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COMPLETE IRREFLEXIVE PREFERENCES
A Definition *

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1. There is a definition of completeness for irreflexive preferences in Bewley (1986), but it depends on the structure of the set on which preferences are defined (namely its topology). We propose a definition which involves only the structure of the preference relation, as is the case in the usual one for reflexive preferences. We show that it is in some sense the appropriate analogue of the latter, and we compare it with Bewley's.

2. Let a preference relation P be defined on a set X, i.e., $P \subset X^2$. If $(x, y) \in X^2$, $(x, y) \in P$ reads 'y is preferred to x'. Let $P(x) =: \{ y \in X \mid (x, y) \in P \}$ and $P_-(x) =: \{ y \in X \mid x \in P(y) \}$. We are dealing with an irreflexive relation, so $x \notin P(x)$, $\forall x \in X$, and assume throughout that P is acyclic.

Consider the following three properties [Gay (1983)]:

Property 1 *(Complete comparability).* $Y \in P(x)$ implies $P(x) \cup P-(y) = X$.

Let $P_- =: \{(x, y) \in X^2 \mid (y, x) \in P\}$, $D_n =: X^2 - (P \cup P_-)$, $E =: \{(x, y) \in X^2 \mid P(x) = P(y)$ and $P_-(x) = P_-(y)\}$, and $C_n =: D_n - E$.

Property 2 *(Equivalence of incomparables).* $C_n = 0$.

Property 3 *(Non-selectivity).* $\forall S \subset X$, $P(y) \cap S = 0$ and $P(x) \cap S \neq 0$ imply $Y \in P(x)$.

Proposition 1. *Properties 1, 2 and 3 are equivalent.*

Since properties 1, 2 and 3 are equivalent, the set of all (acyclic) preferences on X is partitioned into the sets $\mathscr{P}_c$ of preferences satisfying Properties 1–3 and $\mathscr{P}_i$ of those which satisfy none of them. And of course the definition we propose is the following:

Definition 1. P is complete if $P \in \mathscr{P}_c$.

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Remark 1. Notice that if P is complete, it is transitive.

Proof of Proposition. Property 1 $\Rightarrow$ Property 2: Notice that $E \subset D_n$, by acyclicity; so assuming Property 1 let $(x,y) \in D_n$. It is to be shown that $(x,y) \in E$. Suppose $z \in P(x)$. Then $(x,y) \notin P$ implies by Property 1 $(y,z) \in P$ i.e., $P(x) \subset P(y)$. That $P(y) \subset P(x)$ follows in the same way. Next, suppose $z \in P_{-}(x)$, i.e., $(z,x) \in P$. Then $(y,x) \in P$ implies by Property 1 $(z,y) \in P$ i.e., $z \in P_{-}(y)$, so $P_{-}(x) \subset P_{-}(y)$. Similarly $P_{-}(y) \subset P_{-}(x)$.

Property 2 $\Rightarrow$ Property 3: Suppose $C_n \neq 0$; take $(x,y) \in P$ and $z \in X$ arbitrary. Suppose $(x,z) \notin P$; we have to show $(z,y) \in P$. Now either $(z,x) \notin P$, so $(x,z) \in D_n$ and then by hypothesis $(x,z) \in E$ which implies $P(x) = P(z)$ and so $(z,y) \in P$. Or $(z,x) \in P$, in which case we have $(z,x) \in P$ and $(x,y) \in P$. Suppose $(z,y) \notin P$. Then either $(y,z) \notin P$ or $(y,z) \in P$. The latter is excluded by acyclicity, so it should be $(z,y) \in D_n$ and so $(z,y) \in E$, in other words $P(z) = P(y)$ and $P_{-}(z) = P_{-}(y)$. But then $(z,x) \in P$ implies $(y,x) \in P$, a contradiction.

Property 1 $\Rightarrow$ Property 3 is immediate.

Property 3 $\Rightarrow$ Property 1. Take $(x,y) \in P$ and z such that $(x,z) \notin P$, to show $(z,y) \in P$. Let $S = \{x,y,z\}$. z cannot be maximal in S because it does not dominate x which is dominated (this is implied by Property 3). So either $(z,y) \in P$ or $(z,x) \in P$. In the latter case $(z,x) \in P$ and $(x,y) \in P$ so by acyclicity $(y,z) \notin P$. But then y is maximal in S and z is not. So again $(z,y) \in P$ (by Property 3).

3. The above definition is analogous to the usual one for reflexive preferences in the following sense. In this discussion X is a topological space. Following Debreu (1959), a function u on X is a utility for the reflexive relation R if it is continuous and such that

$$(x,y) \in R \quad \text{and} \quad (y,x) \in R \Rightarrow u(x) = u(y),$$

and

$$(x,y) \in R \quad \text{but} \quad (y,x) \notin R \Rightarrow u(x) < u(y). \tag{1}$$

We say that the utility u is 'characterising' if also the converse implications in (1) hold. Then it is easy to see that

Proposition 2. If u is a utility for R, then R is complete if and only if u is characterising.

In a parallel way, u is a utility for the (irreflexive) relation P if it is continuous and such that

$$(x,y) \in P \Rightarrow u(x) < u(y);$$

and a utility u is characterising if also the converse implication holds. Then one finds the analogue of Proposition 2:

Proposition 3. Assume $x \in clP(x)$, $\forall x \in X$ (local non-satiation). If u is a utility for P, then $P \in \mathscr{P}_c$ if and only if u is characterising.

Proof. Suppose u is characterising. We show that $C_n = 0$. So take $(x,y) \in D_n$. Then the hypothesis implies $u(x) = u(y)$. So $(x,z) \in P \Leftrightarrow u(x) < u(z) \Leftrightarrow u(y) < u(z) \Leftrightarrow (y,z) \in P$. Similarly $(z,x) \in P$

$\Leftrightarrow (z,y) \in P$, hence $(x,y) \in E$, as was to be shown. Conversely, suppose $C_n = 0$, and $u(x) < u(y)$. Then $(y,x) \notin P$, so if $(x,y) \notin P$ by hypothesis $(x,y) \in E$. The assertion is then true if we show that $(x,y) \in E$ implies $u(x) = u(y)$. To see this suppose $u(x) < u(y)$. Take $z \in X$ such that $(x,z) \in P$ and $u(x) < u(z) < u(y)$ (possible by continuity of u and local non-satiation of P). Then $(y,z) \notin$ P so $(x,y) \notin E$. Similarly, if $(x,y) \in E$ it is not $u(x) > u(y)$, and the proof is complete.

Remark (Existence of a utility for P). If P is complete and $P(x)$ is open for all $x \in X$, it is easy to see that P is spacious [Peleg (1970); see also below]. So if we add Peleg's 'separability' assumption existence of utility is guaranteed if $P \in \mathscr{P}_c$. The latter is implied, for example, if we assume that X has a countable everywhere dense subset and P is locally non-satiated and such that $P(x)$ and $P_-(x)$ are open.

4. We now compare our definition with Bewley's, which says that P is complete if it satisfies (X is again a topological space)

Property 4. $\forall x \in X$, $cl(P(x) \cup P_-(x)) = X$.

In general Property 4 is weaker than Properties 1–3:

Proposition 4. Assume $x \in cl P(x)$, $\forall x \in X$. Then if $P \in \mathscr{P}_c$, P satisfies Property 4.

Assume that $y \in P(x)$ implies $cl P_-(x) \subset P_-(y)$ (spaciousness), and that $\text{int}(cl P(x)) \subset P(x)$, $\forall x \in X$ (regularity). Then if P satisfies Property 4, $P \in \mathscr{P}_c$.

Proof. If Property 4 is not satisfied, for some x there is a y and a neighbourhood of y, N such that for all $y' \in N$ one has $y' \notin P(x), x \notin P(y')$. But by local non-satiation, for some $y' \in N$ $Y' \in P(y)$. This implies $P \notin \mathscr{P}_c$. Conversely, assume Property 4; take $y \in P(x)$ and $z \in X$ such that $y \notin P(z)$, to show $z \in P(x)$. By spaciousness $y \notin P(z)$ implies $z \in cl P_-(x)$, so there is a neighbourhood N of z such that $N \subset cl P(x)$, by Property 4. Regularity then implies that in fact $N \subset P(x)$, and the result follows. Q.E.D.

In the following example P is transitive, it satisfies the assumptions in Peleg (1970) for existence of a utility, and $P(x)$ is open and convex for all $x \in X$; but regularity is violated. And Property 4 is satisfied although $P \in \mathscr{P}_c$.

Take $X = R^2_+$, with the usual relative topology. We describe P by specifying $P(x)$ for all $x = (x_1, x_2) \in R^2_+$. For $x \in R^2_{++}$, $P(x)$ is some nice open convex set with boundary asymptotic to the axes. For $\{(0,x_2) \mid x_2 > 0\}$, $P((0,x_2)) = R^2_{++}$, and for $\{x_1,0) \mid x_1 \geqslant 0\}$, $P((x_1,0)) = R^2_{++} \cup \{x_1',0) \mid x_1' > x_1\}$.

All properties of P are easily checked. That $P \notin \mathscr{P}_c$ is seen by taking $x = (0,0)$, $y(x_1,0)$ with $x_1 > 0$ and $z = (0,x_1)$ with $x_2 > 0$, for which Property 1 is violated.

In this example it is the structure of X which makes regularity a strong assumption. When for instance X is a linear space (as in Bewley), regularity imposes no substantial restrictions on P, and our definition is essentially equivalent to Property 4.

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