Setup and simple games

In the part on decision theory we have looked at situations where the consequences of our decisions depend on chance (or Nature, or however you want to call it). In games there are more people involved, so the consequences of your actions may depend on chance but crucially also on what the others do. This creates a strategic interaction among the players, and this interaction is the subject of Game Theory, otherwise known as *Interactive Decision Theory.*

Our reference book is: Ariel Rubinstein and Martin Osborne *A course in Game Theory*, which is available from Rubinstein's website. It will be referred to as OR in our course. Another, simple but longer reference is Martin Osborne *An Introduction to Game Theory.*

Definition of games in strategic form, and connection with EU

A game in strategic form is specified by a set of players N, for each player $i \in N$ the actions she can take $a_i \in A_i$, and her preferences $\succcurlyeq_i$ on the set of action *profiles* $a \in A \equiv \times_i A_i$ (Cartesian product). A profile a describes what the players do, so $a = (a_1, \ldots, a_N)$ (we are using N also as the number of players abusing notation). In strategic form games the players move *simultaneously.*[1]

Players have preferences over profiles because a profile a determines a consequence $g(a)$ of interest to them, and player i has preferences $\succcurlyeq_i^*$ over consequences. Then the above $\succcurlyeq_i$ should be interpreted to mean $a \succcurlyeq_i b \iff g(a) \succcurlyeq_i^* g(b)$.

We further assume that $\succcurlyeq_i$ is defined not only on action profiles but also on *lotteries* over profiles, and that it satisfies EU. So there is a vNM utility u_i defined on A, which means that for player i a prospect with outcomes $\{a^1, \ldots, a^n\}$ in A and probabilities $(p_1, \ldots, p_n)$ has value $\sum_j p_j u_i(a^j)$. In conclusion we can describe a strategic form game as

$$\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \, .$$

You may look at the Prisoners Dilemma below to visualize an example. Keep in mind that the numbers we see in the tables below representing games are assumed to be *utility* values.

The tale of King Solomon and a game in extensive form

— You women come here — said King Solomon.

There was an infant whom each of two women claimed as her own son, and the King was ready to resolve the dispute over who should keep the baby. The child was crying on a wooden

[1] This is in contrast to the so called "extensive form games" where players move in turn. An example is provided below when discussing King Solomon problem.

table, and a headsman stood by his side with his ax ready in his hands; everyone around was holding its breath.

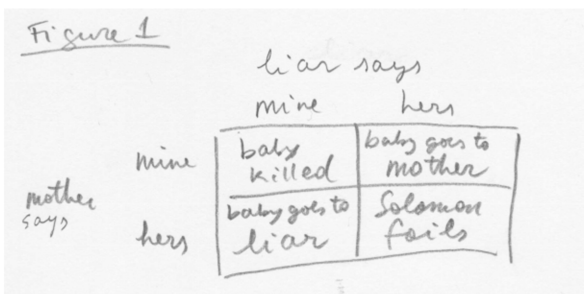
— If you agree on whose child this is, said Solomon, he will live with his mother. But if each of you asserts that this child is her child, then I shall have him cut in two, and each will get a half. —

The women briefly screamed at each other; finally one declared "it's hers", and the other said "it's mine". The King looked at the woman who had withdrawn her claim and said

— This is your baby. Go with him in peace. —

Solomon had thought over the matter. He was sure that the mother would rather see her baby go with the other woman than be killed, while the liar had opposite preferences: she did not care so much for that particular child; she probably just wanted one. Given this, he was sure that in a competition necessarily ending up with agreement or death of the baby the true mother would yield. To be more precise, the King anticipated that the women thought he would assign the baby to the woman they designated as mother, and that they would therefore come to perceive themselves in a situation which we can summarize in Figure 1:

Figure 1: the game as seen by the (naive) women



The legenda is self–explanatory: if both say 'mine' the baby is killed; if liar says 'hers' and mother says 'mine' baby goes to mother; etcetera, each pair of choices determining the outcome described in the cell at the junction of the row corresponding to mother's choice and the column relative to liar's choice (it is immaterial if we put mother on rows and liar on columns or vice versa).

We can check in figure 1 that for the liar 'mine' beats 'hers' *whatever mother does*: if mother says 'mine', liar's 'mine' will result in the baby's death, while 'hers' makes baby go to mother - and the liar prefers the former; in case mother says 'hers', then liar's 'mine' gives her the baby, while 'hers' gives the worse (for her) outcome of Solomon failure. Conclusion, the liar is going to say 'mine'.

The mother will eventually realize that this is really the liar's best choice; but then (look at the the picture) her own choice is between saying 'mine' and having the baby killed and saying 'hers' and letting the other have the baby - alive. Since she prefers the latter outcome she will say 'hers', and they will agree on the liar being the mother. The conclusion - ours and King Solomon's - is that the two women will agree to declare that the liar is the mother.

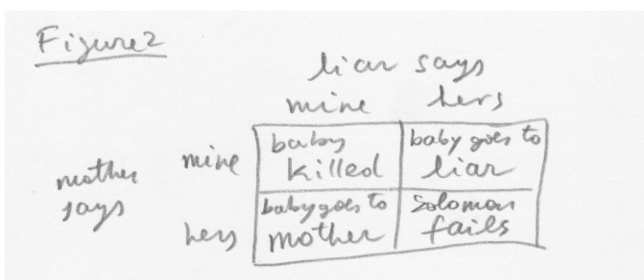
Having figured this out, to ensure justice the King just had to prepare the scene we saw above, wait for the women to designate a mother, and give the baby to the other. And thus he did.

Solomon had been more astute than the women: he anticipated their behaviour, while they did not anticipate his. Indeed remember that the women *wrongly* thought that Solomon would give the baby to the woman they would indicate as the mother (and this is what figure 1 says). But then: what would have happened if the women were 'strategically' more sophisticated? A strategically minded woman might outguess King Solomon by reasoning as follows:

— Solomon is asking us to indicate a mother, else he is going to kill the baby; but is he going to give the baby to the one we indicate? If yes, then we would be playing the game in figure 1, so we would indicate the liar as the mother, and the King would give the baby to the wrong woman; but this very analysis is certainly within his reach; therefore it cannot be his intention to follow our suggestion. Quite the opposite, assuming we are not capable of the analysis I am doing, he must be predicting that we will then agree on the false mother; he is therefore determined to assign the baby to the woman we do *not* indicate to establish justice. —

In short, two smart women would have realized that the game they were playing was not that of the previous figure, but the one in 2:

Figure 2: the game as seen by the sophisticated women



In this game each player has a choice which is best whatever the other does. For the mother: if liar says 'mine', saying 'hers' (resulting in baby given to mother) beats 'mine' (baby killed); if liar says 'hers', saying 'hers' results in Solomon failure while 'mine' results in the baby assigned to liar, so again 'hers' beats 'mine'; thus, for the mother it is best to say 'hers' whatever the other does. And you can similarly check that it is best for the liar to say 'hers' whatever the mother does. Therefore, in this situation both women will say 'hers'.

So, a not-so-sophisticated Solomon against two sophisticated women would not reach his goal: anticipating his intentions, they would both answer 'hers' and thwart his plans. Incidentally: we are assuming that both women are sophisticated, but to 'defeat' Solomon here all is needed is a smart liar: if she says 'hers' the outcome is either Solomon's failure (in case mother says 'hers' too) or that she - the liar! - gets the baby (if mother says 'mine').

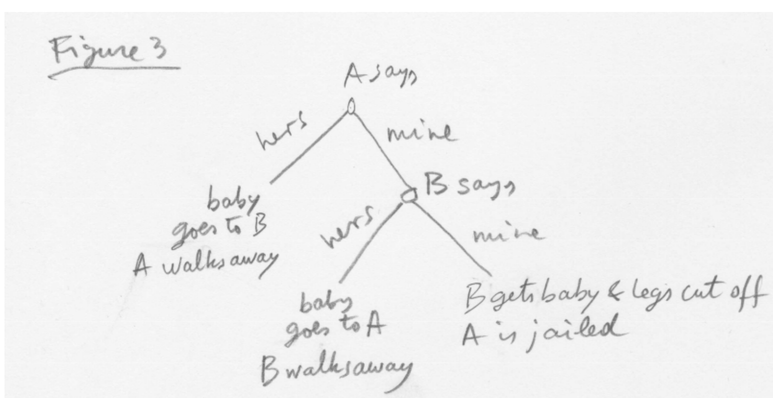
Before we move on to vindicate the King: if you feel that the mother's emotions are so much involved in this situation that it is absurd to pretend that she could use her brain however sophisticated it might be, well, I could not agree more. But keep in mind: a sophisticated liar would propose to indicate the mother as mother, and if the latter is not 'present' enough she could happily agree, thereby losing her baby! So, it is very true that in many 'games' of real interest it is utterly difficult to have your brain 'on'; on the other hand, it is probably the most important thing; having it 'off' may be extremely costly.

Imagine then a sophisticated Solomon: he would in turn anticipate the women's behavior and would never set up such a 'game'. Can he do better? We shall now see that the answer is Yes. Say the two women are called A and B, and consider the King speaking as follows:

— I shall now ask A whose baby this is; if she says 'hers', then B will keep the baby; if she says 'mine' I shall then ask B: then, if she says 'hers' the baby will go to A; if she too says 'mine' then B will have the baby, but also both legs cut off, and A will be jailed as a liar. So, A, whose baby is it? —

Actually, to help the women visualize the situation he would draw the following picture for them (again the legenda is self-evident: A starts and has two choices, 'hers' and 'mine'; the former ends the game with the outcome described below the corresponding play; the latter gives turn to B, who too has two choices, analogous to those of A; the plays ('mine', 'mine') and ('mine', 'hers') lead to the outcomes described correspondingly). The drawing is that of the Figure below:

Figure 3: the extensive form game



With the help of this picture we can be with King Solomon in his analysis of the situation. Since he does not know whether A is the mother or not we must go through the two possible cases separately. We are assuming that the mother - and only her - would prefer to have the baby and the legs cut off to letting the baby go with the other woman.

Suppose first that A is the mother; if she says 'hers' she loses the baby; if she says 'mine' she will examine the liar's (B) position and say to herself, "the liar does not care so much for the baby as to forgo both her legs to have him, so if I say 'mine' she will prefer to say 'hers' and I will get the baby"; thus if A is the mother she should choose between saying 'hers' and losing the baby and saying 'mine' and getting it, so she will say 'mine'; then when it actually comes to B, she cannot do better than saying 'hers', for it is true that she does not care for the baby that much. Conclusion: if A is the mother, she will get the baby (and no one suffers penalties).

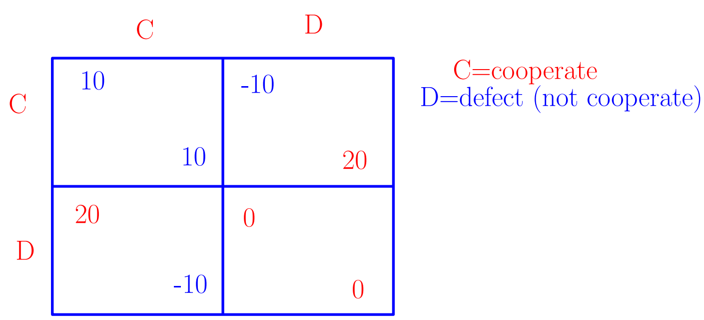
Suppose now A is the liar: if she says 'hers' she loses the baby and that's it. Next, since she knows that B (the mother now) would happily have her legs cut off if that was the price to be paid to have her baby, she (the liar) anticipates that if she says 'mine' B will say 'mine' too and then not only would she not get the baby, but she would also be jailed; better choosing to say 'hers' at the outset. Thus also if A is the liar the baby goes to the mother and no penalties are involved.

Conclusion, by setting up this 'game' the King is sure that whoever is the mother will receive the baby and walk off with him (well: we are *assuming* that the women are 'rational', e.g. whoever is at the last decision node chooses her preferred outcome; and also that each must know that the other is rational - to justify choice at initial position).

So, King Solomon's adventures are well captured by 'games', that is situations where what I get from my action depends on what *your* action is; and the same is for you. So in games it is not individual actions that matter, but *pairs* of actions.[2]

The Prisoners Dilemma

This is probably the most widely known game. A payoff matrix representing the game is this:



If Players cooperate they get 10, if they both defect they get zero; if player 1 (the row player) cooperates but 2 defects - profile CD - then 2 gets 20 and 1 gets -10; analogous situation at the profile DC. In this game it is easy to predict outcome DD, because for each player D is better

[2] In more general, multi-person games it is action profiles that matter.

than C *whatever* the other is doing - which means D is a *dominating strategy.* The big problem is that it is inefficient: individual rationality will not produce the socially desirable outcome.

Note that DD is also the maxmin profile, where each player plays the action yielding the highest minimum payoff over the opponent's actions. Consider for example player 1: if she plays C the minimum she gets is -10; by playing D the corresponding minimum is 0; so the highest of the minima is reached by playing D.

Game of chicken

	stop	go
stop	0, 0	-5, 5
go	5, -5	-10, -10

In this game there is no dominant strategy. Consider player 1 for example: if 2 plays Stop then best response is Go; if 2 plays Go best response is Stop; similarly for player 2. Mutual best responses occurs at profiles (Go, Stop) and (Stop, Go).

We shall see in a moment that this says that these two profiles are *Nash Equilibria.* But this is not as good a prediction as in the PD. Here also the (Stop, Stop) profile can be justified, as we shall see presently.

The central equilibrium concept: Nash Equilibrium

As already indicated, unlike in single person decisions, in games the payoff to each player depends crucially on what *the others* are doing.

As said before we let $u_i(a) = u_i(a_1, a_2, \ldots, a_n)$ the payoff of i at profile a. we denote by a_{-i} the actions taken by all players other than i, so that when we focus on player i we can write a profile a as (a_i, a_{-i}). The main concept of equilibrium in games is due to John Nash (who got the Nobel prize for that in 1994).

Definition. Action profile a^* is Nash equilibrium if *no player wants to deviate.* That is the key idea. So the profile a^* is Nash if for all $i \in N$

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i$$

To emphasize that we are talking about best replies (aka best responses) notice that the definition says that for all i the Nash action a_i^* solves

$$\max_{a_i \in A_i} u_i(a_i, a_{-i}^*)$$

which says that a_i^* is a *best response* to a_{-i}^*.

Letting $B_i(a_{-i})$ denote the set of best responses to a_{-i}, that is

$$B_i(a_{-i}) = \left\{ a_i \in A_i \colon u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \; \forall a_i' \in A_i \right\},$$

we may rewrite the definition thus: a^* is Nash equilibrium if $a_i^* \in B_i(a_{-i}^*)$ for all i. When best responses are unique this reduces to the system of N equations $a_i^* = B_i(a_{-i}^*)$ in the N unknowns a_i^*.

We continue with more examples of well know games.

Game of Chicken: maxmin, "pessimistic" choices

Look again at the game of chicken:

	stop	go
stop	0, 0	−5, 5
go	5, −5	−10, −10

Player asks: "if I stop what is the *worst* that can happen to me?" Answer: -5. What if I go? Answer: -10. If I am "cautious" I may choose to stop. In this case I maximize the minimum utility from my action.

Cautious play equilibrium here is (Stop,Stop). in this game (Stop,Stop) is the result of max-min play (maximize the worst that can happen under any action). It is NOT expected utility maximization - but it is definitely just as reasonable.

Interesting fact (proved later): in the important class of the so-called "zero-sum games" Nash and cautious play coincide.

Matching Pennies

This is the popular game of depicted in the following matrix:

		Mr. D	
		P	D
Mr. P	P	1, -1	-1, 1
	D	-1, 1	1, -1

There is NO Nash Equilibrium here. The way this game is always played is that both choose P or D completely at random (50-50 mix). This kind of play will be covered shortly.

Exercise. Suppose you are player 1. Suppose 2 plays P and D at random, 50-50 probability. Compute expected payoff under P and under D.

Given that 2 plays at random, how would you play? You are indifferent, so you may choose your action at random as well. So we understand that both players playing 50-50 is an "equilibrium" - in quotation marks because we have not formally defined what the game becomes if players can choose to mix between actions - essentially playing *lotteries* over actions.

Remark. This example shows that there is a problem with existence in general: which games are guaranteed to have a Nash equilibrium? There are answers, which are too mathematical for us. We will not be concerned with existence problems.

Stag Hunt (Rousseau, circa 1755)

	C	F
C	4, 4	1, 3
F	3, 1	2, 2

Here C means hunt the Stag, F hunt a hare (smaller animal). Equilibria are CC and FF but maxmin strategies lead to FF.

This game may be interpreted as cold war between US and USSR after WW2. Maxmin predicts observed behavior - both built an arsenal of nuclear weapons.

Its original interpretation is as a *trust dilemma*, but it may also be seen as a *coordination problem*. Another incarnation of this game is the work/shirk game

	work	shirk
work	2, 2	-1, 1
shirk	1, -1	0, 0

The general problem is that there are multiple equilibria, "Pareto ranked" in the sense that in one everybody is better off than in the other.

Coordination again (more cruel)

This is the game (note the typographically simpler description of the payoff matrix):

	L	R
T	$10, 10$	$0, 0$
B	$0, 0$	$10, 10$

This is a pure coordination game: the (big) problem is for the players to guess which of the two equilibria is played.

Battle of Sexes

The game is this:

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

Here again there are two equilibria - the off-diagonal profiles. We may observe that this is not the way the game is played in real boy-girl situations. We will hopefully come back to this.

Risk

Here $a > 0$ may be a large number

	L	R
U	2^*, 2^*	$-a$, 0
D	0, $-a$	1^*, 1^*

Again there are two equilibria: UL and DR, and the former dominates the latter. How will they coordinate on the good equilibrium? When you choose you don't know what the other is choosing. Not being 100% sure you may start being cautious: what if she doesn't play L? I get $-a$ which is *very* bad; if we coordinate on the "bad" equilibrium DR, if anything goes wrong I get zero instead of 1; no big deal.

It is easily checked that the maxmin play here is DR.

Twin sisters

A mother has twin daughters who live in different towns. She tells each to ask for a certain integer number of dollars, at least 1 and at most 100, as a birthday present. If the total of the two amounts does not exceed 101, each will have her request granted. Otherwise each gets nothing.

Both action sets are integer numbers between 1 and 100. Call x the choice (strategy) of the first sister and y the choice of the other.

a. Find the pairs (x, y) which are Nash equilibria of this game.[3]

b. Is there a symmetric equilibrium among these? NO (a symmetric Nash equilibrium is one in which both players use the same strategy).

What kind of bids do you think would be "common" in this game? We would estimate that in real cases the bids would both be close to 50 - but how to *select* such equilibria? It's not trivial to give a sound argument concluding that these profiles will be more likely to be played. And by the way, $(x, y) = (50, 50)$ is *not* an equilibrium.

Remark. In general, equilibrium selection is a difficult problem.

Families of Games

Consider the following family of two-by-two games, with $\lambda, \xi \in \mathbb{R}$. Here C stands for "cooperate" and F for "fight".

[3] Solution: any pair (x, y) such that $x + y = 101$. So there are 100 NE, one for each value of x between 1 and 100.

	C	F
C	$1,1$	ξ,λ
F	λ,ξ	$0,0$

Restrict attention to the region $\lambda+\xi<2$ so that any profile other than CC gives a lower average payoff than CC. In the (λ,ξ) plane there are 4 possible types of games in the family, corresponding to 4 disjoint regions in the plane. The first is "mutual interest", where only CC is equilibrium; the second is Stag Hunt, where CC and FF are equilibria; the third is Chicken, where equilibria are CF and FC; the fourth is Prisoners Dilemma, where only FF is an equilibrium. The first two types we may call "cooperation games" since CC is an equilibrium; the last two are "conflict games", where CC is not an equilibrium.

Exercise. Draw the four regions in question. Hint: compare λ with 1 and ξ with zero.

The mixed extension of a game

It was argued in the matching pennies example that the natural equilibrium in that case is that the players randomize 50-50 between the two choices; we now proceed more formally. The idea is to extend the strategy spaces to *lotteries* on the pure strategies, where we assume that players randomize independently and are expected utility maximizers with vNM utility u_i. We arrive at the "mixed extension" of the original game, and a mixed equilibrium of the original game will then be a Nash equilibrium of the extended game.

Formally the *mixed extension* of game $G=\langle N,(A_i),(u_i)\rangle$ with finite A_i's is the game $G^\Delta=\langle N,(\Delta(A_i)),(U_i)\rangle$ defined as follows:

- Players can mix over A_i: $\Delta(A_i)$ is the set of probability distributions on A_i; its generic element is α_i which assigns probability $\alpha_i(a_i)$ to $a_i\in A_i$, with $\sum_{A_i}\alpha_i(a_i)=1$
- Players mix independently: the profile $\alpha=(\alpha_1,\dots,\alpha_n)\in\Pi_i\Delta(A_i)$ induces the product probability $\tilde{\alpha}\in\Delta(\Pi_iA_i)$, defined by

$$\tilde{\alpha}(a)=\Pi_i\alpha_i(a_i)\quad a=(a_1,\dots,a_n)\in A$$

- Players are expected utility maximizers, so they evaluate α according to expected utility:

$$U_i(\alpha)=\sum\nolimits_{a\in A}\tilde{\alpha}(a)u_i(a)=\sum\nolimits_{a\in A}u_i(a)\prod\nolimits_j\alpha_j(a_j).$$

The equilibrium concept is the same as before: *no player wants to deviate.* So α^* is NE if for all i and $\alpha_i\in\Delta(A_i)$ one has

$$U_i(\alpha^*)\geq U_i(\alpha^*_{-i},\alpha_i).$$

Note that we can identify the original action set A_i with a subset of $\Delta(A_i)$ - namely, we can identify a_i with the mixed strategy α_i such that $\alpha_i(a_i) = 1$ (assigning probability 1 to a_i), thus $\Delta(A_i)$ is indeed an *extension* of A_i.

It is useful to exploit a linearity property of $U_i(\alpha)$[4]. If i plays a_i and the others play α_{-i} player i gets $U_i(a_i, \alpha_{-i}) = \sum_{a_{-i}} \prod_{j \neq i} \alpha_j(a_j) u_i(a_i, a_{-i})$. From this we can deduce that if i plays the general mixed strategy α_i she gets[5]

$$\begin{aligned} U_i(\alpha) &= \sum_a \prod_j \alpha_j(a_j) u_i(a) = \sum_{a_i} \alpha_i(a_i) \sum_{a_{-i}} \prod_{j \neq i} \alpha_j(a_j) u_i(a_i, a_{-i}) \\ &= \sum_{a_i \in A_i} \alpha_i(a_i)\, U_i(a_i, \alpha_{-i}). \end{aligned} \tag{32.2}$$

This is equation (32.2) in OR, which shows that if i plays α_i she gets the lottery with values $U_i(a_i, \alpha_{-i})$ and probabilities $\alpha_i(a_i)$.

Example (Matching Pennies)**.** Recall the game:

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

For player 1 a distribution on (H, T) is of the form $(p, 1-p)$; similarly for 2, $(q, 1-q)$. Therefore we may write a profile of mixed strategies as (p, q). Observe that the strategy space for each player is much larger than before: now it is the whole interval $[0, 1]$. For player 1 playing $p = 1$ effectively means playing the pure strategy H, same with $q = 1$ for player 2. We have (keep in mind the assumption of independent mixing)

$$\begin{aligned} U_1(p, q) &= pq + (1-p)(1-q) - p(1-q) - q(1-p) \\ &= p(2q-1) + (1-p) - 2q\,[1-p] = p(2q-1) + (1-p)(1-2q) \\ &= (1-2q)(1-2p) = 4(1/2 - p)(1/2 - q). \end{aligned}$$

Note incidentally that $u_1(a) = -u_2(a)$ and the same holds in the mixed extension: $U_1(\alpha) = -U_2(\alpha)$ (please check this). This says that matching pennies is a *zero-sum game*. For best

[4]Equation (32.2) in OR

[5]Observing that

$$\sum_a [\cdot] = \sum_{a_i} \sum_{a_{-i}} [\cdot].$$

responses, where $B_1(q)$ is a set of p's, we have (it is obvious)

$$B_1(q) = \begin{cases} p = 0 & q < 1/2 \\ 0 \le p \le 1 & q = 1/2 \\ p = 1 & q > 1/2 \end{cases}$$

You can write down the best response $B_2(p)$ of player 2 (a set of q's) and conclude that the only equilibrium pair (p^*, q^*) such that $p^* \in B_1(q^*)$ and $q^* \in B_2(p^*)$ is $p^* = q^* = 1/2$. That is throw at random. Indeed you can check that

$$B_2(p) = \begin{cases} q = 1 & p < 1/2 \\ 0 \le q \le 1 & p = 1/2 \\ q = 0 & p > 1/2 \end{cases}$$

For example if $p < 1/2$ (1 is more likely to play T than H) then 2 will play H for sure - formally $B_2(p) = \{1\}$. But $B_1(1) = \{1\} \not\ni p$, so $p < 1/2$ is not an equilibrium. On the other hand if (and only if) $p = q = 1/2$ we have $p \in B_1(q)$ and $q \in B_2(p)$.

Structure of equilibria in mixed strategies

We now look at the structure of mixed equilibria. Observe that equation (32.2) implies that

$$\overline{U}_i(\alpha_{-i}) \equiv \max_{a_i} U_i(a_i, \alpha_{-i})$$

is the highest utility you can get against α_{-i}. Indeed for any α_i we have

$$U_i(\alpha_i, \alpha_{-i}) = \sum\nolimits_{a_i \in A_i} \alpha_i(a_i)\, U_i(a_i, \alpha_{-i}) \le \sum\nolimits_{a_i \in A_i} \alpha_i(a_i)\, \overline{U}_i(\alpha_{-i}) = \overline{U}_i(\alpha_{-i})$$

The next result (OR Lemma 33.2) says that in equilibrium you give positive probability only to actions yielding maximum utility:

Proposition (OR Lemma 33.2)**.** *α is Nash if and only if $\alpha_i(a_i) > 0$ implies $U_i(a_i, \alpha_{-i}) = \overline{U}_i(\alpha_{-i})$, for all i.*

Proof. Let $B_i(\alpha_{-i}) = \{a_i \colon U_i(a_i, \alpha_{-i}) = \overline{U}_i(\alpha_{-i})\}$. Observe that α_i is a best response to α_{-i} iff $U_i(\alpha_i, \alpha_{-i}) = \overline{U}_i(\alpha_{-i})$. We have

$$U_i(\alpha_i, \alpha_{-i}) = \sum\nolimits_{a_i \in B_i(\alpha_{-i})} \alpha_i(a_i)\, U_i(a_i, \alpha_{-i}) + \sum\nolimits_{a_i \notin B_i(\alpha_{-i})} \alpha_i(a_i)\, U_i(a_i, \alpha_{-i}),$$

but for any $a_i \notin B_i(\alpha_{-i})$ it is $U_i(a_i, \alpha_{-i}) < \overline{U}_i(\alpha_{-i})$, so $U_i(\alpha_i, \alpha_{-i}) = \overline{U}_i(\alpha_{-i})$ if and only if all the probabilities in the second sum are zero. $\square$

The following direct consequence of the above result greatly simplifies the computation of equilibria.

Corollary. *Any action which i plays with positive probability in equilibrium gives her the same payoff, namely* $\overline{U}_i(\alpha_{-i})$.

Example (Matching Pennies again). We compute the equilibrium making use of the previous observation. Player 1 must be indifferent between H and T, so $q-(1-q)=-q+(1-q)$ whence $q=1/2$; similarly $p=1/2$. That was quick!

Exercise (battle of sexes). Find the mixed equilibrium in battle of sexes (OR p.34). Unlike the two pure-strategy equilibria this is symmetric. But the unpleasant fact about this equilibrium is that the equilibrium probability that they stay together is

$$pq+(1-p)(1-q)=2/9+2/9=4/9<1/2!$$

A different, more natural solution? Toss a coin and let chance decide between BB and SS. This is the real solution. But notice it's not Nash - it is called *Correlated Equilibrium*. This is covered in Section 3.3 of OR, we'll see if we have time to cover it.

Example (the coordination game). We saw this earlier:

	L	R
T	$10,10$	$0,0$
B	$0,0$	$10,10$

Of course there is the mixed equilibrium where $p=q=1/2$ but it suffers from the same problem we have just seen in the battle of sexes: with probability 1/2 both get zero. The natural way to play it is again to toss a coin, and that is the correlated equilibrium.

Finding all the Nash equilibria in a nontrivial case

Consider the following game:

	W	X	Y	Z
A	$3^*,2^*$	$0,0$	$0,0$	$1,1$
B	$0,0$	$2^*,3^*$	$0,0$	$1,1$
C	$0,0$	$0,0$	$0^*,0^*$	$-1,-1$
D	$1,1$	$1,1$	$-1,-1$	$0,0$

Let us find all the Nash equilibria. First there are the starred pure equilibria. We now search for mixed ones.

$A \preccurlyeq_1 D$:

$$\begin{aligned} 3q_1 + 1 - q_1 - q_2 - q_3 &\leq q_1 + q_2 - q_3 \\ q_1 &\leq 2q_2 - 1 \end{aligned}$$

$B \preccurlyeq_1 D$:

$$\begin{aligned} 2q_2 + 1 - q_1 - q_2 - q_3 &\leq q_1 + q_2 - q_3 \\ 1 &\leq 2q_1 \end{aligned}$$

so the two above imply $1/2 \leq q_1 \leq 2q_2 - 1$ which in turn implies $3/4 \leq q_2$; this and $q_1 \geq 1/2$ cannot both hold since $q_1 + q_2 \leq 1$. So for any mixed strategy of 2, either $D \prec_1 A$ or $D \prec_1 B$; therefore D cannot be played in equilibrium. One can analogously exclude Z. So $p_4 = q_4 = 0$.

Suppose q_1, q_2 and q_3 are all strictly positive. Then 1 plays A or B, say with probability p on A; then $Y \succcurlyeq_2 W \implies p = 0$ and $Y \succcurlyeq_2 X \implies p = 1$; therefore 2 cannot play Y with positive probability.

Suppose $q_2, q_3 > 0$ and $q_1 = 0$; then 1 plays B and then 2 plays X (so $q_3 = 0$) - contradiction. Next $q_1, q_3 > 0$ and $q_2 = 0$; then 1 plays A and then 2 plays W (so $q_3 = 0$), another contradiction.

So we are left with only one possibility: $q_1, q_2 > 0$ and $q_3 = 0$, that is probability q on W and $1 - q$ on X; then 1 mixes between A and B - with p on A. And at this point the indifferences for the two players give $3q = 2(1 - q)$ that is $q = 2/5$, and $2p = 3(1 - p)$ or $p = 3/5$.

Back to pure equilibria: first and second price auctions

These are the two basic models for sealed-bid auctions, so-called first price and second price auctions; we start with the common elements. There is an object to be assigned to one of n players in exchange for a payment. Player i's valuation of the object is v_i; order players so that $v_1 > v_2 > \cdots > v_n$. In the simple version we consider here it is assumed that each player knows all the v_i's. Of course the more interesting cases are where the others' valuations are not known, and we will model those as "Bayesian games" later on. Each player submits a bid $b_i \geq 0$, and the object is assigned to the player who submits the highest bid. If more than one player submit the highest bid then the object goes to the one with the highest valuation among them.

First price auction (see exercise 18.2 in OR) Possible actions of player i are bids $b_i \geq 0$ - that is $A_i = [0, \infty)$, which notice is an infinite set. Letting $b = (b_1, \ldots, b_n)$, then

$$u_i(b) = \begin{cases} 0 & \textit{if } i \textit{ does not win} \\ v_i - b_i & \textit{if } i \textit{ wins} \end{cases}$$

and i wins if $b_i \geq b_j$ for all $j \neq i$ and there is no $j < i$ with $b_j = b_i$. In this version the winner pays her bid. Consider the highest valuation player, namely player 1. Her best response to b_{-1} is

$$B_1(b_{-1}) = \begin{cases} \max_{j>1} b_j & \textit{if} \;\; \max_{j>1} b_j < v_1 \\ [0, \max_{j>1} b_j] & \textit{if} \;\; \max_{j>1} b_j = v_1 \\ [0, \max_{j>1} b_j) & \textit{if} \;\; \max_{j>1} b_j > v_1 \end{cases}$$

because: if the others' max bid is less than v_1 player 1 wins by matching that bid and gets a positive payoff; in case $\max_{j>1} b_j = v_1$ she is indifferent between winning and losing; if $\max_{j>1} b_j > v_1$ she does not want to win, goes under the highest bid and gets zero.

The following is an argument showing that the set of Nash equilibria is the set of profiles

$$\left\{ b = (b_1, \ldots, b_n) \colon v_2 \leq b_1 = \max_{j>1} b_j \leq v_1 \right\}.$$

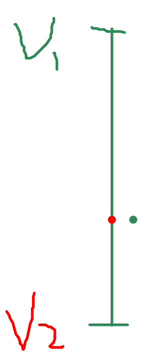
We look for the possible equilibria separately for the cases $\max_{j>1} b_j > v_1$, $\max_{j>1} b_j = v_1$ and $\max_{j>1} b_j < v_1$ (in the last case you should consider player 2 also). First point: $\max_{j>1} b_j > v_1$ is impossible in equilibrium (the highest bidder would get negative payoff).

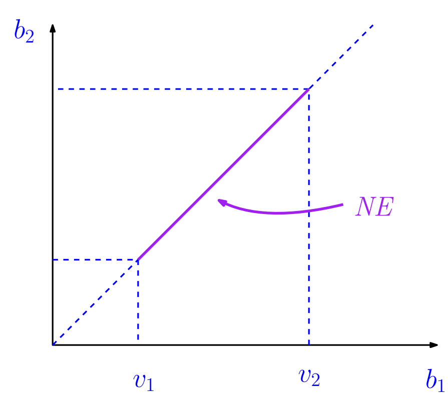
Next, case $\max_{j>1} b_j = v_1$: the bid $b_1 = v_1$ is in player 1's best response; and this is an equilibrium: that is $\max_{j>1} b_j = v_1$ and $b_1 = v_1$.

Lastly: $\max_{j>1} b_j < v_1$: Can it be $\max_{j>1} b_j < v_2$ in equilibrium? No, since then $B_1(b_{-i}) = \max_{j>1} b_j$ and player 2 would deviate (bid a little higher than $\max_{j>1} b_j$, win and get a strictly positive payoff). So let's look at $v_2 \leq b_1 = \max_{j>1} b_j < v_1$. The equality says that player 1 is playing best response. Then given $b_1 \geq v_2$ none of the other players can profitably deviate (they cannot get more than zero by deviating).

This concludes the proof of the statement above about the set of equilibria. Notice that in all equilibria player 1 gets the object.

Let us look at the two-player case $n = 2$:





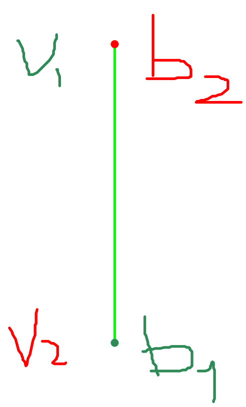
Exercise 18.3 in OR (second price auction) In a second price auction the object is assigned as in the previous case but the winner pays the *second* highest bid: the payoff to i is zero if she does not win and $v_i - \max_{j \neq i} b_j$ if she does. The interesting thing about this type of auction is that $b_i = v_i$ - bidding your value - is an optimal choice *whatever the others are doing.*

To show this consider separately the cases $\max_{j \neq i} b_j \geq v_i$ and $\max_{j \neq i} b_j < v_i$. We just have to work out i's best response to b_{-i}. Let $\bar{b} = \max_{j \neq i} b_j$.

- Suppose $\bar{b} > v_i$. Then $u_i(v_i, b_{-i}) = 0$ (you don't win). And for any $b_i \neq v_i$, either you lose and get zero or you win and get $v_i - \bar{b} < 0$.
- Suppose next $\bar{b} \leq v_i$ and $\exists j < i$ such that $b_j = v_i$. Then $\forall b_i \leq v_i$ you lose and get zero; $\forall b_i > v_i$ you win and get zero.
- Suppose next $\bar{b} \leq v_i$ and $\nexists j < i$ such that $b_j = v_i$. Then for any $b_i \geq v_i$ you win and get $v_i - \bar{b} \geq 0$; for $\bar{b} \leq b_i < v_i$ you win and get again $v_i - \bar{b} \geq 0$; for $b_i < \bar{b}$ you lose and get zero.

Therefore we conclude that for any b_{-i} and any b_i we have $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$.

There are also equilibria where $b_i \neq v_i$. A "perverse" inefficient equilibrium in the case of two players is this:



this is NE in second price auction

In this equilibrium the object is assigned to player 2, who wind and pays $v_2 (= b_1 < b_2)$. For more on second price auctions you may see Osborne's book *An Introduction to Game Theory*, section 3.5.