Examples of extensive form games with perfect information[1]

For extensive form games we follow closely OR chapter 6. Recall the basic definition of an extensive form game with perfect information:[2]

Definition (OR definition 89.1). An extensive form game with players in set N consists of the following three components:

1. A set H of finite or infinite sequences - $h \in H$ is a "history of moves" - such that:
    - The empty sequence $\emptyset$ belongs to H
    - If $\left(a^k\right)_{k=1,\dots,K} \in H$ (where K may be infinite) and $L < K$ then $\left(a^k\right)_{k=1,\dots,L} \in H$ - so that initial segments of histories are histories
    - If an infinite sequence $\left(a^k\right)_{k=1}^{\infty}$ satisfies $\left(a^k\right)_{k=1,\dots,K} \in H$ for all positive K then $\left(a^k\right)_{k=1}^{\infty} \in H$.

We say that history $h = \left(a^k\right)_{k=1,\dots,K}$ is *terminal* if it is infinite or there is no a^{K+1} such that $\left(a^k\right)_{k=1,\dots,K+1} \in H$. The other two components are:

2. A function P - the *player function* - which assigns a player $P(h) \in N$ to any non-terminal history h; and
3. For each player $i \in N$ a *preference relation* $\succsim_i$ on the set of terminal histories.

The set of terminal histories is denoted by Z. At history $h \in H \setminus Z$ player $P(h)$ chooses an action from the set $A(h) = \{a \colon (h, a) \in H\}$, where (h, a) denotes history h followed by a.[3]

The remaining definitions, interpretation of the game and main results will be taken from OR chapter 6 and are to be studied from there. We continue here with some examples which seem interesting enough to be studied.

Inefficiency of equilibrium

First let us remark that, as in normal form games, also in sequential games the outcome may be inefficient. The following is a simple example where the subgame perfect equilibrium seems the only reasonable way to play the game:

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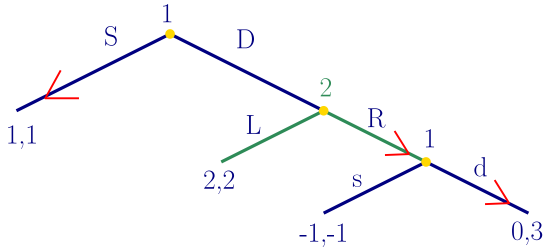
[1] S Modica 2024

[2] The qualification on information refers to the fact that each player, whenever it is her turn to play, knows the moves chosen by the players who have played before her.

[3] In several textbooks the definition of extensive form games is different than the one above (although of course equivalent to it). It is covered, just in case you need to know it, in the Appendix at the end of the present file.



The history marked with arrows is the unique subgame perfect equilibrium of the game, yielding only $(1,1)$ to the players. The payoff $(2,2)$ is obtained with the strategy profile (Ds, L), which as you can easily verify is a Nash equilibrium.

Cournot vs Stackelberg (two players)

We consider a market with two firms $i = 1, 2$ to keep things simple. Choices are non-negative quantities q_i. We assume that demand and cost are linear: demand price is $p(q_1, q_2) = \alpha - \beta(q_1 + q_2)$, production cost is $c_i(q_i) = cq_i$ for $i = 1, 2$. We assume $\alpha > c$ (otherwise no firm would produce). As usual $Q := q_1 + q_2$.

Cournot (simultaneous choices)

As we know payoffs are the profits: letting $\sigma = (\alpha - c)/\beta$ we have

$$\pi_i(q_1, q_2) = q_i \left[\alpha - \beta Q - c\right] = \beta q_i \left[\sigma - Q\right].$$

The Nash equilibrium is given by the condition that firm i maximizes π_i given q_j. As we know $\pi_i = \beta q_i \left[(\sigma - q_j) - q_i\right]$ is a parabola in q_i with maximum $(\sigma - q_j)/2$. In other words i's best response is

$$B_i(q_j) = \frac{\sigma - q_j}{2}$$

Nash equilibrium solves the system $q_i = B_i(q_j)$ for $i \neq j = 1, 2$ that is

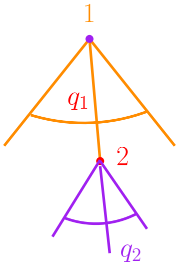
$$q_1 = \frac{\sigma - q_2}{2}, \quad q_2 = \frac{\sigma - q_1}{2}.$$

Subtract the second from the first to get $q_1 - q_2 = (q_1 - q_2)/2$ that is $q_1 = q_2$; then from the first we obtain

$$q_i^C = \frac{1}{3}\sigma \qquad Q^C = \frac{2}{3}\sigma.$$

Stackelberg (sequential game)

In this case player 1 chooses first. Note that the resulting game has finite horizon but it is not finite.



Backward induction. Given q_1 player 2 solves $\max \pi_2(q_1, q_2)$ with solution (as above)

$$q_2(q_1) = \frac{\sigma - q_1}{2}.$$

Then, in the Stackelberg equilibrium, player 1 solves $\max \pi_1(q_1, q_2)$ subject to the above value of q_2, that is maximizes

$$\begin{aligned} q_1\left[\alpha - \beta(q_1 + q_2(q_1)) - c\right] &= q_1\left[\alpha - \beta\left(q_1 + \frac{\sigma - q_1}{2}\right) - c\right] \\ &= \beta q_1\left[\sigma - \left(q_1 + \frac{\sigma - q_1}{2}\right)\right] = \frac{\beta}{2} q_1\left[\sigma - q_1\right] \end{aligned}$$

which is a parabola with maximum $q_1 = \sigma/2$. Note that here player 1 takes into account that player 2 is maximizing. The result is

$$q_1^S = \frac{1}{2}\sigma \qquad q_2^S = \frac{1}{4}\sigma \qquad Q^S = \frac{3}{4}\sigma$$

Compare

Quantities:

$$q_1^S > q_1^C \qquad q_2^S < q_2^C < q_1^S \qquad Q^S > Q^C.$$

Observe that

$$\frac{d}{dq_1}\pi_1(q_1, q_2(q_1)) = \frac{\partial \pi_1}{\partial q_1} + \frac{\partial \pi_1}{\partial q_2}\frac{dq_2}{dq_1} > \frac{\partial \pi}{\partial q_1}$$

because in the last term both factors are negative. So taking 2's reaction into account player 1 produces more than he would if she did not. Indeed

$$q_1^S = \frac{1}{2}\sigma > B_1(q_2^S) = \frac{\sigma - q_2^S}{2}$$

and as a matter of fact $q_1^S = B_1(0)$: player 1 behaves as if he was a monopolist.

Profits: in the Cournot equilibrium we have

$$\pi_i = q_i\left[\alpha - \beta Q - c\right] = \frac{1}{9}\beta\sigma^2 \qquad \pi_1 + \pi_2 = \frac{2}{9}\beta\sigma^2 \approx 0.22\beta\sigma^2$$

In Stackelberg equilibrium

$$\pi_1 = \frac{1}{8}\beta\sigma^2 \qquad \pi_2 = \frac{1}{16}\beta\sigma^2 \qquad \pi_1 + \pi_2 = \frac{3}{16}\beta\sigma^2 \approx 0.19\beta\sigma^2$$

Therefore the increase in profits for the leader is more than offset by the decrease in profits for the follower. Since total production is higher the selling price is lower than in Cournot.

The role of information

Here again player 2's information lowers her payoff. The problem for player 2 is that 1 knows that 2 knows q_1. *Therefore here again more information hurts,* and the reason is again that your opponent knows you possess this information.

Exercise. We know that the Stackelberg outcome is not a Nash Equilibrium of the Cournot game. Who would want to deviate? (just read well the above, the answer is there)

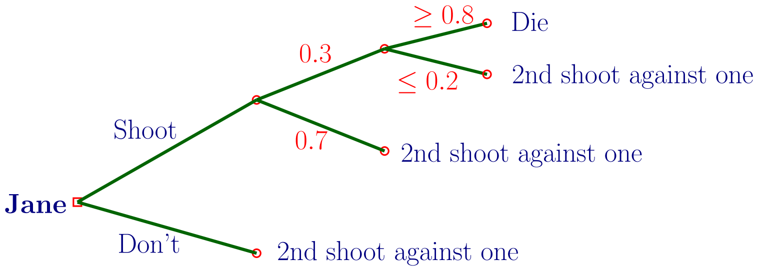
Three-way duels (the game of your life?)

There are three players in a duel: *Jane*, a young girl, who kills with probability 0.3 if she shoots (that's you); *Cowboy* who kills with probability 0.8; and *Killer*, who kills with probability 1. There are two rounds, in each of which each alive player has a shot; first Jane, then Cowboy then Killer. At any round the player who plays may choose to shoot at any alive player or not to shoot. At the end of the second round the players who are alive share a prize M. Players have equal preferences, represented by a u such that $u(0) < u(M/3) < u(M/2) < u(M)$, where $u(0)$ is what you get if you are not alive at the end of the game.

We look for the subgame perfect equilibrium of the game. In particular, we are interested in what Jane has to do in the first round. We begin by showing that in the first stage, after Jane's turn, if all are alive then Cowboy will shoot at Killer, and if he fails Killer will kill him. This implies in particular that at the end of the first round one and only one between C and K is alive, whatever J does.

We use initials. Start from the second assertion: if J and C are alive K kills C. This is because the one left will shoot at him and C has higher probability of killing him. The first assertion is that C will shoot at K rather than J, and the argument is the following. If both remain alive he's dead (killed by K), so he'll try to kill one. And if only one remains alive he or she will shoot at him (K in the first stage or J in the second). Again he prefers higher probability of remaining alive, therefore he will try to kill K.

Now look at Jane at the beginning of the game. If she does not shoot only one of C and K will remain alive at the end of the first stage and she will have a second shoot against him. Suppose on the contrary that she shoots. With probability 0.7 she fails, and again she will have a second shoot against the one alive. If instead she kills C or K (probability 0.3) then the one alive will shoot at her, and she will face the following prospect: die with probability $\geq .8$, or have a second shoot at him (with probability $\leq .2$). Dying is obviously worse than a second shoot, so this prospect is worse than a second shoot for sure. Therefore: if she does not shoot she has a second shoot for sure; if she does she is worse off, because the prospect she faces is: with probability 0.7 second shoot for sure, with probability 0.3 a dispreferred prospect. Jane's decision problem is depicted in the figure below:



In conclusion, in the first stage it is in Jane's interest not to kill, and she won't shoot. The rest of the equilibrium is clearly as follows: in stage 2 Jane shoots at the single other player alive, and if she fails he shoots back.

Exercise. In the equilibrium found above compute, for Jane and Killer, the probability of remaining alive at the end of the game (*Answer:* 41.2% and 14% respectively!) and the probability of getting M (*Answer:* 30% and 14% respectively). Pretty surprising. The computation is in footnote.[4]

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[4] Probability of remaining alive for Killer:

$$0.2[\text{prob. Cowboy fails}] * 0.7[\text{prob. Jane fails}] = 0.14$$

this is also equal to the probability that he wins M. For Jane we compute the probability of dying (starting at her second turn):

$$0.2[\text{Cowboy misses Killer}] * 0.7[\text{she misses Killer}] * 1[\text{Killer kills her}] \\ + 0.8[\text{Cowboy kills Killer}] * 0.7[\text{she misses Cowboy}] * 0.8[\text{Cowboy kills her}]$$

that is $0.14 + 0.56 \cdot 0.8 = 0.588$ so that Jane has $1 - 0.588 = 0.412 = 41.2\%$ chance of remaining alive. The probability of winning M is

$$0.2[\text{Cowboy misses Killer}] * 0.3[\text{she kills Killer}] \\ + 0.8[\text{Cowboy kills Killer}] * 0.3[\text{she kills Cowboy}] = 0.30$$

that is more than twice as much as the Killer.

Moral of the Story and a Modification of the game

When you, young and small, start moving in a working environment with lots of bigger guys around, don't rush to raise your voice too much. It may be better to stay covered for a while. Unless... well, unless the situation is such that if you waive your first chance you will not have another one, as in the following modification of the above game.

The modification is simply that if you choose not to shoot in the first round you will not be given the move in the second. In this case Jane has to shoot in the first round. *If you want to try, these suggestions are for you:* It suffices to show that shooting for example at Killer is better than no shooting. So compute expected payoff from not shooting (*Sol.* $0.84u(0) + 0.16u(M/2)$) and the probability of dying if you shoot at Killer (*Sol.* 0.6852). Conclude from this that shooting is better.

Solution. If you don't shoot in the first round, since the rest of the game is the same for the others, you get

$$\begin{aligned} 0.8[\text{Cowboy kills Killer}] * \Big(0.8u(0)[\text{Cowboy kills you}] + 0.2u(M/2)[\text{Cowboy misses you}]\Big) \\ + 0.2u(0)[\text{Cowboy misses Killer and Killer kills you}] \end{aligned}$$

that is $0.84u(0) + 0.16u(M/2)$. If you shoot at Killer the probability of dying is

$$\begin{aligned} 0.3[\text{u kill K}] * \Big[0.8[\text{Cb kills u}] + 0.2[\text{Cb misses u}] * 0.7[\text{u miss Cb}] * 0.8[\text{Cb kills u}]\Big] \\ + 0.7[\text{u miss K}]\Big[0.8[\text{Cb kills K}]\big(0.7[\text{u miss Cb}] * 0.8[\text{Cb kills u}]\big) \\ + 0.2[\text{Cb misses K}] * 0.7[\text{u miss K \& K kills u}]\Big] \\ = 0.3\Big[0.8 + 0.2 * 0.7 * 0.8\Big] + 0.7\Big[0.8 * 0.7 * 0.8 + 0.2 * 0.7\Big] \\ = 0.24 + 0.14 * 0.24 + 0.64 * 0.49 + 0.14 * 0.7 = 0.6852 \end{aligned}$$

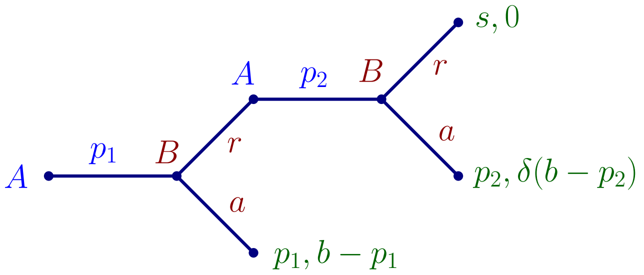
so you get $0.68u(0) + 0.32u(M/2)$.

A two-stage bargaining game

This example takes into account that future benefits may be discounted (one Euro now is better than one Euro next year). The message of the example is this: when you bargain, try to be patient.

Alice wants to sell a good to Bob. She values the good at s, Bob at $b \in \{b_l, b_h\}$ with $0 < b_l < b_h$; $b = b_h$ with probability π and $b = b_l$ with probability $1 - \pi$. Assume $b_l > s$ so that it is efficient to trade. If Alice has a single chance she should choose between two alternatives: propose a selling price $p = b_l$ or or a selling price $p = b_h$. Indeed, proposing $p > b_h$ would mean Bob refuses for sure; $p < b_l$ would be accepted for sure but so would $p = b_l$;[5] and $b_l < p < b_h$ would be accepted only if $b = b_h$ so better propose $p = b_h$. If she proposes $p = b_l$ she gets b_l for sure; if she proposes $p = b_h$ she gets $\pi b_h + (1 - \pi)s$. Let us assume that $\pi b_h + (1 - \pi)s \leq b_l$, so that if Alice has a single chance she offers the good at $p = b_l$ and Bob accepts.

Let us now suppose that Alice has two chances: she can propose a first selling price p_1, and if Bob refuses she can try one more time with a p_2 proposal. If Bob accepts p_1 then payoffs are $(p_1, b - p_1)$. If he refuses then Alice proposes p_2; if Bob refuses then payoffs are $(s, 0)$; if Bob accepts payoffs are $(p_2, \delta(b - p_2))$ - where $0 < \delta < 1$ is the discount factor which Bob applies to future payoffs. The game can be described as in the picture below (a for accept, r for refuse):



If $p_1 = b_l$ Bob accepts for sure (because $p_2 \geq b_l$). Looking for a SPNE we work backwards and see what happens if $p_1 > b_l$ and Bob refuses p_1. We have to compute the updated probability $P(b_h \mid r)$; given $p_1 > b_l$ we have $P(r \mid b_l) = 1$, so by Bayes rule we obtain

$$P(b_h \mid r) = \frac{P(b_h)P(r \mid b_h)}{P(b_h)P(r \mid b_h) + P(b_l)P(r \mid b_l)} = \frac{\pi P(r \mid b_h)}{\pi P(r \mid b_h) + 1 - \pi} \leq \pi$$

since $P(r \mid b_h) \leq 1$. But recall that with $P(b_h) = \pi$ and a single shot Alice would offer the good at $p = b_l$; a fortiori she would do the same with $P(b_h \mid r) \leq \pi$. We conclude that if $p_1 > b_l$ and Bob refuses, in the second stage Alice will set $p_2 = b_l$, Bob will accept and the resulting payoff vector is $(b_l, \delta(b - b_l))$.

Let us now look at Bob's best response in stage 1. If $b = b_l$ he will only accept $p_1 \leq b_l$. Suppose $b = b_h$: if he refuses p_1 he knows that $p_2 = b_l$ which he will accept getting $\delta(b_h - b_l)$; if he accepts he gets $b_h - p_1$. So if $b = b_h$ Bob accepts p_1 as long as $b_h - p_1 \geq \delta(b_h - b_l)$ that is

$$p_1 \leq b_h - \delta(b_h - b_l) = \delta b_l + (1 - \delta)b_h \equiv p^*.$$

Thus if $b = b_h$ Bob accepts $p_1 \leq p^*$. Note that $p^* > b_l$ since $\delta < 1$.

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[5] If $b = b_l$ he is indifferent between accepting and refusing; we assume he accepts.

Now if Alice offers $p_1 = b_l$ she gets b_l for sure, while if she offers p^* she gets $\pi p^* + (1-\pi) b_l > b_l$ (of course other offers yield lower payoffs). This is the unique subgame perfect equilibrium of the game: Alice offers $p_1 = p^*$ and Bob accepts if $b = b_h$. If $b = b_l$ Bob refuses and in the second round Alice proposes $p_2 = b_l$. So: if $b = b_l$ trade occurs in the second period and payoffs are $b_l, 0$; if $b = b_h$ trade occurs in the first period and payoffs are $p^*, b_h - p^*$.

The point here is that a buyer with high valuation of the good ($b = b_h$) will get better trade terms (lower proposed p^*) the more patient he is. Indeed for low δ (impatient buyer) p^* is close to b_h, while $p^* \to b_l$ as $\delta \to 1$.

Mother and Son

First scenario

Son chooses the amount s of his income y to save for college. Mother *then* observes s and chooses the amount t out of her income $Y > y$ to transfer to her son. Utilities u, v are strictly increasing concave, and δ is a discount factor. Payoffs for Son and Mother are respectively

$$\begin{aligned} \pi^S(s,t) &= v(y-s) + \delta v(s+t) \\ \pi^M(s,t) &= u(Y-t) + \alpha \pi^S(s,t), \end{aligned}$$

where α is a constant weight measuring how much the mother cares about the son's welfare.

We look for Subgame Perfect Equilibrium by backward induction: given s find Mother's choice $t(s)$; then given this function find Son's optimal choice s^*. We assume interior solutions: for any $0 \leq s \leq y$ it is $0 < t(s) < Y$, and that given $t(s)$ it is $0 < s^* < y$.

Then Mother's choice $t(s)$ is characterized by the first order condition

$$u'(Y - t(s)) = \alpha \delta v'(s + t(s))$$

where observe that $t'(s) < 0$:

$$\begin{aligned} -u''(Y - t(s)) t'(s) &= \alpha \delta v''(s + t(s))(1 + t'(s)) \\ t'(s) &= -\frac{\alpha \delta v''(s + t(s))}{[\alpha \delta v''(s + t(s)) + u''(Y - t(s))]} < 0 \end{aligned}$$

Son maximizes $\pi^S(s, t(s))$; the solution is given by the first order condition

$$v'(y - s^*) = \delta v'(s^* + t(s^*)) \left[1 + t'(s^*)\right].$$

The two conditions above characterize the SPNE $s^*, t^* = t(s^*)$. The equilibrium is Pareto inefficient. To see this observe that efficient savings should equate marginal benefit and marginal cost, that is $v'(y-s^*) = \delta v'(s^* + t(s^*))$, while in equilibrium marginal benefit is $\delta v'(s^* + t(s^*))\,[1 + t'(s^*)] < \delta v'(s^* + t(s^*))$ - so there is too little savings.[6] Formally

$$\frac{\partial \pi^S(s^*, t^*)}{\partial s} = -v'(y - s^*) + \delta v(s^* + t^*) = -\delta v'(s^* + t^*)t'(s^*) > 0$$

so there is $\hat{s} > s^*$ such that $\pi^S(\hat{s}, t^*) > \pi^S(s^*, t^*)$ and consequently also $\pi^M(\hat{s}, t^*) > \pi^M(s^*, t^*)$. Son does not save enough because of the negative effect that saving has on his Mother's contribution.

Maximize social welfare $\pi = u(Y-t) + v(y-s) + \delta v(s+t)$, and assume interior maximum.[7] Derivatives:

$$\frac{\partial \pi}{\partial s} = -v'(y-s) + \delta v'(s+t) \qquad \frac{\partial \pi}{\partial t} = -u'(Y-t) + \delta v'(s+t)$$

$\partial \pi / \partial s = 0$ gives efficient savings; then by substituting into the second we get $u'(Y-t) = v'(y-s)$: marginal costs for Mother and Son are equal. Denote optimum by $\left(\hat{s}, \hat{t}\right)$. If Mother promises $\hat{t}$ then Son will choose $\hat{s}$ because given t social welfare depends only on the Son's payoff.

The other question is: is the Nash t^* too small or too large? Given s the Nash t satisfies $u'(Y-t) = \alpha\delta v'(s+t)$, while the efficient t is given by $u'(Y-t) = \delta v'(s+t)$. So if $\alpha = 1$ fixing t at the Nash value is optimal; if $\alpha < 1$ [resp. $\alpha > 1$] Nash gives $u'(Y-t) < \delta v'(s+t)$ [resp. $u'(Y-t) > \delta v'(s+t)$] so that Nash t is too small [resp. too high].

Second scenario (the Rotten Kid Theorem)

Here incomes of Mother and Son depend on Son's effort level $e > 0$: $Y = Y(e), y = y(e)$. Son chooses e; *then* Mother observes $Y(e)$ and $y(e)$ and chooses a transfer t to pass to Son, where $0 \leq t \leq Y(e)$. The functions Y, y are strictly increasing concave.

The mother cares about Son but he does not requite. So payoffs are

$$\pi^S(e,t) = v(y(e) + t), \qquad \pi^M(e,t) = u(Y(e) - t) + \alpha\pi^S(e,t).$$

Concavity and interiority assumptions are maintained. Therefore Nash equilibrium is found as before by backward induction. Given e Mother's choice $t(e)$ is given by the FOC

$$u'(Y(e) - t(e)) = \alpha v'(y(e) + t(e)).$$

Son maximizes $v(y(e) + t(e))$ for which FOC is $v'(y(e) + t(e))\,[y'(e) + t'(e)] = 0$ which since

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[6] Similar to monopolist who produces too little because of $p' < 0$.

[7] Notice that π is the mother's payoff with $\alpha = 1$.

v is strictly increasing implies that optimal e^* is given by $y'(e^*) + t'(e^*) = 0$. Equilibrium transfer is then $t(e^*)$.

In this equilibrium by differentiating the Mother's FOC and inserting the Son's FOC $y'(e^*) + t'(e^*) = 0$ we get (omitting the asterisk)

$$u''(Y(e) - t(e))(Y'(e) - t'(e)) = \alpha v''(y(e) + t(e))(y'(e) + t'(e)) = 0$$

so that (since u is strictly concave) $0 < Y'(e) = t'(e)$. Substituting into Son's FOC we obtain $Y'(e^*)+y'(e^*) = 0$ which is the condition characterizing maximum family income $Y(e)+y(e)$. Thus in this case the equilibrium coincides with the social optimum, notwithstanding the Son's egoistic behavior. Here interests are aligned because $Y'(e) = t'(e)$: at the margin the Mother's income is entirely transferred to the Son.

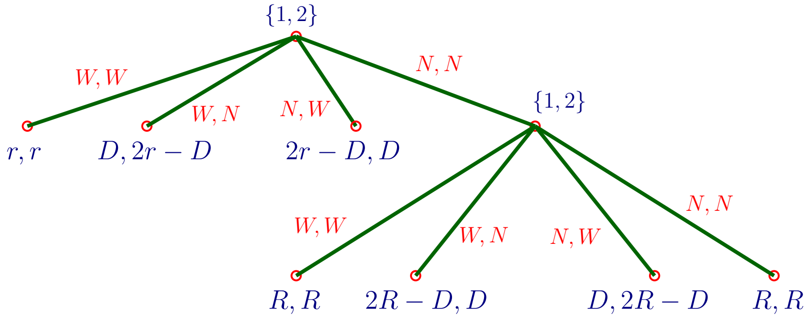
Compare

Note the difference between the two scenarios. In the latter effort can be rewarded: in equilibrium $t'(e) = Y'(e) > 0$. In the former "effort" is savings which is detrimental to transfers: in equilibrium $t'(s) < 0$.

A Bank Deposit Game ($P(h) = N \ \forall h$)

This game tries to capture the possibility of bank runs. There are two players/depositors with total deposits $2D$ (D each). There are two stages, both players play in each stage and they can either withdraw, action W or not withdraw, action N. Total yield is $2R$ if there is no withdrawal in stage 1, otherwise $2r$ with $r < D < R$. The other payoffs are as specified in the following game:

Figure 1: A Deposit Game



If they both withdraw at the first stage they get $r < D$ each. If neither does and they perform the same action in stage 2 they get $R > D$ each. In the first stage the single player who withdraws takes $D > r$, the other the remaining yield $2r - D < r$; in the second stage it is the opposite: the single one who leaves the money in the bank gets D, the other gets $2R - D > R$ (for the proceeds can be collected then).

We look for subgame perfect equilibria, so start at history NN. There we have the following simultaneous move subgame, with unique equilibrium WW.

	W	N
W	R, R	$2R - D, D$
N	$D, 2R - D$	R, R

Therefore we can replace the above subgame with the resulting equilibrium payoff R, R, and get the following game at stage 1:

	W	N
W	r, r	$D, 2r - D$
N	$2r - D, D$	R, R

This game has *two* Nash equilibria, WW and NN. Therefore the two-stage game has two subgame perfect equilibria: one is the profile (WW, WW), where both players play WW that is withdraw at each stage, the other is (NW, NW) where both only withdraw at the second stage. The second yields R each, which is higher than the payoff r they get in the first equilibrium. The latter models a “bank run” - it is inefficient but it is still an equilibrium.

Appendix: OR def. 89.1 and Trees

The other definition is based on the concept of Tree.

Definition. Binary relation $\succ$ is *asymmetric* if $x \succ y \Rightarrow \neg(y \succ x)$; it is a *total order* if $x \neq y \Rightarrow (x \succ y) \vee (y \succ x)$. A *least element* for $\succ$ is an x such that $x' \succ x$ for all $x' \neq x$. By $x \prec y$ we mean $y \succ x$.

A pair $(H, \succ)$, where H is a set and $\succ$ is a binary relation on it, is a *tree* if $\succ$ has a least element $\emptyset$ and for each $h \in H$ the set $B(h) = \{h' : h' \prec h\}$ is totally ordered by $\succ$.

The set H in definition 89.1 becomes a tree $(H, \succ)$ as follows. For $h = (a^1, a^2, \ldots a^k) \in H$ and $1 \leq m \leq k$ define its m-truncation as $\tau_m(h) = (a^1, a^2, \ldots a^m)$. Then define $\succ$ on H as follows: (i) $\emptyset \prec h$ for all $h \neq \emptyset$; (ii) for all $h, h' \neq \emptyset$, $h' \prec h$ if there is $m < k$ such that $h' = \tau_m(h)$.

That's all. So you can call h a "node" if you like.