue of the global property which the Arrow–Pratt coefficient r_u has (namely, that for u, v vonNeumann–Morgenstern utilities, $r_u > r_v$ if and only if $u = k(v)$ with k increasing concave)? The answer to this is no: neither of the transformations k and h defined respectively by $u = k(v)$ and $u' = h(v')$ behave as one would hope. As for k, the relation $u = k(v)$ with k increasing concave with positive third derivative, which is the natural candidate for an equivalence with $d_u > d_v$, is in fact stronger than it. Precisely, one can show that k is increasing concave with positive third derivative iff $r_u \geq r_v$ and

$$d_u - d_v \geq 3r_v(r_u - r_v).$$

Looking at the transformation $u' = h(v')$ one finds that $d_u > d_v$ is stronger than $r_u > r_v$ and weaker than $p_u > p_v$, where p_u is the prudence coefficient defined by Kimball [7] as $p_u = -u'''/u''$. But again, the global properties of h equivalent to $d_u > d_v$ are untidy; we shall not present the details.

However, the coefficient d_u does have some interesting local properties, which we find worth reporting.

3. Local analysis: d_u and downside risk premium

We relate d_u to a downside risk premium, paralleling the result connecting the Arrow–Pratt coefficient of risk aversion to the classical risk premium. *Notation*: a lottery wich takes values $x_1, \ldots, x_n$ with respective probabilities $p_1, \ldots, p_n$ will be written as $(p_1, x_1; p_2, x_2; \ldots; p_n, x_n)$.

3-convex stochastic order. Consider the class of the random variables X (interpreted as portfolios) with support contained in an interval $[-a, a]$ with mean zero and second moment fixed at μ_2 ($\leq a^2$). On this set consider the 3-convex order $\succcurlyeq$ defined via preference by all downside risk averse u's, that is $X \succcurlyeq Y$ if $\mathbb{E}u(X) \geq \mathbb{E}u(Y)$ for all u with convex derivative. By Denuit, De Vylder and Lefèvre [1] Proposition 4.1, there are $\succcurlyeq$–best and $\succcurlyeq$–worst elements in the given set, call them X_M, X_m (dependence on a and μ_2 suppressed). Precisely,

$$X_M = \left(\tfrac{a^2}{a^2+\mu_2}, -\tfrac{\mu_2}{a}; \tfrac{\mu_2}{a^2+\mu_2}, a\right), \quad X_m = \left(\tfrac{\mu_2}{a^2+\mu_2}, -a; \tfrac{a^2}{a^2+\mu_2}, \tfrac{\mu_2}{a}\right).$$ [3]

Since X_M and X_m are the lotteries with least and most downside risk, the downside risk aversion of u should be reflected in the strength of preference of X_M over X_m. To obtain such a link we construct a *downside risk premium* as follows: take a downside risk averse u, and define $\pi_u(w, a, \mu_2)$ as the premium which makes

[3] In [1] an equivalent definition of $\succcurlyeq$ in terms of finite differences is given. Incidentally, notice that given fixed first and second moments, the order is equivalently defined by the class of *increasing concave* downside risk averse u's.

$X_M - \pi$ and $X_m + \pi$ indifferent at income level w, i.e. as the number π solving the equation

$$\mathbb{E}u(w + X_M - \pi) = \mathbb{E}u(w + X_m + \pi). \qquad (2)$$

The following relation between π_u and d_u then holds:

Proposition 2. *Fix income w and sufficiently small a. Then $d_u(w) \geq d_v(w)$ iff $\pi_u(w, a, \mu_2) \geq \pi_v(w, a, \mu_2)$ for all $\mu_2 \leq a^2$.*

Proof. Letting $Y = X_M = -X_m$, equation (2) reads $\mathbb{E}u(w+Y-\pi) = \mathbb{E}u(w+\pi-Y)$; expand this around w up to third order to find $d_u(w) \cong 6\pi/\mathbb{E}(Y-\pi)^3$. By Menezes–Geiss–Tressler [10], Theorem 2 and Proposition 2, $\mathbb{E}Y^3 > 0$; so $\mathbb{E}(Y - \pi)^3/\pi = \mathbb{E}Y^3/\pi - 3\mu_2 - \pi^2$ decreases with π, and the proposition follows. □

Betting interpretation. Consider the following two fair bets on a 0–1 valued coin which falls 1 with probability θ (and 0 with probability $1-\theta$). The first, which we call B, for 'buying', is betting θ on the coin falling 1, that is, winning (net) $1 - \theta$ with probability θ and losing θ with probability $1 - \theta$; in lottery notation, $B = (\theta, 1 - \theta; 1 - \theta, -\theta)$. The second is $S = -B$, 'selling', where one bets $1 - \theta$ on 0: $S = (\theta, -(1 - \theta); 1 - \theta, \theta)$. For θ small B involves losing little with high probability, S losing much with low probability; for large θ the roles are reversed. Now suppose θ is small: do you prefer B or S?

It turns out that neither of B, S 1st– or 2nd–order stochastically dominates the other, and B 3rd–order dominates S iff $\theta \leq 1/2$, the opposite occuring for $\theta \geq 1/2$. So (Menezes–Geiss–Tressler [10] Proposition 3) all downside risk averse individuals would prefer B (resp. S) for $\theta \leq 1/2$ (resp. $\geq 1/2$); that is, they would always prefer to put down, and stand to lose, the minimum between θ and $1 - \theta$.

Taking $\theta \leq 1/2$ to fix ideas, it is again natural to say that the downside risk effect is more marked the stronger the preference for B over S. To make this precise consider, at fixed $\theta \leq 1/2$, betting on 1 the amount $p > \theta$, i.e. the lottery $B(\theta, p) := (\theta, 1 - p; 1 - \theta, -p)$; the larger p, the worse this becomes; and the better becomes $S(\theta, p) = -B(\theta, p)$. By downside risk aversion $B(\theta) \equiv B(\theta, \theta)$ is preferred to $S(\theta) \equiv S(\theta, \theta)$; this preference is stronger the higher the price $p(\theta) = \theta + \pi(\theta)$ which makes $B(\theta, p)$ indifferent to $S(\theta, p)$; so the premium $\pi(\theta)$ appears to be the appropriate downside risk premium in this context. Notice incidentally that $B(\theta, \theta + \pi) = B(\theta) - \pi$, and $S(\theta, \theta + \pi) = S(\theta) + \pi$. The proposition which follows shows that modulo rescaling, the $\pi(\theta)$ just defined is the same as the π_u defined previously via the 3–conxex order. [4]

Lemma. *$X_M(a, \mu_2) = \alpha B(\theta)$ and $X_m(a, \mu_2) = \alpha S(\theta)$, with (a, μ_2) and (α, θ) in the one–to–one correspondence (for $\theta \leq 1/2$) given by*

$$\alpha = a + \mu_2/a, \quad \theta = \mu_2/(a^2 + \mu_2).$$

Proof. Just check that supports and probabilities are the same for the lotteries claimed to be equal. Notice $\theta \leq 1/2$ (since $\mu_2 \leq a^2$). □

Defining $\pi_u(w, \alpha, \theta)$ as the π such that $w + \alpha B(\theta) - \pi$ is u–indifferent to $w + \alpha S(\theta) + \pi$, we then have the following direct consequence of the lemma and the previous proposition:

Proposition 3. *Fix income w and α sufficiently small. Then $\pi_u(w, \alpha, \theta)$ is higher for all $\theta \leq 1/2$ iff $\pi_u(w, a, \mu_2)$ is higher for all $\mu_2 \leq a^2$ (in u–space).*

[4]The case $\theta \geq 1/2$ is in all symmetric. In fact $B(1 - \theta, p) = S(\theta, 1 - p)$ and $S(1 - \theta, p) = B(\theta, 1 - p)$; so also, via the indifference $B(\theta, p(\theta)) \sim_u S(\theta, p(\theta))$, $p(1 - \theta) = 1 - p(\theta)$, and therefore $\pi(1 - \theta) = -\pi(\theta)$. Hence the right half–interval is a copy of the left one, and we will restrict attention to the latter in the text.

Links to Literature. Hanson and Menezes [3] characterize the *sign* of the third derivative of u through preference between pairs of two–point lotteries in a family, namely of $X_1(h) = (.25, 2h; .75, 0)$ over $X_2(h) = (.75, h; .25, -h)$. We characterize its *size* using the family $\alpha B(\theta), \alpha S(\theta)$.

A pair of two-point lotteries was used by John Hicks ([4], p.118) to point out the relevance of skewness (i.e. as we now know of third derivatives), namely: $X_1 = (.9, 4; .1, 14)$, $X_2 = (.9, 6; .1, -4)$; this is an $(\alpha B(\theta), \alpha S(\theta))$ pair. Reference to Hicks' work is in footnote 5 of Hanson–Menezes [3].

4. The coefficient d_u and local risk vulnerability

We conclude by observing that decreasing d_u is equivalent to the concept of local risk vulnerability introduced by Gollier–Pratt in [2]. Recall that risk vulnerability means that any unfair background risk increases risk aversion; the *local* version refers to sufficiently small risks (at each income level). Gollier–Pratt [2] show that local risk vulnerability is equivalent to $2rr' - r'' \leq 0$ and $r' \leq 0$. Since $d = r^2 - r'$, this is the same as $d' \leq 0$ and $r' \leq 0$; now by elementary analysis one can show that under positive and bounded risk aversion $d' \leq 0$ implies $r' \leq 0$. So we have:

Proposition 4. *Assume $r_u(x) \geq 0$ and bounded as $x \to \infty$. Then decreasing d_u is equivalent to local risk vulnerability.*

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