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Unawareness and Partitional Information Structures*

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We claim first that simple uncertainty is not an adequate model of a subject's ignorance, because a major component of it is the inability to give a complete description of the states of the world, and we provide a formal model of unawareness. In Modica and Rustichini (1994) we showed a difficulty in the project, namely that without weakening of the inference rules of the logic one would face the unpleasant alternative between full awareness and full unawareness. In this paper we study a logical system where non full awareness is possible, and prove that a satisfactory solution to the problem can be found by introducing limited reasoning ability of the subject. A determination theorem for this system is proved, and the appearance of partitional informational structures with unawareness is analysed. *Journal of Economic Literature Classification Numbers*: D80, D83.

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1. UNCERTAINTY AND IGNORANCE

Decision theory under uncertainty models the behavior of a subject who has to take an action, but does not know which of a list of possible states of the world is the true state. In the theory as presently structured, however, the ignorance of the subject is limited to this lack of knowledge: the description of the world in his mind is in fact correct and exhaustive. It is perhaps obvious, but still important to note that this is by no means necessarily true; rather it is an assumption on the rationality of the decision maker. The assumption is that he may be uncertain about the true state, but he has no ignorance about the state space.

The purpose of this paper is to model situations where such ignorance is possible. We provide a model where some of the facts that determine which state of nature occurs are not present to the subject's mind, and this lack of awareness makes him incapable of giving a complete description of the ‘‘objective'' state space. The idea which we formalize here is that the individual's ‘subjective' description of the world is constructed on the basis of those events of which the subject has awareness.

Our formal definition of awareness, first introduced in Modica and Rustichini (1994), is based on the idea that there are three possible states of a subject's knowledge regarding the truth/falsity of a proposition p (for example, ‘‘it rains''). First, he may be certain of the truth value (true or false) of p; second, he may be consciously uncertain about it, in the sense of not knowing and knowing of not knowing. These are the two possibilities that are usually considered in Decision Theory. The third possibility we want to consider is the following: he may not know p, not know he does not know it, not know he does not know he does not know it, and so on *ad infinitum*. The last possibility represents the situation of ‘‘not having in mind,'' which we call unawareness. The occurrence of p, or even of its opposite, will be a surprise for that decision maker.

We can be more precise using the logical symbols $\wedge$, $\vee$, $\neg$ of ‘‘and, or, not'' and the knowledge operator k, so that kp is interpreted as ‘‘the subject knows that p,'' and $\neg kp \wedge k \neg kp$ is not knowing and knowing of not knowing. Modica and Rustichini (1994) define the awareness operator a as *union of certainty and conscious uncertainty*: $ap := kp \vee (\neg kp \wedge k \neg kp)$; and introduce an axiom of symmetric awareness: $ap \leftrightarrow a \neg p$, (where ‘‘ $\leftrightarrow$ '' is ‘‘if and only if''), which reflects the idea that if p is not present to mind the same must be true of $\neg p$ (this is axiom A, reported below). Unawareness is the negation of awareness, and it is $\neg ap \leftrightarrow \neg kp \wedge \neg k \neg kp$; using the symmetry axiom one proves that $\neg ap$ is equivalent to all the sequences of ‘‘ not knowing of not knowing of not knowing . . . p.'' This last might therefore be equivalently taken as the definition of ‘‘being unaware.''

In Sect. 2 below we state the basic properties that an awareness operator should have. For instance, if the subject is aware of some thing, then he should be aware of this very fact, indeed he should know it.

Partitions and Possibility Correspondences

Information partitions have arisen in models where information is represented as a correspondence from states to states (which to each state associates a set of states seen as possible) and the range of this correspondence is a partition of the space. This class of models "corresponds" to a logical system known as $S5$, which consists of a "weaker" system $S4$ plus the so-called axiom of negative introspection: $\neg kp \rightarrow k \neg kp$. It is easily seen that this is equivalent to ap; hence, schematically $S5 = S4 + ap$, so that in $S5$ there is full awareness. Modica and Rustichini (1994) prove that in fact $S5 = S4 + (ap \leftrightarrow a \neg p)$ (seemingly weaker). This implies that if one wishes to maintain the symmetry axiom $ap \leftrightarrow a \neg p$, to get *un*awareness one should weaken $S4$. Moreover, it suggests what to weaken: for it says that full awareness can be achieved without the deduction $\neg kp \rightarrow k \neg kp$, thus, it must be implied by some other deductive process; therefore one has to concentrate on $S4$'s deduction rules. The nature of unawareness now hints to the direction of the weakening; because intuitively, it has nothing to do with deductive power, for it is not at all the case that a subject who is aware of fewer things than another must necessarily be less capable of logical reasoning than the latter; on the other hand, the former may not be able to make some deductions exactly because of unawareness: for example if he is unaware of q, he will not be able to deduce knowledge of $p \vee q$ from knowledge of p, which he would otherwise do, for he cannot conceive of $p \vee q$. With this motivation, we give the subject the same deductive power as an $S5$ subject, but only *within his domain of awareness*; and this results in the logical system $\mathscr{U}$ introduced in the present paper. Once one has a system which admits unawareness but is "locally" like $S5$, locally in the sense of "within the domain of awareness," it is a small step to guess that the corresponding models will exhibit a partitional information structure in some local sense; and the result is the following. We know that in $S5$ there is a possibility correspondence from states to states, whose range is a partition; in $\mathscr{U}$, if in a state σ the subject is aware of the set of sentences $a^{-}(\sigma)$, then at σ the *conceivable* states will be those which he can describe using the sentences in $a^{-}(\sigma)$; among them, on the basis of his knowledge he will identify a subset of *possible* states; therefore the possibility correspondence P is from states to *conceivable* states, and $P(\sigma)$ and $P(\tau)$ are in the same space iff $a^{-}(\sigma) = a^{-}(\tau)$; so given σ, as the state varies over the set of τ's such that $a^{-}(\tau) = a^{-}(\sigma)$, the images $P(\tau)$ all lie in a fixed space (that of the states conceivable at σ); the result is that these images describe a partition of the perceived space.

States of the World

There is more; to continue let us distinguish between a *state of the world* and an *epistemic state* (the states of the foregoing paragraphs were meant to be epistemic states). A state of the world is described in terms of elementary events, like the event "it rains;" at Savage puts it, it is a "description of the world, leaving no relevant aspect undefined" (Savage 1954), p. 9). In this paper these elementary events are described by atomic sentences, p, q and so on, which are the basic components of a formal language. So a state of the world corresponds to an assignment of truth value to each atomic sentence; with countably many atomic sentences (which is our case), fixing once and for all a correspondence between the set of atoms of the language and the set $\mathbb{N}$ of the natural numbers, a state of the world corresponds to an element of the product space $\{0, 1\}^{\mathbb{N}}$. An epistemic state on the other hand is a complete description of the world *and* of the knowledge that the decision maker has, both of the world itself and of his own knowledge (one has of course to think that such description is available only to an outside theorist). The introduction of epistemic states makes it possible to study the subject's capability of logical deduction, and to analyse the correspondence between a class of models and a logical system; but one is also interested in the induced information structure on the space of states of the world; in this regards we may assert the following. If at epistemic state σ the subject is aware of the set Q of atoms and this corresponds to $N \subseteq \mathbb{N}$, the perceived space of states of the world at σ will be $\{0, 1\}^N$; within this there will be a subset, say $\rho(\sigma)$, that the subject sees as possible at σ; with σ fixed, consider epistemic states τ's with $a^-(\tau) = a^-(\sigma)$ (hence, with same perceived spaces as σ); as τ varies across a set of states where the subject can observe the occurrence of a *given* set of events (formally knows the truth value of the corresponding sentences), $\rho(\tau)$ describes a partition of the perceived states of the world. In the particular σ's where there is full awareness (they correspond to $S5$ states) the set Q is the whole set, it corresponds to $\mathbb{N}$, and the partitioned space of the states of the world is the real one $\{0, 1\}^{\mathbb{N}}$.

To Know and to Believe

As a final comment, we would like to discuss a possible objection to our concept of awareness, which has been raised by Robert Stalnaker. This objection may be formulated as follows (we hope we are fair to his argument):

"According to the definitions and theorems of this paper one has

(I) I am aware of something if and only if I am aware of the negation of that something; in addition I am aware of something if and only if I am aware that this something is possible.

These two facts indeed are respectively the symmetry axiom, and a consequence of the assumptions. But on the other hand:

(II) I am aware of the fact that $1 = 1$, and I am certainly not aware that $1 \neq 1$ is possible.

Hence, this concept of unawareness is philosophically uninteresting."

This point raises two very interesting issues. To do full justice to the first, we need to introduce the distinction between knowledge and belief. Our logic will be based on *knowledge*: if the subject knows something, then that something is true. This excludes the possibility that the agent "knows" or, better, "believes," something which is false. This distinction opens a question: the one of defining and analysing awareness with *belief*, rather than knowledge. In the formal language of the next sections, this would consist in dropping the axiom T from our system. This is an important question, that we do not discuss here.

The second issue is what "possible" means in the present context.

In the conclusions (see Sec. 5), we will argue that in (I) and (II) above the term *possible* is used in two different meanings. This, we believe, provides an answer to the objection; it does, in addition, indicate different interesting variations on the concept of awareness. To do that, however, we will have to first set the notation and definitions necessary for a formal discussion. We proceed to do so, and refer for the moment the reader to the conclusions.

The Main Result

Technically, the main result of the paper is a determination theorem for the logical system $\mathscr{U}$ here introduced. A determination theorem for a system gives a complete characterization of the class of models that validate it; in intuitive terms, this gives a complete description of the states of the world and the information structure over them that can arise when the subject has the reasoning ability assumed by the system. Once more we recall what happens with $S5$ (full awareness case): $S5$ is determined by a class of so-called standard models, namely the "partitional" ones. Our result is that $\mathscr{U}$ is determined by a class of "generalized standard models" (introduced in Sec. 3), again the partitional ones.

The Literature

In the literature of Artificial Intelligence, a model of unawareness is presented in Fagin and Halpern (1988); they define an awareness operator independent of knowledge, and have it satisfy some desirable properties by definition; we do not introduce a second modal operator (we define awareness in terms of knowledge), and prove desirable properties under assumptions. The issue of partial awareness has also been considered in the literature of economic theory. Something completely different from our approach is the illuminating analysis of a related issue in Lipman (1992) and (1993). Even more critical of the standard approach to the description of the state space is the line of research in Gilboa and Schmeidler (1992), (1993a) and (1993b). Perhaps closer, at least from the formal point of view, is the sequence of papers studying nonpartitional structures of information, started with Brown and Geanakoplos (1988); see also Geanakoplos (1989), (1992) or Samet (1990). In fact, weakening of the system $S5$ is usually associated, or even identified, with the introduction of structures of this type. The problem which is typically analysed in this literature is the if and how nonpartitional structures may affect results like agreeing to disagree, in the classical sense of Aumann (1976), or no trade theorems, as in Milgrom and Stokey (1982); see Rubinstein and Wolinsky (1990) for an unifying view. In this paper we take a different route, for the system we introduce is in fact strong enough to produce, on the set of states of the world of which the subject is aware, a partitional structure.

Organization of the Paper

The sequel of the paper is as follows. In Sec. 2 we set up notation, define the system $\mathscr{U}$, and prove basic properties of the awareness operator. Section 3 introduces the concept of Generalized Standard Models, which contain the determining class of $\mathscr{U}$, and discusses their basic properties. The determination theorem is stated in Sec. 4. In the Conclusions (Sec. 5) we outline some directions of future research in the topic. The main theorem is proved in the appendix.

2. THE LOGICAL SYSTEM OF UNAWARENESS

We first report a few basic notions of modal logic, which are essential for the understanding of what follows; we refer to Chellas (1980) for a detailed exposition.

Basic Concepts

A language is fixed throughout the paper, and is denoted by Λ. It consists of a set of atomic sentences, of five propositional operators, and two modal operators.

The countable set of *atomic sentences* $p, q, \ldots,$ is denoted by L; they are interpreted as the elementary "facts."

The *operators* are $\top$, $\bot$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$. The first two are interpreted as "true" and "false;" the third is a one-place operator, that transforms a single sentence p into $\neg p$, and is interpreted as "negation;" the last four are two-place operators, interpreted as, respectively, "or," "and," "implies," and "implies and is implied by."

Finally we have two *modal operators*. The first is k; the sentence $k\varphi$ is to be read "the subject knows φ." The second modal operator is $\Diamond$; it is roughly to be interpreted as a subjective possibility operator, so that $\Diamond\phi$ should be read as "the subject considers ϕ possible;" but a more precise discussion of the interpretation of $\Diamond$ in this paper is given later, when we present our system.

Sentences in which at least one of the two modal operators appear are called modal sentences; the others are called nonmodal sentences. For details in the use of connectives, parentheses, and the rest of basic propositional calculus, the reader is referred to any textbook on propositional logic, e.g. Van Dalen (1983), Chap. 1. Propositional Logic is denoted by PL. For any subset Q of L, $\Lambda(Q)$ is the language built on the atomic sentences in Q. For further discussion, see Chellas (1980), Sec. 2.1.

Awareness

We define awareness of a sentence φ in terms of knowledge as certainty or (conscious) uncertainty, by setting

$$a\varphi \equiv k\varphi \vee (\neg k\varphi \wedge k\neg k\varphi), \varphi \in \Lambda,$$

where $a\varphi$ reads "aware of φ." As we mentioned, this definition is adopted to emphasize the two distinct ways in which a subject can be aware of a sentence; but $k\varphi \vee (\neg k\varphi \wedge k\neg k\varphi)$ is PL equivalent to $k\varphi \vee k\neg k\varphi$, which therefore is an equivalent definition of $a\varphi$. This last is in fact the definition that will be used most frequently in the rest of the paper.

We recall that a system of modal logic is a set of sentences in the language, closed with respect to inference according to Propositional Logic. The sentences of a system are also called its *theorems*.

The System S5

One system in particular has a distinguished importance in modelling knowledge, the system $S5$. Since it is a model of full awareness, and can be used for comparison with the system we introduce, we recall briefly his definition.

We recall that an *axiom* is a sentence that is assumed in the system; examples of axioms are the sentences M, C, and T in the system $S5$ defined immediately below. An *inference rule* has the form:

$$\frac{A_1, A_2, \ldots, A_n}{A},$$

where the sentences $A_1, \ldots, A_n$ are *hypotheses* of the rule and A is the *conclusion*. A well known example is MP (*modus ponens*):

$$MP. \quad \frac{\varphi, \varphi \to \psi}{\psi};$$

a second example is the rule RM, defined in the next section: the reader will find there a simple illustration and a discussion of the application of an inference rule.

$S5$ is generated by the following axioms and inference rules:

$$\begin{array}{rl}
\text{PL.} & \text{the set of all tautologies} \\
Df\Diamond. & \Diamond\varphi \leftrightarrow \neg k \neg \varphi \\
M. & k(\varphi \wedge \psi) \to k\varphi \wedge k\psi \\
C. & k\varphi \wedge k\psi \to k(\varphi \wedge \psi) \\
T. & k\varphi \to \varphi \\
4. & k\varphi \to kk\varphi \\
5. & a\varphi \\
\text{MP.} & \dfrac{\varphi, \varphi \to \psi}{\psi} \\
\text{RE} & \dfrac{\varphi \leftrightarrow \psi}{k\varphi \leftrightarrow k\psi}.
\end{array}$$

As we shall see, the system $\mathscr{U}$ is essentially obtained from $S5$ by weakening RE; by replacing 5 with two weaker axioms, A and AM below; and finally by reformulating $Df\Diamond$.

The presentation of $\mathscr{U}$ is divided into subsections. We begin with the two most specific features of the system: the inference rules and the symmetry axiom A.

Definition of $\mathscr{U}$

The inference rules of the system $\mathscr{U}$ are defined by imposing an additional restriction on inference rules as traditionally defined. This restriction is motivated by our intuitive notion of awareness. Consider for example the inference rule RM (which stands for Rule of Monotonicity)

$$\text{RM.}\ \frac{\varphi \to \psi}{k\varphi \to k\psi}$$

If a system has this rule, and $\varphi \to \psi$ is one of its theorems, then also $k\varphi \to k\psi$ is a theorem of the system. But consider now what this implies from the point of view of awareness of a sentence. Assume that $k\varphi$; then if $\varphi \to \psi$ is a theorem, $k\psi$ follows. If for instance $\varphi \to \psi$ is $p \to (p \vee q)$, then from kp we can infer $k(p \vee q)$. Since it is reasonable to think that if a subject knows that $p \vee q$ then he must be aware of p and q (we will have it as a formal result), awareness of q is here generated by mere knowledge of p and logical implication. And this is stronger than one would want, for clearly *un*awareness of q is in principle perfectly compatible with the subject knowing p and being able to use the inference rule MP. The reformulation of inference rules that we adopt eliminates this kind of "unwarranted" generation of awareness.

First we introduce the inference rule that we want to modify:

$$RK.\ \frac{\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \to \varphi}{k\varphi_1 \wedge k\varphi_2 \wedge \cdots \wedge k\varphi_n \to k\varphi},\ n \geq 1.$$

We want to distinguish sentences on the basis of the atoms that appear in them; so we say that two sentences φ, ψ *have the same atomic sentences* if and only if there is a subset Q of L such that both φ and ψ are in $\Lambda(Q)$, and $\varphi \notin \Lambda(Q')$, $\psi \notin \Lambda(Q')$ for any proper subset Q' of Q. Then we can formally introduce the following definition.

2.1. DEFINITION.

$$RK_{sa}.\ \frac{\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \to \varphi}{k\varphi_1 \wedge k\varphi_2 \wedge \cdots \wedge k\varphi_n \to k\varphi},\ n \geq 1$$

where $\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi$ have the same atomic sentences; that is, (for $n \geq 1$) RK_{sa} is RK where the involved propositions all have the same atomic sentences.

The inference rules RM_{sa} and RE_{sa} are defined analogously: they are the inference rules corresponding to RM and RE, with the additional condition that all the sentences involved have the same atomic sentences, similarly to what the above definition requires for RK_{sa}. There is, obviously, no correspondent for the rule RN, which we recall together with axioms K and N:

$$RN. \quad \frac{\varphi}{k\varphi}$$

$$K. \quad k(\varphi \to \psi) \to (k\varphi \to k\psi)$$

$$N. \quad k\top .$$

Some relations among inference rules are given in the following proposition. We recall that a system is said to be *closed* under an inference rule if and only if it contains all the conclusions of the rule when it contains its hypotheses.

2.2. PROPOSITION.

(i) *A system of modal logic closed under* RM_{sa} *and containing K is closed under* RK_{sa};

(ii) *A system closed under* RE_{sa} *and M is closed under* RM_{sa};

(iii) *A system closed under* RM_{sa} *is closed under* RE_{sa}.

Proof. (i) We proceed by induction on n. RM_{sa} is $n = 1$. Suppose that the assertion true for $n - 1$, and assume $\varphi_1, \ldots, \varphi_n, \varphi$ have the same atoms and $\vdash_{\mathscr{S}} \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \to \varphi$. Then by PL $\vdash_{\mathscr{S}} \varphi_1 \wedge \cdots \wedge \varphi_{n-1} \to (\varphi_n \to \varphi)$, and by inductive hypothesis $\vdash_{\mathscr{S}} k\varphi_1 \wedge \cdots \wedge k\varphi_{n-1} \to k(\varphi_n \to \varphi)$. By K, $\vdash_{\mathscr{S}} k(\varphi_n \to \varphi) \to (k\varphi_n \to k\varphi)$, so by PL we have the desired conclusion.

(ii) the proof of the first assertion is as in Chellas p. 236—notice that φ and $\varphi \wedge \psi$ have the same atoms if φ and ψ do.

(iii) follows as in Chellas p. 115. ■

Remark. A system which is closed under RK_{sa} does not necessarily contain K nor M or C (it does if it has RK).

The second specific feature of the system $\mathscr{U}$ is a pair of axioms which are weakenings of the axiom 5 of the system $S5$. The first is the symmetry axiom A, which was first introduced in Modica and Rustichini (1994):

$$A. \; a\varphi \leftrightarrow a\neg\varphi.$$

In intuitive terms, A says that if an agent is aware of a sentence, he must also be aware of its negation. The meaning of the axiom will probably be clearer if we keep in mind that the intuitive notion of awareness that is adopted in this paper is that of "being present to mind." When for instance φ represents a physical fact ("it rains"), being aware of φ means conceiving the possibility that φ might occur, so it seems natural to require that it imply conceiving the possibility that φ might *not* occur. A comparison with axiom 5, which as we have seen may be written as $a\varphi$, shows clearly why A is a weaker axiom.

The second axiom is introduced here for the first time:

$$AM.\ a(\varphi \wedge \psi) \rightarrow a\varphi \wedge a\psi.$$

The intuitive content of the axiom AM should be clear: if an agent is aware of the conjunction of two sentences, then he is aware of each of the two sentences separately; in other words, the axiom AM is simply the correspondent of the axiom M for the operator a.

The last element we need is the modified definition of $DF\Diamond$:

$$Df'\Diamond.\ \Diamond\varphi \leftrightarrow a\varphi \wedge \neg k \neg \varphi.$$

It is useful to contrast $Df'\Diamond$ with the standard definition $Df\Diamond$. $Df'\Diamond$ requires explicitly awareness of the sentence φ, and seems more appropriate to define possibility in the present context. In fact, an agent might not know not φ and be unaware of φ (and $\neg\varphi$), in which case it would be awkward to say that he considers φ possible.

The formal definition of the system $\mathscr{U}$ is the following:

2.3. DEFINITION. The system $\mathscr{U}$ is generated by the following axioms and inference rules: RPL (i.e., PL and MP), $Df'\Diamond, M, C, T, 4, A, AM, N$, and RE_{sa}.

Note that by Proposition 2.2(ii) one could equivalently use RM_{sa} instead of RE_{sa} in the above definition.

Awareness, Basic Properties

We begin by showing that the axioms and inference rules of the system we have defined imply some basic properties of awareness. The following notation will be used:

Notation. $\mathscr{S}(\ldots)$ denotes any system containing the axioms and inference rules in parenthesis, in addition to RPL and $Df'\Diamond$. Given a system $\mathscr{S}$, we shall sometimes write e.g., $\varphi \rightarrow_{\mathscr{S}} \psi \leftrightarrow_{\mathscr{S}} \xi$ to mean that both $\vdash_{\mathscr{S}} \varphi \rightarrow \psi$ and $\vdash_{\mathscr{S}} \psi \leftrightarrow \xi$ hold. We use $\vdash_{\mathscr{S}} \varphi$ and $\varphi \in \mathscr{S}$ interchangeably: they both mean that φ is a theorem of $\mathscr{S}$.

2.4. LEMMA.

(i) $a\varphi \leftrightarrow k\varphi \vee k \neg k\varphi \in \mathscr{S}$

(ii) $k\varphi \rightarrow k \neg k \neg \varphi \in \mathscr{S}(T, RM_{sa})$

(iii) $\mathscr{S}(T, Re_{sa})$ *has the rule*: *for all* φ, ψ *having the same atomic* *sentences,*

$$\frac{\varphi \leftrightarrow \psi}{a\varphi \leftrightarrow a\psi}$$

(iv) $k \neg k\varphi \leftrightarrow k \neg kk\varphi \in \mathscr{S}(T, 4, RE_{sa})$.

Proof is similar to that of Lemmas 2.2 and 2.3 in Modica and Rustichini (1994). In (i) above $\mathscr{S}$ is any system containing RPL and Df′◇.

As we have seen, axiom A requires symmetry of awareness. It is interesting to note that in a large class of systems this axiom has an equivalent formulation, which makes the comparison with the axiom 5 more transparent:

$$A'.\ \neg k\varphi \wedge k \neg k \neg \varphi \rightarrow k \neg k\varphi.$$

2.5. PROPOSITION. *A system $\mathscr{S}(T, RM_{sa})$ contains A iff it contains A'.*

Again, the proof follows the lines of proposition in Modica and Rustichini (1994); in fact only the *sa*-version of RM is used there.

The next (readily proved) proposition says that being aware of φ is equivalent to knowing φ's truth value or being consciously uncertain about it. This, we contend, reflects the idea of being present to mind.

2.6. PROPOSITION. *A system $\mathscr{S}(T, A, RM_{sa})$ contains*

$$a\varphi \leftrightarrow k\varphi \vee k \neg \varphi \vee (k \neg k\varphi \wedge k \neg k \neg \varphi).$$

The following proposition too is conceptually important; it establishes conditions under which if an agent does not know a sentence, and he does not know that he does not know this, then it cannot happen that at some iteration he knows his own ignorance. To state it recall that for a given system $\mathscr{S}$ a set of sentences is said to be $\mathscr{S}$-consistent if the false $\perp$ cannot be deduced from that set, using axioms and inference rules in $\mathscr{S}$. Also we need the concept of *maximally consistent set of sentences* of a system $\mathscr{S}$, abbreviated $\mathscr{S}$-maximal, that is a set of sentences that is $\mathscr{S}$-consistent and does not have proper $\mathscr{S}$-consistent extensions (see, e.g., Chellas (1980), Chap. 2). Intuitively, the $\mathscr{S}$-maximals are the possible worlds consistent with $\mathscr{S}$; for instance, if $\mathscr{S}$ contains the axiom $k\varphi \rightarrow \varphi$, then there will be no $\mathscr{S}$-maximal containing $k\varphi$ and $\neg \varphi$.

2.7. PROPOSITION. *Let $\Gamma \subseteq \Lambda$ be a maximally consistent set of $\mathscr{S}(T, 4, A, RM_{sa})$, and let $\varphi \in \Lambda$. If $\neg k\varphi \wedge \neg k \neg k\varphi \in \Gamma$, then $(\neg k)^n\varphi \in \Gamma$ for all $n \geq 1$.*

This too is proved the same way as in Modica and Rustichini (1994). We finally come to the basic properties of the awareness operator.

2.8. PROPOSITION.

(i) $a\varphi \leftrightarrow ak\varphi \in \mathscr{S}(T, 4, RE_{sa})$
(ii) $k\varphi \leftrightarrow ak\varphi \in \mathscr{S}(4)$
(iii) $a\varphi \leftrightarrow a\Diamond\varphi \in \mathscr{S}(T, 4, A, RE_{sa})$
(iv) $a\varphi \leftrightarrow ka\varphi \in \mathscr{S}(T, 4, RM_{sa})$
(v) $a\varphi \leftrightarrow aa\varphi \in \mathscr{S}(T, 4, RM_{sa})$
(vi) $\neg a\varphi \leftrightarrow \neg a \neg a\varphi \in \mathscr{S}(T, 4, A, RM_{sa})$.

The proof follows the lines of Proposition 2.4 in Modica and Rustichini (1994). Another property of a describes implications of the joint awareness of two sentences:

2.9. PROPOSITION. (i) $a\varphi \wedge a\psi \rightarrow a(\varphi \wedge \psi) \in \mathscr{S}(T, M, C, RE_{sa})$
(ii) $a\varphi \wedge a\psi \rightarrow a(\varphi \vee \psi) \in \mathscr{S}(T, M, C, A, RE_{sa})$

Proof. (i) By Lemma 2.4(i) and PL,

$$\mathscr{S} \ni a\varphi \wedge a\psi \leftrightarrow (k\varphi \wedge k\psi)$$
$$\vee \left[(k\varphi \wedge k\neg k\psi) \vee (k\psi \wedge k\neg k\varphi) \vee (k\neg k\varphi \wedge k\neg k\psi)\right].$$

By C in turn, $[\cdot] \rightarrow k(\varphi \wedge \neg k\psi) \vee k(\psi \wedge \neg k\varphi) \vee k(\neg k\varphi \wedge \neg k\psi)$. Now using RM_{sa}, we have

$$\frac{\varphi \wedge \neg k\psi \rightarrow \neg k\varphi \vee \neg k\psi}{k(\varphi \wedge \neg k\psi) \rightarrow k(\neg k\varphi \vee \neg k\psi)}$$

and analogously with $\psi \wedge \neg k\varphi \rightarrow \neg k\varphi \vee \neg k\psi$ and $\neg k\varphi \wedge \neg k\psi \rightarrow \neg k\varphi \vee \neg k\psi$. Therefore $\vdash_{\mathscr{S}} [\cdot] \rightarrow k(\neg k\varphi \vee \neg k\psi)$. But by M, $\vdash_{\mathscr{S}} k(\varphi \wedge \psi) \rightarrow k\varphi \wedge k\psi$, so by PL $\vdash_{\mathscr{S}} \neg k\varphi \vee \neg k\psi \rightarrow \neg k(\varphi \wedge \psi)$; so by RM_{sa} $\vdash_{\mathscr{S}} k(\neg k\varphi \vee \neg k\psi) \rightarrow k\neg k(\varphi \wedge \psi)$. By looking at this last implication and the one where $[\cdot]$ last appears, we see that $\vdash_{\mathscr{S}} [\cdot] \rightarrow k\neg k(\varphi \wedge \psi)$. Now by C the result is direct.

(ii) Proof unchanged from Modica and Rustichini (1994), via Lemma 2.4(iii). ■

In the appendix we provide different characterizations of the axiom *AM*.

Two Systems for Unawareness

We will see in Proposition 6.4 that the system $\mathscr{U}$ has the property that for any maximally consistent σ, one has $a^{-}(\sigma) = \Lambda(Q)$ for some $Q \subseteq L$. In fact it follows from Proposition 5.1 of Modica and Rustichini (1994) that also the system $\mathscr{A}$ defined there has the same property; in that case, it is always the case that $Q = \varnothing$ or $Q = L$ (this follows from the above quoted proposition), a particularly unpleasant implication. Here we show that this is not the case for $\mathscr{U}$.

Notation. For any $Q, \emptyset \subseteq Q \subseteq L$, we agree that $\top \in \Lambda(Q)$. In particular $\top \in \Lambda(\emptyset)$.

The following proposition will follow from soundness results presented later, but is reported here to emphasize a major difference between the two systems. A direct proof also exists, but is omitted to avoid duplications.

2.10. PROPOSITION. *For any* $Q \subseteq L$, *the set* $\Gamma =: \{a\varphi | \varphi \in \Lambda(Q)\} \cup \{\neg a\psi | \psi \notin \Lambda(Q)\}$ *is* $\mathscr{U}$*-consistent*.

3. GENERALIZED STANDARD MODELS

In this section we define a basic concept of our theory: the Generalized Standard Models (GSM for short, called, thus, because they generalize the concept of Standard Models—Chellas (1980) Chap. 3). The main result of this paper, proved in the next section, is that the system $\mathscr{U}$ is determined by a class of such models, the ‘‘partitional’’ GSM’s.

*The Standard Model for S*5

A good way to introduce the Generalized Standard Model is to start from the Standard Model for the system $S5$ that we have defined earlier. We take a fixed enumeration of the sentences in L, so we may write $L \equiv \{p_1, p_2, \ldots\}$.

A model in this case is a structure:

$$\mathscr{M} = \langle \Sigma, P, \wp \rangle.$$

The three elements $\Sigma, P, \wp$ of this structure have the following interpretation. Σ is a set, the set of states. $\wp$ is map from the integers to subsets of Σ; for a given atomic sentence p_n, the set $\wp(n)$ is the set of states where p_n is true. Finally, P is a map from Σ into subsets of Σ, that is a correspondence, called *possibility correspondence*.

P is assumed to be *reflexive* (that is, $\sigma \in P(\sigma)$ for all $\sigma \in \Sigma$), *symmetric* (that is, if $\sigma \in P(\tau)$ then if $\tau \in P(\sigma)$, for all $\sigma, \tau \in \Sigma$), and *transitive* (that is, if $\sigma \in P(\rho)$ and $\tau \in P(\sigma)$ then $\tau \in P(\rho)$, for all $\rho, \sigma, \tau \in \Sigma$.)

This model defines *truth conditions* on the sentences: they provide the bridge between the model and the logical system. The truth conditions are easy to understand in the case of nonmodal sentences. For instance, for the atomic sentence p_n,

$$\mathscr{M}, \sigma \models p_n \text{ if and only if } \sigma \in \wp(n)$$

since in fact $\wp(n)$ is defined to be the set of states where p_n is true. For any two sentences ϕ and ψ:

$$\mathscr{M}, \sigma \models \phi \wedge \psi \text{ if and only if } \mathscr{M}, \sigma \models \phi \text{ and } \mathscr{M}, \sigma \models \psi.$$

The truth conditions for the other nonmodal sentences are similar. For modal sentences we say:

$$\mathscr{M}, \sigma \models k\varphi \text{ if and only if } \mathscr{M}, \tau \models \varphi, \text{ for all } \tau \in P(\sigma),$$

$$\mathscr{M}, \sigma \models \Diamond\varphi \text{ if and only if there is a } \tau \in P(\sigma), \mathscr{M}, \tau \models \varphi.$$

The first truth condition says that one knows ϕ at a state σ if and only if ϕ holds in all the states that he considers possible at σ; the second, that he considers ϕ possible at σ if ϕ is true in at least one of the states that he considers possible at σ. We illustrate these notions in a simple example.

A Simple Example of Standard Model

The language of the system we consider has only two sentences, $p_1 = p$ and $p_2 = q$. We consider a subject who does not know whether p or q is true; as a consequence, the possibility correspondence is rather dull:

$$P(\sigma_i) = \Sigma$$

for every i. Note that since the system is $S5$, the subject is aware of the two atomic sentences. In fact more is true: he knows he does not know them. And in fact the model validates this. Take for instance $k \neg kp$. If we unravel the definitions, we have that $\mathscr{M}, \sigma \models k \neg kp$ if and only if it is not true that $\mathscr{M}, \tau \models p$ for all the $\tau \in \Sigma$, which is certainly the case.

The definition of GSM will have a similar structure, with one important difference: to introduce the possibility of unawareness, we will have to introduce the distinction between states that are conceivable by someone who is aware of all the possible events, and someone who is not.

Definition of GSM

To illustrate the construction we begin by noting that a state of the world is described as the intersection of events (so for instance the state in which it rains and there is no war is the intersection of the two events); and therefore described as a conjunction of sentences. Then in general the set of states of the world that an agent can (subjectively) conceive depends on the sentences of which he is aware. To each possible set of sentences corresponds a possible set of states of the world. There are however many different possible such sets; hence, the entire set of states of the world is given by the disjoint union of sets. Each of these sets corresponds to the set of the atomic sentences of which the subject is aware. This family of subsets is the first component of our state space. Together with it we define a projection from the "objective" state to the "subjective" state, where the "subjective" state space corresponds to the set of sentences of which the agent is aware. At each state the sentences of which the subject is not aware are irrelevant to the description of what he knows, and are therefore excluded by this projection; correspondingly at each subjective state the states considered possible are a subset of those which are conceivable.

Before we proceed with the formal definition, we fix once and for all an *enumeration* of the atomic sentences, that is a map $\mathcal{Q}$ from the set of natural numbers $\mathbb{N}$ into L. We let $p_n = \mathcal{Q}(n)$, and $\mathcal{Q}(N) = \{p_n : n \in N\}$ for $N \subseteq \mathbb{N}$.

We can now introduce more precisely the different elements of a GSM. An illustration of each of them is given in the simple example **1** in the next section. The reader may find it useful to refer to that example as he reads through the definition.

First, the state space Σ is:

$$\Sigma = \bigcup \{\Sigma_N | N \subseteq \mathbb{N}\} \tag{1}$$

where the union is disjoint. Each set Σ_N is the set of states where the subject is aware of the propositions in $\Lambda(\mathcal{Q}(N))$. These may be thought of as "objective states," states from the point of view of someone who is aware of all the atomic sentences in the language.

To each Σ_N corresponds its subjective version, a set Σ'_N. These are the states conceivable by an agent who is aware of the atomic sentences in $Q(N)$. We let

$$\Sigma' \equiv \bigcup \{\Sigma'_N | N \subseteq \mathbb{N}\},$$

where the union is disjoint. This is the second element.

The third element is a projection:

$$\pi : \Sigma \to \Sigma' \tag{2}$$

which is onto, so that letting $\Sigma'_N = \pi(\Sigma_N)$,

$$\Sigma' = \bigcup \{\pi(\Sigma_N) | N \subseteq \mathbb{N}\},$$

and π is such that given $\sigma, \tau \in \Sigma_N$, if $\pi(\sigma) = \pi(\tau)$ then for all $n \in N$ either both $\sigma, \tau \in \wp(n)$ or both $\sigma, \tau \notin \wp(n)$.

The reason for this restriction on π is that without it, the definition of $\wp_N$ in the truth condition that we give later would leave open the possibility that $\pi(\sigma) \in \wp_N(n)$ although $\sigma \notin \wp(n)$ (which occurs if there exists $\tau \in \Sigma_N$ such that $\tau \in \wp(n), \pi(\tau) = \pi(\sigma)$).

The fourth element is a function (possibility correspondence) $P: \Sigma \to 2^{\Sigma'}$ such that for all $N \subseteq \mathbb{N}$ and $\sigma \in \Sigma_N$, $P(\sigma) \subseteq \Sigma'_N \equiv \pi(\Sigma_N)$. The two functions P and π are such that for all $\sigma, \tau \in \Sigma_N$, if $\pi(\sigma) = \pi(\tau)$, when $P(\sigma) = P(\tau)$. This restriction says that possible states at σ only depend on what the subject sees, i.e., $\pi(\sigma)$; it makes the definition of P_N in the truth condition below unambiguous.

The truth conditions are defined as follows. Those for nonmodal sentences as usual. For $k\varphi$ and $\Diamond\varphi$ we define the standard models $\mathscr{M}_N = \langle \Sigma'_N, P_N, \wp_N \rangle$, for every $N \subseteq \mathbb{N}$, where for $\sigma' = \pi(\sigma) \in \Sigma'_N$, $P_N(\sigma') = P(\sigma)$ and $\wp_N(n) = \pi(\Sigma_N \cap \wp(n))$, $n \in N$. Notice that the domain of p_N is N, so $\mathscr{M}_N$ is a model or $\Lambda(\mathscr{Q}(N))$. Then define, for $\sigma \in \Sigma_N$:

$$\mathscr{M}, \sigma \vDash k\varphi \text{ if and only if } \mathscr{M}_N, \tau' \vDash \varphi, \text{ for all } \tau' \in P(\sigma),$$

$$\mathscr{M}, \sigma \vDash \Diamond\varphi \text{ if and only if there is a } \tau' \in P(\sigma), \mathscr{M}_N, \tau' \vDash \varphi.$$

Finally $\wp$ (the fifth element) is the usual evaluation of atomic sentences, namely a correspondence from $\mathbb{N}$ into Σ; $\wp(n)$ in the set of states, that is elements of Σ, in which the atomic sentence p_n is true.

Now that we have all the components, we may state the formal definition:

3.1. Definition (Generalized Standard Models). A GSM is a model

$$\mathscr{M} = \langle \Sigma, \Sigma', \pi, P, \wp \rangle.$$

As the reader will have noticed, the spirit of the truth conditions for $k\varphi$ and $\Diamond\varphi$ in GSM's is the same as in standard models. A GSM reduces to a standard model when: $\Sigma_N = \emptyset$ for all $N \subset \mathbb{N}$ (where $\subset$ denotes strict inclusion), $\Sigma' = \Sigma$, and $\pi = identity$.

Two Examples of GSM's

1. The first example is based on the one we presented in the previous section to illustrate standard models. There are the two atomic sentences, $p_1 = p$ and $p_2 = q$; we consider a subject who is aware of the atom p and not of q.

The set $\Sigma_{\{1\}}$ is $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$; $\wp(1) = \{\sigma_1, \sigma_2\}$, $\wp(2) = \{\sigma_1, \sigma_3\}$. We indicate under each σ_j the sentences which are true at σ_j:

$$\begin{array}{cccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ pq & p\neg q & \neg pq & \neg p \neg q. \end{array}$$

Next we define the projection: $\pi(\Sigma_{\{1\}}) = \Sigma'_{\{1\}} = \{\alpha, \beta\}$; $\pi(\sigma_1) = \pi(\sigma_2) = \alpha$, $\pi(\sigma_3) = \pi(\sigma_4) = \beta$. That is, σ_1 and σ_2 are not distinguished by the subject, and are perceived as α, and σ_3 and σ_4 are perceived as β. The atomic sentence p is true at σ_1 and σ_2, but from the point of view of the agent is true just at α: $\wp_{\{1\}}(1) = \pi(\Sigma_{\{1\}} \cap \wp(1)) = \{\alpha\}$. The world conceivable by the subject in any $\sigma \in \Sigma$ is now

$$\begin{array}{cc} \alpha & \beta \\ p & \neg p \end{array},$$

where the sentence q, of which the subject is not aware, does not appear. What specifies the subject's information about the world he perceives is the correspondence P. Let us suppose for example that $P(\sigma_i) = \Sigma'_{\{1\}}$, $i = 1, 2, 3, 4$, so the subject never knows whether p, the only atom of which he is aware, is true or false. Then for σ_i, $i = 1, \dots, 4$

$$\mathscr{M}, \sigma_i \vDash \neg kp \wedge k \neg kp \wedge \neg aq.$$

This follows from the definitions. For example, it is not the case that $\mathscr{M}, \sigma_i \vDash kq$. In fact, since $P(\sigma_i) = \{\alpha, \beta\}$ and it is not the case that $\mathscr{M}_{\{1\}}, \gamma \vDash q \; \forall \gamma \in \{\alpha, \beta\}$, then $\mathscr{M}, \sigma_i \vDash \neg kq$. It is also not the case that $\mathscr{M}_{\{1\}}, \gamma \vDash \neg q$, so analogously $\mathscr{M}, \sigma_i \vDash \neg k \neg q$; etc. In the same way one checks that for any $\varphi \notin \Lambda(\{p\})$, $\mathscr{M}, \sigma_i \vDash \neg a\varphi$. Note that the subject is aware of p and is not aware of q: this possibility was excluded in the system of unawareness $\mathscr{A}$ studied in Modica and Rustichini (1994).

Obviously for a subject who is not aware of q we have two states γ and δ, say, with projection $\pi'(\sigma_1) = \pi'(\sigma_3) = \gamma$ and $\pi'(\sigma_2) = \pi'(\sigma_4) = \delta$.

One way of looking at this is that the first agent is "confusing" the two states σ_1 and σ_2 into the state α (and σ_3 and σ_4 into β); while the second is "confusing" the states σ_1 and σ_3 into γ, and σ_2 and σ_4 into δ.

2. In this example we use a GSM to model the story reported in Geanakoplos (1989), and discussed in Modica and Rustichini (1994). The story is this: there are two states, σ and τ; at σ fact p ("the dog barks") is true and the subject (Sherlock Holmes' assistant) hears it so that he knows it is true; at τ, p is false and the subject not only does not hear it, but he does not even think of the possibility that the dog might be there: fact p is not present to his mind—from our point of view, he is not aware of p. Geanakoplos uses in his analysis a standard model which we contended to be inappropriate for the situation, for the story calls for two different perceived spaces at σ and τ; and this is possible in a GSM but not in a standard model.

To describe the GSM for this example a premise is necessary. We know how to define a standard model on $\Lambda(Q)$ for arbitrary nonempty Q. We now have to define it for $Q = \emptyset$, recalling the convention of Sec. 2 that $\top \in \Lambda(\emptyset)$. We define a standard model on $\Lambda(\emptyset)$ as

$$\mathscr{M} = \langle \{\sigma\}, P \rangle \text{ with } P(\sigma) = \{\sigma\}.$$

There is no $\wp$, since there are no atomic sentences. Then for example $\mathscr{M}, \sigma \vDash \top$ (this follows from the definition of model—Chellas (1980) p. 35), $\mathscr{M}, \sigma \vDash k \top$, etc.

The GSM that captures the present story is then

$$\mathscr{M} = \left\langle \Sigma_{\{1\}} \cup \Sigma_{\varnothing}, \Sigma', \pi, P, \wp \right\rangle$$

with

$$\Sigma_{\{1\}} = \{\sigma\}, \Sigma_{\varnothing} = \{\tau\}; \Sigma'_{\{1\}} = \{\alpha\}, \Sigma'_{\varnothing} = \{\beta\};$$

$$\pi(\sigma) = \alpha, \pi(\tau) = \beta; P(\sigma) = \{\alpha\}; P(\tau) = \{\beta\};$$

$$\wp(1) = \{\sigma\} (\text{so } \mathscr{M}, \tau \vDash \neg p), \wp(n) \text{ arbitrary for } n \neq 1, n \in \mathbb{N};$$

$$\mathscr{M}_{\{1\}} = \left\langle \{\alpha\}, P_{\{p\}}, \wp_{\{p\}} \right\rangle \text{ on } L = \{p\} \text{ with } P_{\{p\}}(\alpha) = \{\alpha\}$$

and $\wp_{\{p\}}(p) = \{\alpha\}$;

$$\mathscr{M}_{\varnothing} = \left\langle \{\beta\}, P_{\varnothing} \right\rangle \text{ on } L = \varnothing \text{ with } P_{\varnothing}(\beta) = \{\beta\}$$

(according to the definition above).

Then $\mathscr{M}, \sigma \vDash kp \wedge \neg a\psi$ for all $\psi \notin \Lambda(\{p\})$, and $\mathscr{M}, \tau \vDash \neg a\psi$ for all $\psi \notin \Lambda(\varnothing)$.

The model at τ looks so strange because we are assuming that there is only one thing that this subject may have in mind (namely, "the dog is barking"), so if he does not have it in mind, as in τ, he "merely exists," in that his knowing $\top$ may be interpreted perhaps as self-awareness, a *cogito ergo sum*, but his mind is otherwise empty. Of course this is an extreme case.

GSM's, Basic Properties

In the following $\mathscr{M}$ will denote a GSM, and N a subset of $\mathbb{N}$.

3.2. PROPOSITION. *Let* $\sigma \in \Sigma_N$. *If* $\mathscr{M}, \sigma \vDash a\varphi$, *then* $\varphi \in \Lambda(Q(N))$.

Proof. This follows by applying definitions: if $\varphi \notin \Lambda(Q(N))$, for any $\alpha \in \Sigma'_N$, it is neither the case that $\mathscr{M}_N, \alpha \vDash \varphi$ nor that $\mathscr{M}_N, \alpha \vDash \neg\varphi$, so in particular $\mathscr{M}, \sigma \vDash \neg k\varphi$. Also observe that if $\varphi \notin \Lambda(Q(N))$, then $\neg k\varphi \notin \Lambda(Q(N))$. ■

3.3. PROPOSITION. *Let* $\sigma \in \Sigma_N$ *and* $\varphi \in \Lambda(Q(N))$. *Then*

$$\mathscr{M}, \sigma \vDash \varphi \text{ iff } \mathscr{M}_N, \pi(\sigma) \vDash \varphi.$$

Proof. The proof is by induction on the complexity of φ. Suppose $\varphi = p \in Q(N)$. If $\mathscr{M}, \sigma \vDash p$, then $\mathscr{M}_N, \pi(\sigma) \vDash p$ by definition of $\wp_N$. The converse follows from the property of π in the last part of (ii) of Definition 3.1. The cases of $\neg \varphi$ and $\varphi \wedge \psi$ with inductive hypothesis on φ, ψ follow by applying the definitions of $\mathscr{M}, \sigma \vDash \neg \varphi, \mathscr{M}, \sigma \vDash \varphi \wedge \psi$ and $\mathscr{M}_N, \pi(\sigma) \vDash \neg \varphi$, $\mathscr{M}_N, \pi(\sigma) \vDash \varphi \wedge \psi$. For $k\varphi$ the result follows directly by definition. ■

The following corollary is noteworthy; it says that if the subject is aware of everything then he sees things as they really are, i.e., effectively $\Sigma'_{\mathbb{N}} = \pi(\Sigma_{\mathbb{N}})$ is the same thing as $\Sigma_{\mathbb{N}}$.

3.4. COROLLARY. *Assume there are no duplicates in $\mathscr{M}$, in the sense that any pair of disjoint $\sigma, \tau \in \Sigma$ disagree on some $\varphi \in \Lambda$, i.e., for any such σ, τ there exists $\varphi \in \Lambda$ such that $\mathscr{M}, \sigma \vDash \varphi$, $\mathscr{M}, \tau \vDash \neg \varphi$. Then the restriction of π to $\Sigma_{\mathbb{N}}$ is one to one onto $\Sigma'_{\mathbb{N}}$, and corresponding states agree on all sentences of Λ.*

Proof. π is onto by part (i) of Definition 3.1, and the rest is immediate consequence of the proposition 3.3. ■

4. SYSTEM $\mathscr{U}$ AND PARTITIONAL GSM'S

In this section the main result of the paper is stated. Recall that a system of modal logic is determined by a class of models if it is sound and complete with respect to this class, i.e., every theorem of the system is valid in the class, and conversely every sentence valid in the class is a theorem of the system. The result is that $\mathscr{U}$ is determined by the class of *partitional* GSM's (Definition 4.1 below).

We call *partitional* a correspondence from Σ to subsets of Σ that is reflexive, transitive and symmetric.

4.1. DEFINITION. A GSM is reflexive, resp. transitive, partitional if for all $N \subseteq \mathbb{N}$, P_N is reflexive, resp. transitive, partitional.

It may be useful to express these properties of P_N in terms of P and compare them with their counterparts in standard models. Take for example reflexivity. In a standard model with possibility correspondence P reflexivity means that $\sigma \in P(\sigma)$, i.e., among the possible states there is the true one; we shall see in a moment that in a GSM with possibility correspondence P, reflexivity means that $\pi(\sigma) \in P(\sigma)$: among the possible states there is the true one *as the subject sees it*. To put it loosely, in the case of standard models the subject is nondeluded in the sense that if he is informed that the state is σ he can say "Indeed I thought σ was possible''; in the case of generalized standard models he is nondeluded as far as his awareness lets him be. So when he is told ''σ'' he really understands ''$\pi(\sigma)$'' (because this is all he can conceive, on the basis of what he is aware of) so he can say ''Indeed I thought $\pi(\sigma)$ was possible.''

4.2. PROPOSITION. *Let $\mathscr{M}$ be a GSM.*

(i) *$\mathscr{M}$ is reflexive if and only if for all $\sigma \in \Sigma$, $\pi(\sigma) \in P(\sigma)$;*

(ii) *$\mathscr{M}$ is transitive if and only if: if $\beta \in P(\sigma)$ and for some (hence, all by part (iv) of Definition* 3.1) *$\tau \in \Sigma$ such that $\pi(\tau) = \beta$ one has $\gamma \in P(\tau)$, then $\gamma \in P(\sigma)$;*

(iii) *$\mathscr{M}$ is symmetric iff $\pi(\tau) \in P(\sigma)$ implies $\pi(\sigma) \in P(\tau)$.*

Proof. (i) checks directly by inspecting part (v) of Definition 3.1, which says that $P_N(\pi(\sigma)) = P(\sigma)$.

(ii) is really just reading definitions: P_N is transitive iff for all $\alpha, \beta, \gamma \in \Sigma'_N = \pi(\Sigma_N)$, $\beta \in P_N(\alpha)$ and $\gamma \in P_N(\beta)$ imply $\gamma \in P_N(\alpha)$. But say $\alpha = \pi(\sigma)$, $\beta = \pi(\tau)$, where σ and τ can be chosen arbitrarily as long as their projections remain α and β. Then again by part (v) of Definition 3.1, $\beta \in P_N(\alpha)$ means $\beta \in P(\sigma)$; $\gamma \in P_N(\beta)$ means $\gamma \in P(\tau)$; and $\gamma \in P_N(\alpha)$ is $\gamma \in P(\sigma)$. The result is now obvious.

(iii) Symmetry of P_N is that $\forall \alpha, \beta \in \pi(\Sigma_N)$, $\beta \in P_N(\alpha)$ implies $\alpha \in P_N(\beta)$; but with $\alpha = \pi(\sigma)$ and $\beta = \pi(\tau)$, the result is again direct. ■

4.3. THEOREM. *$\mathscr{U}$ is determined by the class of partitional GSM's.*

Proof is in the appendix. Two observations about the class of partitional GSM's:

4.4. PROPOSITION. (i) *the class of partitional GSM's does not validate axiom* 5;

(ii) *for any $\mathscr{M}$ in this class, let $\sigma \in \Sigma_N$. Then $\mathscr{M}, \sigma \vDash a\varphi$ iff $\varphi \in \Lambda(Q(N))$.*

Proof. To prove (i) it suffices to take an $\mathscr{M}$ in the class with Σ such that for some $N \subset \mathbb{N}$, $\Sigma_N \neq \emptyset$. For (ii), ''only if'' is contained in Proposition 3.2, ''if'' follows from the fact that for any $N \subseteq \mathbb{N}$, P_N is partitional, and from the result on the standard model $\mathscr{M}_N$ (Chellas theorem 3.5 and exercise 3.31). ■

5. CONCLUSIONS

We begin with the objection to our concept of awareness that we reported in the introduction.

In (I), "ϕ *is possible*" is understood as a subjective statement, as "I do not know the negation of that something holds." In formal terms, it translates into $\Diamond\phi$, that is $\neg k \neg \phi$, (our definition of $\Diamond$ is slightly different, but this is immaterial for the present discussion). If we use this in (II), and take $1 \neq 1$ as $\perp$, the false, then the sentence "I am aware that the false is possible," $a\Diamond\perp$, reduces to $k \neg k \neg k \top$; which is really not strange, if one does not find strange that $k \top$ is a theorem.

It is true however that I may accept as obvious that "I am not aware that $1 \neq 1$ is possible" if I read ϕ *is possible* as "ϕ is not always false." Formally, this translates into $\neg(\phi \leftrightarrow \perp)$, but this is not what (I) says.

One may however express the desire for a concept of awareness that is free of the ambiguity. We observe that this can be done with an easy modification of our theory. First, we define "being Aware of something" as "being aware of something which is not identically false," that is we set

$$A\phi \equiv a\phi \wedge \neg(\phi \leftrightarrow \perp).$$

It is easy to see that $A\phi \leftrightarrow A\Diamond\phi$. The symmetry axiom is no longer natural for the operator A, as we have seen in the introduction, but it has a natural reformulation, namely:

$$(A\phi \leftrightarrow A\neg\phi) \leftrightarrow (\neg(\phi \leftrightarrow \perp) \wedge \neg(\phi \leftrightarrow \top)),$$

that is, the symmetry axiom holds for all the sentences that are not equivalent to the true or to the false.

Then all the theorems that we have presented give, with the appropriate modifications, a complete characterization of A.

We now consider some of the issues whose investigation is the natural development of the present line of research.

The first is the model of learning and updating from the point of view of the present theory. We have defined ignorance as a more general concept than uncertainty, because it includes the possibility of unawareness. Correspondingly, learning as resolution of the uncertainty, as refinement of the information on a given set of states should have the additional dimension of *becoming aware*: the ignorance of the decision maker may decrease either because he reduces, thanks to additional information, the set of states that he considers possible, or because he extends this set, thanks to increased awareness. The Generalized Standard Models introduced in this paper are a natural starting point for a dynamic model that combines learning and becoming aware.

Second, it is an open issue how to model awareness of unawareness. In fact, the subject's state of knowledge with respect to his own subjective model is not covered in the present theory. For instance, we cannot formally express, and therefore even less prove consistency in well defined system, the idea that the subject is aware of the possibility that he might be ignoring some fact. In other words, the subject's view of his own model is not formalized. This specific issue, and its relationship with a decision theoretic model of "awareness of not being aware" seems an important open question.

Other ideas concern unawareness in multi-agent settings. For a pictorial example, in the language of unawareness one can naturally express the idea of "taking by surprise:" firm A takes B by surprise when it exploits knowledge of a φ knowing that B is unaware of it (hence, of the fact that A may know it). More substantially, consider the rationale of the prototype no-trade theorem: trader A rejects B's proposal because the latter reveals information by offering; with unawareness, A might not value the information revealed by B by thinking "B thinks he is going to make a good deal just because, wrongly, he only sees the good side of it," and consequently might accept the proposal. In games, one might contend that an important issue is that of unexpected moves (or even players ...), and start thinking about interactive rationality in such a context.

6. APPENDIX

The Axioms A, AM, and DE

In Modica and Rustichini (1993) we prove that in any system that contains (M, C, T, RE_{sa}) the axiom AM is equivalent to the following:

$$DE.\ \neg k\varphi \wedge k(\neg k\varphi \vee \neg k\psi) \to k \neg k\varphi.$$

The intuitive content of DE is less transparent: to help the reader we may note that this axiom says, "Suppose you know that $\neg k\varphi$ or $\neg k\psi$; then you may ask yourself about which one is true (by 'positive' introspection). If $\neg k\varphi$ is true, then you will know it is." The system $\mathscr{A}$ of Modica and Rustichini (1994) has DE, by Proposition 5.1 there. We now spell out some basic properties of awareness in $\mathscr{U}$ and weaker systems.

As we said earlier, in the system $\mathscr{U}$ the axiom 5 of $S5$ is replaced by the two axioms AM and A. Modica and Rustichini (1993) prove that:

6.1. LEMMA. *Any $\mathscr{S}(M, C, T, RE_{sa})$ contains DE iff it contains AM.*

Both DE and A are, clearly, weaker axioms than 5. In the two cases the weakening is of similar nature: axiom 5 imposes that a pure lack of knowledge produces awareness, in the form $k \neg k\varphi$; on the contrary, DE and A require awareness as a condition in the hypothesis.

6.2. LEMMA. *$\mathscr{S}(M, C, T, A, A, RE_{sa})$ contains AM if and only if it contains $a(\varphi \vee \psi) \to a\varphi \wedge a\psi$.*

Proof. Only if. In the following chain we use in turn A, RE_{sa} (Lemma 2.4(iii)), A, the hypothesis, and A: $a(\varphi \vee \psi) \leftrightarrow_{\mathscr{S}} a(\neg\neg(\varphi \vee \psi)) \leftrightarrow_{\mathscr{S}} a(\neg(\neg\varphi \wedge \neg\psi)) \leftrightarrow_{\mathscr{S}} a(\neg\varphi \wedge \neg\psi) \to_{\mathscr{S}} a\neg\varphi \wedge a\neg\psi \leftrightarrow_{\mathscr{S}} a\varphi \wedge a\psi$.

If. Using M, C and RM_{sa} one obtains $\vdash_{\mathscr{S}} k(\varphi \wedge \psi) \vee k\neg k(\varphi \wedge \psi) \to k\varphi \vee k(\neg k\varphi \vee \neg k\psi)$ (for: by C, $\vdash_{\mathscr{S}} k\varphi \wedge k\psi \to k(\varphi \wedge \psi)$; so by PL, $\vdash_{\mathscr{S}} \neg k(\varphi \wedge \psi) \to \neg k\varphi \vee \neg k\psi$; so by RM_{sa}, $\vdash_{\mathscr{S}} k\neg k(\varphi \wedge \psi) \to k(\neg k\varphi \vee \neg k\psi)$). But, using in turn PL, the hypothesis, A, and Proposition 2.8(i) (which uses 4) we have the following: $k(\neg k\varphi \vee \neg k\psi) \to_{\mathscr{S}} a(\neg k\varphi \vee \neg k\psi) \to_{\mathscr{S}} a\neg k\varphi \leftrightarrow_{\mathscr{S}} ak\varphi \leftrightarrow_{\mathscr{S}} a\varphi$. Thus, $\vdash_{\mathscr{S}} a(\varphi \wedge \psi) \to a\varphi$. Since by RE_{sa} $\vdash_{\mathscr{S}} a(\varphi \wedge \psi) \leftrightarrow a(\psi \wedge \varphi)$, one has $\vdash_{\mathscr{S}} a(\varphi \wedge \psi) \to a\varphi \wedge a\psi$. ■

The following proposition provides a characterization of AM:

6.3. PROPOSITION. *$\mathscr{S}(M, C, T, 4, A, RE_{sa})$ contains AM iff it contains $a\varphi \wedge a\psi \leftrightarrow a(\varphi \wedge \psi)$ and $a\varphi \wedge a\psi \leftrightarrow a(\varphi \vee \psi)$.*

Proof. Direct corollary to Lemmas 6.1 and 6.2 and Proposition 2.9. ■

Notation. We introduce two notational conventions. The collection of all the maximally consistent sets of a system $\mathscr{S}(\cdot)$ (cfr. p. 8) will be denoted by $\Sigma(\mathscr{S}(\cdot))$, or simply Σ if possible. With σ denoting any set of sentences in Λ, we let $a^{-}(\sigma) = \{\varphi \in \Lambda | a\varphi \in \sigma\}$, the awareness set of σ.

A and AM together are characterized by existence of a "subjective language" at any maximally consistent set, built on the atomic sentences of which the subject is aware:

6.4. PROPOSITION. *$\mathscr{S}(M, C, T, 4, RE_{sa})$ contains A and AM iff for any $\sigma \in \Sigma$, $a^{-}(\sigma)$ is of the form $\Lambda(Q)$ for some $Q \subseteq L$.*

Proof. We prove the statement for any system $\mathscr{S}(M, C, T, 4, RE_{sa})$ that contains A and DE, in view of our result 6.1. So assume A and DE. We show that $\forall\sigma \in \Sigma$, $a^{-}(\sigma) = \Lambda(Q)$ with $Q = a^{-}(\sigma) \cap L$. First, $a^{-}(\sigma) \subseteq \Lambda(a^{-}(\sigma) \cap L)$. For, let $\xi \in a^{-}(\sigma)$. We use induction on ξ. If $\xi \in L$, assertion is true. If assertion is true for φ of complexity n, then let ξ be of complexity $n + 1$, i.e., $\xi = \neg\varphi$, $\varphi \wedge \psi$ or $k\varphi$ with φ, ψ of complexity $\leq n$. Suppose $\xi = k\varphi \in a^{-}(\sigma)$, i.e., $ak\varphi \in \sigma$. By Proposition 2.8(i) and maximality of $\sigma, a\varphi \in \sigma$, i.e., $\varphi \in a^{-}(\sigma)$. By inductive hypothesis, $\varphi \in \Lambda(a^{-}(\sigma) \cap L)$, so also $k\varphi = \xi \in \Lambda(a^{-}(\sigma) \cap L)$. The other cases are analogous using A and DE respectively. Hence, $a^{-}(\sigma) \subseteq \Lambda(a^{-}(\sigma) \cap L)$. To show the converse inclusion, take $\xi \in \Lambda(a^{-}(\sigma) \cap L)$; then $\xi \in a^{-}(\sigma)$ just because $a^{-}(\sigma)$ is closed under $\wedge$, $\neg$, and k (resp. Proposition 2.9(i), A, and Proposition 2.8(i)).

Conversely, assume that for each $\sigma \in \Sigma$ it is $a^{-}(\sigma) = \Lambda(Q)$ for some $Q \in L$. Then clearly $\forall \sigma \in \Sigma$, $a\varphi \in \sigma$ iff $a \neg \varphi \in \sigma$; hence, by consistency of σ and Chellas' Theorem 2.20(ii), $\vdash_{\mathscr{S}} a\varphi \leftrightarrow a \neg \varphi$. Analogously, $\vdash_{\mathscr{S}} a(\varphi \wedge \psi) \rightarrow a\varphi \wedge a\varphi$, which by Lemma 6.1 is equivalent to DE in our system. ■

6.1. PROOF OF THEOREM 4.3

The proof of the theorem consists of two parts, one for soundness and one for completeness, which will be taken up in turn. Together they will yield the main theorem.

We recall that a sentence φ is said to be valid in $\mathscr{M}$ if $\mathscr{M}, \sigma \vDash \varphi$ for all σ's in $\mathscr{M}$; this is denoted $\mathscr{M} \vDash \varphi$. Whenever unspecified, $\mathscr{M}$ will denote a GSM.

Soundness

To prove soundness we have to check that all axioms of $\mathscr{U}$ are valid in the class of partitional GSM's, and that this class is closed under its inference rules. We refer to Definition 2.3, where $\mathscr{U}$ is defined, and do the checking one by one. Soundness will follow from the lot. To anticipate, we shall see that any GSM is closed with respect to rules RPL and RE_{sa} and validates M, C and N; reflexive GSM's validate T, transitive GSM's validate 4, and partitional GSM's also validate A, AM and $Df'\Diamond$; or equivalently A, DE and $Df'\Diamond$. It is clear that in any GSM, hence, in any class, rule RPL preserves validity, since any class of model is closed with respect to RPL.

($Df'\Diamond$). *The class of partitional* GSM's *validates* $Df'\Diamond$.

Proof. Let $\sigma \in \Sigma_N$. One has $\mathscr{M}, \sigma \vDash \neg k \neg \varphi \wedge a\varphi$ iff $[\mathscr{M}, \sigma \vDash \neg k \neg \varphi$ and $\mathscr{M}, \sigma \vDash a\varphi]$ iff—by Proposition 4.4(ii)—$[\mathscr{M}, \sigma \vDash \neg k \neg \varphi$ and $\varphi \in \Lambda(\mathscr{Q}(N))]$ iff—apply definition—$[\varphi \in \Lambda(\mathscr{Q}(N))$ and $\exists \beta \in P(\sigma)\mathscr{M}_N, \beta \vDash \varphi]$ iff—by definition—$\mathscr{M}, \sigma \vDash \Diamond\varphi$. ■

(M, *C and N*). *Any GSM validates M, C and N, so any class of* GSM's *does*.

Proof. For N, by applying the definition one sees that $\top$ is valid in any state of any GSM. For M: suppose $\mathscr{M}, \varphi \vDash k(\varphi \wedge \psi)$, and say $\sigma \in \Sigma_N$. Then by Proposition 3.3, $\mathscr{M}_N, \pi(\sigma) \vDash k(\varphi \wedge \psi)$, hence, by the result for standard models (Chellas exercise 3.5) $\mathscr{M}_N, \pi(\sigma) \vDash k\varphi \wedge k\psi$, so by Proposition 3.3 again, $\mathscr{M}, \sigma \vDash k\varphi \wedge k\psi$. For C it is the same thing. ■

(RE_{sa}). *In any* GSM, *hence in any class, rule RE_{sa} preserves validity.*

Proof. Suppose that the atomic sentences of φ, $At(\varphi)$, satisfy $At(\varphi) = At(\psi) = \mathscr{Q}(N)$, so that $\varphi, \psi \in \Lambda(\mathscr{Q}(N))$, and that $\mathscr{M} \vDash \varphi \leftrightarrow \psi$. Then if $\sigma \notin \Sigma_N$, $\mathscr{M}, \sigma \vDash \neg k\varphi \wedge \neg k\psi$, so $\mathscr{M}, \sigma \vDash k\varphi \leftrightarrow k\psi$. If $\sigma \in \Sigma_N$, observe that (by Proposition 3.3 and the fact that π on Σ_N is onto Σ'_N) one has $\mathscr{M}_N \vDash \varphi \leftrightarrow \psi$, so by the result for standard models (Chellas exercise 3.7(c)) $\mathscr{M}_N \vDash k\varphi \leftrightarrow k\psi$; hence, again from Proposition 3.3, for all $\sigma \in \Sigma_N$ it is $\mathscr{M}, \sigma \vDash k\varphi \leftrightarrow k\psi$. ■

Remark. A GSM does not necessarily validate rule *RE*. For an example, take a model $\mathscr{M}$ with $\Sigma = \Sigma_{\{p\}} = \sigma$, $\Sigma' = \Sigma'_{\{p\}} = \{\pi(\sigma)\}$, $P(\sigma) = \pi(\sigma)$, $\wp(p) = \{\sigma\}$. Take $\varphi = p$, $\psi = p \wedge (q \vee \neg q)$. Then $\mathscr{M} \vDash \varphi \leftrightarrow \psi$, but $\mathscr{M}, \sigma \vDash k\varphi$ and $\mathscr{M}, \sigma \vDash \neg k\psi$ (for $\psi \notin \Lambda(\{p\})$), hence, it is not that case that $\mathscr{M} \vDash k\varphi \leftrightarrow k\psi$.

(DE). *The class of partitional* GSM's *validates DE.*

Proof. Let $\sigma \in \Sigma_N$, and suppose $\mathscr{M}, \sigma \vDash \neg k\varphi \wedge k(\neg k\varphi \vee \neg k\psi)$. The fact that $\mathscr{M}, \sigma \vDash k(\neg k\varphi \vee \neg k\psi)$ implies $\varphi, \psi \in \Lambda(\mathscr{Q}(N))$, in particular $\neg k\varphi \in \Lambda(\mathscr{Q}(N))$, hence, $\mathscr{M}, \sigma \vDash \neg k\varphi$ implies by Proposition 3.3 $\mathscr{M}_N, \pi(\sigma) \vDash \neg k\varphi$; but since $\mathscr{M}_N$ is partitional this implies $\mathscr{M}_N, \pi(\sigma) \vDash k\neg k\varphi$, and by Proposition 3.3 again, $\mathscr{M}, \sigma \vDash k\neg k\varphi$, as was to be shown. ■

(T). *The class of reflexive* GSM's *validates axiom T.*

Proof. Obvious by Proposition 3.3 and Chellas Theorem 3.5. ■

(4). *The class of transitive* GSM's *validates axiom* 4.

Proof. Take any transitive GSM $\mathscr{M}$, and suppose $\mathscr{M}, \sigma \vDash k\varphi$, that is, for all $\beta \in P(\sigma), \mathscr{M}_N, \beta \vDash \varphi$ $(\sigma \in \Sigma_N)$. To show $\mathscr{M}, \sigma \vDash kk\varphi$, i.e., $\forall \beta \in P(\sigma), \mathscr{M}_N, \beta \vDash k\varphi$, i.e., $\forall \beta \in P(\sigma), \forall \gamma \in P_N(\beta), \mathscr{M}_N, \gamma \vDash \varphi$. Now with $\tau \in \Sigma$ such that $\pi(\tau) = \beta$ (so necessarily $\tau \in \Sigma_N$), $P_N(\beta) = P(\tau)$. Since $\mathscr{M}$ is transitive, by Proposition 4.2(ii) any $\gamma \in P(\tau)$ is such that $\gamma \in P(\sigma)$. From $\mathscr{M}, \sigma \vDash k\varphi$, for all such γ's $\mathscr{M}_N, \gamma \vDash \varphi$. Conclusion follows. ■

(A). *The class of partitional* GSM's *validates axiom A (besides T*, 4).

Proof. Let $\sigma \in \Sigma_N$ and suppose $\mathscr{M}, \sigma \vDash a\varphi$. Then (Proposition 4.4) easily $\varphi \in \Lambda(Q)$, and $\forall \psi \in \Lambda(\mathscr{Q}(N)), \mathscr{M}_N, \pi(\sigma) \vDash a\psi$ (since P_N is partitional on $\pi(\Sigma_N)$, this follows from Chellas theorem 3.5). So $\mathscr{M}_N, \pi(\sigma) \vDash a\neg\varphi$, and by Proposition 3.3 $\mathscr{M}, \sigma \vDash a\neg\varphi$. Thus $\mathscr{M}, \sigma \vDash a\varphi \rightarrow a\neg\varphi$. The opposite implication is analogous. ■

This completes the soundness part.

Completeness

The method followed for the proof of the completeness part is that of canonical models, which we briefly recall. A canonical model $\mathscr{M}$ for a system $\mathscr{S}$ on a language Λ is so designed that for all $\varphi \in \Lambda$,

$$\mathscr{M} \vDash \varphi \text{ iff } \vdash_{\mathscr{S}} \varphi. \tag{1}$$

So a canonical model constitutes by itself a determining class for the system; and the system is complete with respect to any class containing such a model. The aim of a completeness proof that uses canonical models is just to produce one in the desired class.

To build a model which satisfies the relation (1) one uses a key result on systems of modal logic, namely:

> *Given a system $\mathscr{S}$ on a language Λ, and any sentence $\varphi \in \Lambda$, φ is a theorem of $\mathscr{S}$ iff it belongs to all the $\mathscr{S}$-maximally consistent sets, i.e., $\vdash_{\mathscr{S}} \varphi$ iff $\varphi \in \Gamma$ for all $\Gamma \in \Sigma(\mathscr{S})$.*

(see for instance Chellas (1980), Theorem 2.20(2)). To achieve (1) one then takes as state space Σ of $\mathscr{M} = \langle \Sigma, \dots, \wp \rangle$ the set $\Sigma(\mathscr{S})$ of $\mathscr{S}$-maximally consistent sets and designs truth conditions such that for all $\sigma \in \Sigma$, and $\varphi \in \Lambda$

$$\mathscr{M}, \sigma \vDash \varphi \text{ iff } \varphi \in \sigma. \tag{2}$$

(2) and the Theorem quoted above give (1). To get (2) in turn, one first sets $\wp(n) = \{\sigma \in \Sigma(\mathscr{S}) | p_n \in \sigma\} \equiv |p_n|_{\mathscr{S}}$ (the $\mathscr{S}$-proof set of p_n), and this gives (2) for all nonmodal φ's; so the whole task reduces to define truth conditions for $k\varphi$ (and $\Diamond\varphi$, but we will ignore this for the moment) so as to have (2) for this type of sentences too.

Let us see for example how this leads to the construction of standard models and canonical standard models for "normal" systems (for a definition of normal system, see Chellas). The result used in this case is the following:

> *Let $\mathscr{S}$ be a normal system on Λ and take $\sigma \in \Sigma(\mathscr{S})$. Let $k^{-}(\sigma) = \{\varphi \in \Lambda | k\varphi \in \sigma\}$. Then for any $\varphi \in \Lambda$,*
>
> $$k\varphi \in \sigma \textit{ iff } \varphi \in \tau \textit{ for all } \tau \in \Sigma(\mathscr{S}) \textit{ such that } k^{-}(\sigma) \subseteq \tau.$$

(see e.g. Chellas (1980), Theorem 4.30.(1)). This result, on the basis of hypothesis (2) for sentences smaller than $k\varphi$, says: $k\varphi \in \sigma$ iff $\mathscr{M}, \tau \vDash \varphi$ for some special τ's in $\Sigma(\mathscr{S})$. Then it is natural to *define* $\mathscr{M}, \sigma \vDash k\varphi$ iff $\mathscr{M}, \tau \vDash \varphi$ for a set of τ's "related" to σ, i.e. a $P(\sigma)$, and complete the canonical model $\mathscr{M} = \langle \Sigma, \dots, \wp \rangle$ (where Σ and $\wp$ are always specified as above) by letting

$$P(\sigma) = \{\tau \in \Sigma | k^{-}(\sigma) \subseteq \tau\}.$$

In fact with these definitions one steps on hypothesis (2) for smaller sentences and obtains: $\mathscr{M}, \sigma \models k\varphi$ iff—by definition—$\mathscr{M}, \tau \models \varphi \forall \tau \in P(\sigma)$ iff—hypothesis—$\varphi \in \tau \ \forall \tau \in P(\sigma)$ iff—Chellas' theorem 4.30(1) and definition of $P(\sigma)$—$k\varphi \in \sigma$. This gives (2) for all φ's, as wanted. Then (1) gives completeness of any normal system with respect to the class of all standard models.

To sum up: one takes the "skeleton" canonical model $\mathscr{M} = \langle \Sigma, \dots, \wp \rangle$ which gives (2) for nonmodal sentences; looks for suitable truth conditions for $\mathscr{M}, \sigma \models k\varphi$ so as to achieve (2) for all φ's; and then uses (2) and Chellas' theorem 2.20(2) to get (1). We now proceed to see how this search goes for $\mathscr{U}$, thus outlining the proof to follow.

Notation. Let σ be any set of sentences in Λ. We let $L(\sigma) = a^{-}(\sigma) \cap L$, and $\Lambda(\sigma) = \Lambda(L(\sigma))$, the language built on the atomic sentences of which the subject is aware in σ. Notice that in $\mathscr{U}$, $a^{-}(\sigma) = \Lambda(\sigma)$, by Proposition 6.4. Always, $k^{-}(\sigma) = \{\varphi \in \Lambda | k\varphi \in \sigma\}$.

A system $\mathscr{S}$ is always a set of sentences in a language, and so far all systems considered were on Λ, in the sense that the axiom schemata and inference rules concerned sentences in Λ; in what follows we shall also consider systems on subsets of Λ, namely on $\Lambda(Q)$, $Q \subseteq L$. By $\mathscr{S}(Q)$ we shall denote the system generated on $\Lambda(Q)$ by the same axioms and inference rules of $\mathscr{S}$ (for example, $S5$ on Q will be denoted by $S5(Q)$). For $Q = L$ the argument is usually omitted: $\mathscr{S} = \mathscr{S}(L)$. As before $\Sigma(\mathscr{S}(Q))$ denotes the family of maximally consistent sets of $\mathscr{S}(Q)$.

If $\sigma \in \Sigma(\mathscr{U})$, $k^{-}(\sigma) \subseteq \Lambda(\sigma)$ (for knowledge implies awareness and by Proposition 6.4 $a^{-}(\sigma) = \Lambda(\sigma)$), therefore $k^{-}(\sigma) = k^{-}(\sigma) \cap \Lambda(\sigma)$; and it is not difficult to see (6.10 below) that, letting $\pi(\sigma) = \sigma \cap \Lambda(\sigma)$ (the set of sentences in σ of which the subject is aware, by Proposition 6.4), one has $k^{-}(\sigma) \cap \Lambda(\sigma) = k^{-}(\pi(\sigma))$. Thus, $\varphi \in k^{-}(\sigma)$ iff $\varphi \in k^{-}(\pi(\sigma))$. Now in fact $\pi(\sigma)$ is $S5(L(\sigma))$ maximal (6.8); thus $\varphi \in k^{-}(\pi(\sigma))$, i.e., $k\varphi \in \pi(\sigma)$, iff—by Chellas' 4.30(1) applied to $\pi(\sigma)$—φ is true at all states possible at $\pi(\sigma)$ in the $S5(L(\sigma))$-canonical model. This condition, equivalent to $k\varphi \in \sigma$, suits our purposes. For if we make this condition the definitory condition for validity of $k\varphi$ at $\sigma \in \Sigma(\mathscr{U})$, we then get what we want, that $k\varphi$ is true at σ iff $k\varphi \in \sigma$.

The relation between $\Sigma(\mathscr{U})$ and the canonical models of $S5(L(\sigma))$, $\sigma \in \Sigma(\mathscr{U})$, is given by the structure of $\Sigma(\mathscr{U})$, which is a union whose generic component is a set of σ's where the subject is aware of the same things, $\Sigma(\mathscr{U}) = \bigcup\{\Sigma_N(\mathscr{U}) | N \subseteq \mathbb{N}\}$ (6.7); and by the fact that π on each of these subsets is not only into (6.8), but also onto: $\pi(\Sigma_N(\mathscr{U})) = \Sigma(S5(Q(N)))$ (6.9). This is why a GSM is defined as it is, with a projection π and the truth condition for $k\varphi$ which refers to the models obtained as projections by π.

Now the details of the proof. To prove completeness we will produce a model, denoted by $\mathscr{M}(\mathscr{U})$, which is a canonical model for $\mathscr{U}$, so satisfies (1), i.e.,

$$\mathscr{M}(\mathscr{U}) \vDash \varphi \textit{ iff } \vdash_{\mathscr{U}} \varphi, \varphi \in \Lambda,$$

and is a partitional GSM. We begin with two lemmas on logical systems, then go on with the various points which prove the theorem.

6.5. LEMMA. *Let $\mathscr{S}$ be a system and $Q \subseteq Q' \subseteq L$. Then $\mathscr{S}(Q') \cap \Lambda(Q) = \mathscr{S}(Q)$.*

Proof. $\mathscr{S}(Q) \subseteq \mathscr{S}(Q') \cap \Lambda(Q)$ is clear; for the reverse inclusion, consider any $\varphi \in \mathscr{S}(Q') \cap \Lambda(Q)$ and let $\{\varphi_1, \varphi_2, \dots, \varphi_k\}$ be a proof of φ, i.e., a set Γ of sentences in $\Lambda(Q')$ such that $\varphi_k = \varphi$, and for every $j \leq k$, φ_j is either an axiom of $\mathscr{S}(Q')$ or is derived from a subset of Γ, $\{\varphi_{j_i}\}$, $j_i \leq j$ and an inference rule of $\mathscr{S}(Q')$. Let $Q'' = Q' \setminus Q$, and for a fixed $q \in Q$ define the map ρ: $Q' \to Q$ as: $\rho(p) = p$ if $p \in Q$, $\rho(p) = q$ if $p \in Q''$. This map defines a translation of sentences in $\Lambda(Q')$ into sentences of $\Lambda(Q)$ if we define it recursively on $\Lambda(Q')$ by: $\rho(\neg \varphi) = \neg \rho(\varphi)$, $\rho(\varphi \wedge \psi) = \rho(\varphi) \wedge \rho(\psi)$ and so on for the other nonmodal connectives, plus $\rho(k\varphi) = k\rho(\varphi)$ and $\rho(\Diamond\varphi) = \Diamond\rho(\varphi)$. Consider now any axiom of $\mathscr{S}(Q')$. Its translation is clearly an axiom of $\mathscr{S}(Q)$. Similarly, an inference rule of $\mathscr{S}(Q')$ is mapped into an inference rule of $\mathscr{S}(Q)$, and so a proof of $\varphi \in \Lambda(Q)$ in $\mathscr{S}(Q')$ is mapped into a proof of $\rho(\varphi) = \varphi$ in $\mathscr{S}(Q)$. ■

We recall that given a system $\mathscr{S}$ on some language and a set Γ of sentences in the language, a sentence φ is said to be $\mathscr{S}$ deducible from Γ, written $\Gamma \vdash_{\mathscr{S}} \varphi$, if there is a finite set of sentences $\varphi_1, \varphi_2, \dots, \varphi_n \in \Gamma$ such that $\vdash_{\mathscr{S}} \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \to \varphi$ (Chellas p. 47). In the following we shall use a different but equivalent definition of deducibility. The equivalence is formally stated in the following well known lemma:

6.6. LEMMA (Deduction Lemma). *Let $\mathscr{S}$ be a system of modal logic on a language Λ, and $\Gamma \subseteq \Lambda$. Then $\Gamma \vdash_{\mathscr{S}} \varphi$ if and only if there exists a finite sequence of sentences $(\gamma_1, \gamma_2, \dots, \gamma_m = \varphi)$ such that for all $i \leq m$, $\gamma_i \in \Gamma \cup \mathscr{S}$, or $\gamma_k = \gamma_j \to \gamma_i$ for some $j, k < i$.*

Lemmas 6.7–6.10 describe the structure of $\Sigma(\mathscr{U})$.

6.7. LEMMA. $\Sigma(\mathscr{U}) = \bigcup\{\Sigma_N(\mathscr{U}) | N \subseteq \mathbb{N}\}$, *with union disjoint, where*

$$\emptyset \neq \Sigma_N(\mathscr{U}) =: \{\sigma \in \Sigma(\mathscr{U}) | a^-(\sigma) = \Lambda(\mathscr{Q}(N))\}.$$

Remark. The condition $a^-(\sigma) = \Lambda(\mathscr{Q}(N))$ is equivalent to $L(\sigma) = \mathscr{Q}(N)$, because in $\mathscr{U}$ awareness is closed with respect to all connectives, by axiom A and Propositions 2.8(i) and 2.9(i) (cfr. Proposition 6.4).

Proof of 6.7. That $\Sigma(\mathscr{U})$ is a union as above follows from Proposition 6.4. That for each $N \subseteq \mathbb{N}$, $\Sigma_N(\mathscr{U}) \neq \emptyset$ follows from Proposition 2.10. Disjointness is obvious. ■

Next result: for any $\sigma \in \Sigma_N(\mathscr{U})$, $\sigma \cap \Lambda(\mathscr{Q}(N))$ is $S5(\mathscr{Q}(N))$ maximal.

6.8. LEMMA. *For* $\sigma \in \Sigma(\mathscr{U})$ *define* $\pi(\sigma) =: \sigma \cap \Lambda(\sigma)$. *Then* $\pi(\sigma) \in \Sigma(S5(L(\sigma)))$.

Proof. 1. First we prove a preliminary result. Let $\phi, \psi, \phi \to \psi \in \pi(\sigma)$. Note that ϕ and ψ may not have the same atomic sentences. Still we claim that $k\phi \to k\psi \in \pi(\sigma)$. In fact $a\phi, a\psi \in \pi(\sigma)$; we now give a proof in $\mathscr{U}$ of $k\phi \to k\psi$. By PL $\phi \wedge a\psi \to \psi \wedge a\phi$; so by RM_{sa} also $k(\phi \wedge a\psi) \to k(\psi \wedge a\phi)$. Now from M and C we have $k\phi \wedge ka\psi \to k\psi$. But by proposition 2.8 above $a\psi \leftrightarrow ka\psi$, and so $k\phi \to k\psi$ follows. Now we proceed with the proof of our main statement.

2. We first prove that $\pi(\sigma)$ is $S5(L(\sigma)))$ consistent, by contradiction. Suppose not, then there is a $\varphi \in \Lambda(\sigma)$ such that $\pi(\sigma) \vdash_{S5(L(\sigma)))} \varphi$ and $\pi(\sigma) \vdash_{S5(L(\sigma)))} \neg\varphi$ (Chellas' theorem 2.16(10)). Take the former, and consider those γ_i's in the sequence from the above Deduction Lemma such that $\gamma_i \in \pi(\sigma) \cup S5(L(\sigma))$. We claim that $\gamma_i \in \pi(\sigma) \cup \mathscr{U}$. In fact suppose $\gamma_i \in S5(L(\sigma))$. If it is an axiom of $\mathscr{U}$ we have concluded. On the other hand, the only axioms of $S5(L(\sigma))$ which are not axioms of $\mathscr{U}$ are 5 and $Df'\Diamond$. Take 5, so suppose $\gamma_i = a\psi$; this is not in $\mathscr{U}$, but since $\psi \in \Lambda(\sigma)$, it is in $\pi(\sigma)$—hence, $\gamma_i \in \pi(\sigma) \cup \mathscr{U}$. The argument for $Df'\Diamond$ is similar, based on the maximality of σ. Since $\mathscr{U}$ has *MP*, our preliminary result (1. above) and the Deduction Lemma imply $\pi(\sigma) \vdash_{\mathscr{U}} \varphi$. Similarly one has $\pi(\sigma) \vdash_{\mathscr{U}} \neg\varphi$; so since $\pi(\sigma) \subseteq \sigma$, we conclude that σ is $\mathscr{U}$-inconsistent. This is a contradiction.

3. $\pi(\sigma)$ is $S5(L(\sigma)))$ maximal. σ is $\mathscr{U}$-maximal, so $\forall\varphi \in \Lambda$ either $\varphi \in \sigma$ or $\neg\varphi \in \sigma$ (Chellas theorem 2.18(5)). Therefore $\forall\varphi \in \Lambda(\sigma)$ either $\varphi \in \pi(\sigma)$ or $\neg\varphi \in \pi(\sigma)$. This and $S5(L(\sigma)))$-consistency imply $S5(L(\sigma)))$-maximality of $\pi(\sigma)$ (Chellas exercise 2.48). ■

In fact, the map π is onto (not only into) from each $\Sigma_N(\mathscr{U})$ to $\Sigma(S5(\mathscr{Q}(N)))$:

6.9. LEMMA. *With* π *and* $\Sigma_N(\mathscr{U})$ *as in* 6.7 *and* 6.8:

$$\pi(\Sigma_N(\mathscr{U})) =: \{\pi(\sigma) | \sigma \in \Sigma_N(\mathscr{U})\} = \Sigma(S5(\mathscr{Q}(N))).$$

Proof. $\subseteq$ from 6.8; to prove $\supseteq$. For $N = \mathbb{N}$ result again from 6.8: because $\Sigma(S5) = \Sigma_{\mathbb{N}}(\mathscr{U})$ (clearly $\subseteq$; $\supseteq$ by 6.8); and $\pi(\Sigma_{\mathbb{N}}(\mathscr{U})) = \Sigma_{\mathbb{N}}(\mathscr{U})$. For $N = \emptyset$, any consistent set of sentences of any $\mathscr{S}(\emptyset)$ is formed by $\top$ and/or some of its PL-equivalent sentences in $\Lambda(\emptyset)$. Therefore, using

Proposition 2.10, we can complete any $\alpha \in \Sigma(S5(\emptyset))$ into a $\sigma \in \Sigma_{\emptyset}(\mathscr{U})$ such that $\pi(\sigma) = \alpha$. Take then $\emptyset \subset N \subset \mathbb{N}$. Any $\alpha \in \Sigma(S5(\mathscr{Q}(N)))$ is $\mathscr{U}(\mathscr{Q}(N))$-consistent, hence, $\mathscr{U}$-consistent from Lemma 6.5 (for suppose α were not $\mathscr{U}$-consistent. Then there would be $\varphi_1, \dots, \varphi_n \in \alpha$ such that $\varphi_1 \wedge \cdots \wedge \varphi_n \to \perp \in \mathscr{U}$. But $\alpha \subseteq \Lambda(\mathscr{Q}(N))$ so the last sentence is in fact in $\mathscr{U} \cap \Lambda(\mathscr{Q}(N)) = \mathscr{U}(\mathscr{Q}(N))$, i.e., α would be $\mathscr{U}(\mathscr{Q}(N))$-inconsistent). We now show that for such α also the set

$$\Gamma = \alpha \cup \{\neg a\psi | \psi \notin \Lambda(\mathscr{Q}(N))\}$$

is $\mathscr{U}$-consistent. From this the result follows: for then we can complete Γ into a $\mathscr{U}$-maximally consistent set, and this process will add no sentences of $\Lambda(\mathscr{Q}(N))$ to Γ (by maximality of α), so the projection π gives back α. To complete the proof we have to show: *not* $\Gamma \vdash_{\mathscr{U}} \perp$, i.e., that for $\varphi_1, \dots, \varphi_m \in \alpha$, $\psi_1, \dots, \psi_n \notin \Lambda(\mathscr{Q}(N))$, *not* $\vdash_{\mathscr{U}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_m \wedge \neg a\psi_1 \wedge \cdots \wedge \neg a\psi_n)$. For this it suffices to find a model $\mathscr{M}$ of $\mathscr{U}$ in a class of models sound with respect to $\mathscr{U}$, and a σ in $\mathscr{M}$ such that

$$\mathscr{M}, \sigma \models \varphi_1 \wedge \cdots \wedge \varphi_m \wedge \neg a\psi_1 \wedge \cdots \wedge \neg a\psi_n.$$

We know from the previous subsection that $\mathscr{U}$ is sound with respect to the class of partitional GSM's. In this class is the following GSM:

$$\mathscr{M} = \left\langle \bigcup_{R \subseteq \mathbb{N}} \Sigma_R, \bigcup_{R \subseteq \mathbb{N}} \Sigma'_R, \pi, P, \wp \right\rangle$$

with

$$\Sigma_R = \Sigma'_R = \emptyset \text{ for } R \neq N$$

$$\Sigma_N = \Sigma(S5(\mathscr{Q}(N))) = \Sigma'_N$$

$$\pi = \mathit{identity}$$

$P(\sigma) = \{\tau \in \Sigma | k^-(\sigma) \subseteq \tau\}$ as in the canonical standard model of $S5(\mathscr{Q}(N))$, and

$$\wp(n) = \begin{cases} |p_n|_{S5(\mathscr{Q}(N))} & n \in N \\ \mathit{arbitrary} & n \notin N, \end{cases}$$

where $|p_n|_{S5(\mathscr{Q}(N))} = \{\sigma \in \Sigma(S5(\mathscr{Q}(N))) | p_n \in \sigma\}$, the $S5(Q(N))$-proof set of p_n.

In this model, there exists a σ such that $\mathscr{M}, \sigma \models \varphi_1 \wedge \cdots \wedge \varphi_m$, by completeness of $S5(\mathscr{Q}(N))$ with respect to $\mathscr{M}_N$ and $S5(\mathscr{Q}(N))$ consistency of α; and for any $\sigma \in \mathscr{M}$, $\mathscr{M}, \sigma \models \neg a\psi_1 \wedge \cdots \wedge \neg a\psi_n$ by construction. Result now follows. ■

6.10. LEMMA. *Let* $\sigma \in \Sigma(\mathscr{U})$, *and* $\pi(\sigma) = \sigma \cap \Lambda(\sigma)$ *as before. Then* $k^-(\sigma) = k^-(\sigma) \cap \Lambda(\sigma) = k^-(\pi(\sigma))$.

Proof. By Proposition 6.4 and its proof $\Lambda(\sigma) = a^-(\sigma)$, and by maximality of σ, $k^-(\sigma) \subseteq a^-(\sigma)$; hence, $k^-(\sigma) = k^-(\sigma) \cap \Lambda(\sigma)$. Now

1. $k^-(\sigma) \cap \Lambda(\sigma) \subseteq k^-(\sigma \cap \Lambda(\sigma))$. For since $\vdash_{\mathscr{U}} a\varphi \to ak\varphi$, $\varphi \in \Lambda(\sigma)$ implies $k\varphi \in \Lambda(\sigma)$. So $k^-(\sigma) \cap \Lambda(\sigma) = \{\varphi \in \Lambda(\sigma) | k\varphi \in \sigma\} = \{\varphi \in \Lambda(\sigma) | k\varphi \in \sigma \cap \Lambda(\sigma)\} \subseteq \{\varphi \in \Lambda | k\varphi \in \sigma \cap \Lambda(\sigma)\} = \{k^-(\sigma \cap \Lambda(\sigma)\}$.

2. Conversely: if $\varphi \in k^-(\pi(\sigma))$, i.e., if $k\varphi \in \pi(\sigma)$, then since σ is $\mathscr{U}$-maximal, by axiom T also $\varphi \in \pi(\sigma) \subseteq \Lambda(\sigma)$. On the other hand since $\pi(\sigma) \subseteq \sigma$, $k\varphi \in \pi(\sigma)$ implies $k\varphi \in \sigma$, i.e., $\varphi \in k^-(\sigma)$. So $\varphi \in k^-(\pi(\sigma))$ implies $\varphi \in k^-(\sigma) \cap \Lambda(\sigma)$, as was to be shown. ■

Now we define $\mathscr{M}(\mathscr{U})$, and then prove three claims of which completeness is direct consequence.

6.11. DEFINITION. $\mathscr{M}(\mathscr{U}) = \langle \Sigma, \Sigma', \pi, P, \wp \rangle$ with:

$$\Sigma = \Sigma(\mathscr{U}) = \bigcup \{\Sigma_N(\mathscr{U}) | N \subseteq \mathbb{N}\} \text{ (see 6.7)}$$

$$\wp(n) = |p_n|_{\mathscr{U}}, n \in \mathbb{N} \text{ (as in all canonical models)}$$

$$\pi(\sigma) = \sigma \cap \Lambda(\sigma), \sigma \in \Sigma(\mathscr{U})$$

$$\Sigma' = \bigcup \{\Sigma'_N | N \subseteq \mathbb{N}\} = \bigcup \{\pi(\Sigma_N(\mathscr{U})) | N \subseteq \mathbb{N}\}$$

$$P(\sigma) = \{\beta \in \pi(\Sigma_N(\mathscr{U})) | k^-(\sigma) \subseteq \beta\}, \sigma \in \Sigma_N(\mathscr{U}), N \subseteq \mathbb{N}.$$

The $\mathscr{M}_N$'s of $\mathscr{M}(\mathscr{U})$ we call $\mathscr{M}_N(\mathscr{U})$.

Claim 1. $\mathscr{M}(\mathscr{U})$ is a GSM.

Proof. To check the conditions in (ii), (iii) and (iv) of Definition 3.1. The first two are evident by inspection. As to the last one, if $\pi(\sigma) = \pi(\tau)$, then using 6.10 $k^-(\sigma) = k^-(\pi(\sigma)) = k^-(\pi(\tau)) = k^-(\tau)$, so by definition $P(\sigma) = P(\tau)$. ■

Claim 2. For any $N \subseteq \mathbb{N}$, $\mathscr{M}_N(\mathscr{U}) = \langle \Sigma'_N, P_N, \wp_N \rangle$ is the (proper) canonical standard model of $S5(\mathscr{Q}(N))$, so P_N is an equivalence relation.

We have put "proper" in parenthesis, for we had not used the term before. The only canonical model that we have seen is the proper one (Chellas, p. 173), and it is characterized by the definition of P stated previously.

Proof of Claim 2. $\Sigma'_N = \pi(\Sigma_N(\mathscr{U})) = \Sigma(S5(\mathscr{Q}(N)))$ by 6.9, as it should be.

$$P_N(\pi(\sigma)) = P(\sigma) = \{\beta \in \Sigma(S5(\mathscr{Q}(N))) | k^-(\pi(\sigma)) \subseteq \beta\}$$

using 6.10, as it should be. And for $n \in N$,

$$\wp_N(n) = \pi(\Sigma_N \cap \wp(n)) = \pi(\Sigma_N(\mathscr{U}) \cap |p_n|_{\mathscr{U}})$$
$$= \{\alpha = \sigma \cap \Lambda(\sigma) | \sigma \in \Sigma_N(\mathscr{U}) \cap |p_n|_{\mathscr{U}}\}$$

so $\alpha \in \wp_N(n)$ iff $p_n \in \alpha$, i.e., $\wp_N(n) = |p_n|_{S5(\mathscr{Q}(N))}$, as it should be. ■

Claim 3. For any $\varphi \in \Lambda$,

$$\mathscr{M}(\mathscr{U}) \vDash \varphi \text{ iff } \vdash_{\mathscr{U}} \varphi.$$

Proof. For nonmodal φ's the result is clear (see Chellas exercise 2.53). For $k\varphi$, with $\varphi \in \Sigma_N(\mathscr{U})$:

$$\begin{array}{ll} \mathscr{M}(\mathscr{U}), \sigma \vDash k\varphi & \text{iff (Definition 3.1(v))} \\ \mathscr{M}_N(\mathscr{U}), \beta \vDash \varphi \ \forall \beta \in P(\sigma) & \text{iff (Definition 3.1(v))} \\ \mathscr{M}_N(\mathscr{U}), \beta \vDash \varphi \ \forall \beta \in P_N(\pi(\sigma)) & \text{iff (definition for standard models)} \\ \mathscr{M}_N(\mathscr{U}), \pi(\sigma) \vDash k\varphi & \text{iff (2)} \\ k\varphi \in \pi(\sigma) & \text{iff (6.10)} \\ k\varphi \in \sigma. & \end{array}$$

For $\Diamond\varphi$ the situation is analogous; we omit the details. ■

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