Going beyond probability in reasoning under uncertainty

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Models for dynamic reasoning under partial knowledge to make interpretable decisions

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Overview of the talk

1. Representing uncertainty
2. Some applications
3. From De Finetti's coherence to AI calculi
4. Imprecise stochastic processes in Dempster-Shafer theory
5. A financial application: dynamic bid-ask option pricing

Reasoning and making decisions under uncertainty

Human beings in their daily activities **make decisions** whose final result is not known in advance, i.e., they act under **uncertainty**.

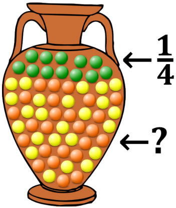
 $\Longleftarrow$ Artificial agent

Descriptive approach: the goal is to understand how a real decision maker decides.

Normative approach: the goal is to fix a set of **axioms** that rule decisions of a "rational" decision maker.

"Rationality" is important in the AI context since it favors **interpretability** of decisions.

Dealing with uncertainty: Ellsberg example

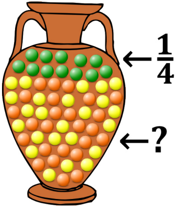


$P(G) = \frac{1}{4}$ $G =$ 'draw a green ball'

$P(Y) =$ **?** $Y =$ 'draw a yellow ball'

$P(O) =$ **?** $O =$ 'draw an orange ball'

Dealing with uncertainty: Ellsberg example



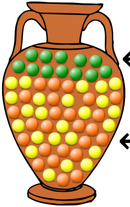
The probability distributions over (G, Y, O)

$$P_\theta = \left\{ \left(\frac{1}{4}, \theta, \frac{3}{4} - \theta \right) \ : \ \theta \in [0, \frac{3}{4}] \right\}$$

generate a convex set of probability measures such that

$$\begin{array}{cc} P(G) = \frac{1}{4} & G = \text{ 'draw a green ball'} \\ 0 \leq P(Y) \leq \frac{3}{4} & Y = \text{ 'draw a yellow ball'} \\ 0 \leq P(O) \leq \frac{3}{4} & O = \text{ 'draw an orange ball'} \\ \frac{1}{4} \leq P(G \cup Y) \leq 1 & \frac{1}{4} \leq P(G \cup O) \leq 1 \\ P(Y \cup O) = \frac{3}{4} & \end{array}$$

Dealing with uncertainty: Ellsberg example



The probability distributions

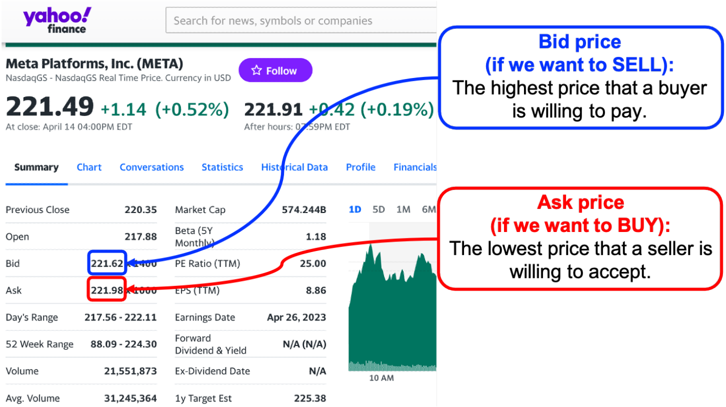
$$P_\theta = \{(\frac{1}{4}, \theta, \frac{3}{4} - \theta \ : \ \theta \in [0, \frac{3}{4}]\}$$

generate a (coherent) lower probability (the lower envelope)

$$\begin{array}{cc} \nu(G) = \frac{1}{4} & \nu(Y \cup O) = \frac{3}{4} \\ \nu(Y) = \nu(O) = 0 & \nu(G \cup Y) = \nu(G \cup O) = \frac{1}{4} \\ \nu(\emptyset) = 0 & \nu(G \cup Y \cup O) = 1 \end{array}$$

Actually, ν is an inner measure.

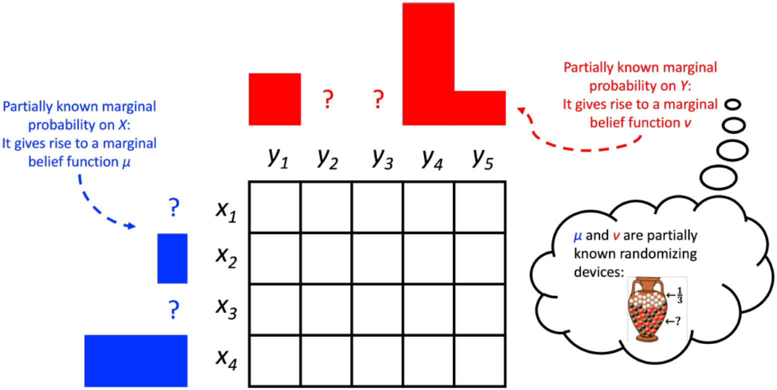
Financial application



- Market consistent dynamic bid-ask option pricing[1]
- Behavioral dynamic portfolio selection under ambiguity[2]

[1] Cinfrignini et al (2024)
[2] Petturiti-Vantaggi (2024)

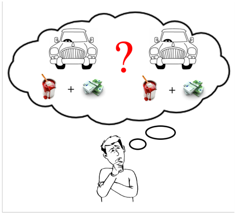
Optimal transport under partially known probabilities

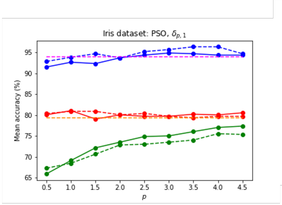


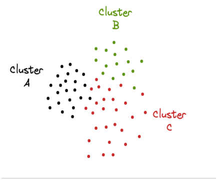
- Stackelberg-Cournot games under ambiguity [3]

[3] Lorenzini et al. (2025), Caprio (2025)

Similarity measures and information fusion

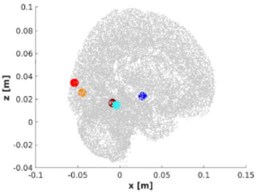


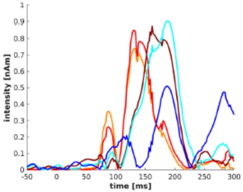


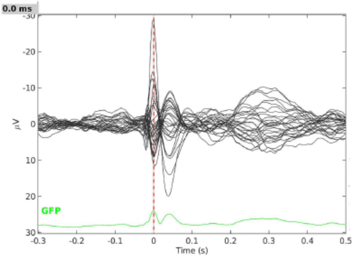


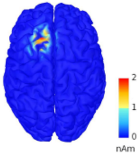
- Similarity learning: entropic regularizations for identifiability and XAI
- Non-supervised ML techniques for fuzzy data: fuzzy clustering
- Roboust statistical matching

Reconstruction of the neural activity in the brain from MEG or EEG data[4]









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[4] Calvetti et al (2015, 2023)

Towards probability measures

The most known uncertainty measures are **probability measures**. How to justify **additivity**? From behavioural point of view...

Betting scheme (de Finetti (1937))

A **combination of bets** on $E_1, \ldots, E_n$ with fixed **stakes** $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ produces the **random gain**

$$G(\omega) = \sum_{i=1}^{n} \lambda_i(\mathbf{1}_{E_i}(\omega) - P(E_i))), \quad \forall \omega \in \Omega.$$

Coherence and consistency

Definition

$P : \mathcal{E} \to [0,1]$ is **coherent** iff for all choices of stakes $\lambda_1, \ldots, \lambda_n$ we do not have a sure loss or a sure win (i.e., a **Dutch book**), that is

$$\min_{\omega \in \Omega} G(\omega) \leq 0 \leq \max_{\omega \in \Omega} G(\omega). \quad \longleftarrow \text{Normative axiom}$$

Theorem

Given a probability assessment $P : \mathcal{E} \to [0,1]$, the following statements are equivalent:

(i) P is coherent;

(ii) there exists a (finite additive) probability measure P' extending P, i.e., $P'(E_i) = P(E_i)$, $E_i \in \mathcal{E}$.

$\Longrightarrow$ In general, we have a class $\mathcal{P}$ of probability measures extending P

Coherent extensions and non-additivity (de Finetti (1937))

Lower/upper probabilities (Walley 1981)

Given $P : \mathcal{E} \to [0,1]$, for any algebra $\mathcal{F}$ with $\mathcal{E} \subseteq \mathcal{F}$, we get two **non-additive set functions**

$$\begin{array}{ccccc} \underline{P}, \overline{P} & : & \mathcal{F} & \to & [0,1] \\ & & \Cup & & \Cup \\ & & E & \mapsto & \begin{array}{l} \underline{P}(E) = \min\limits_{P' \in \mathcal{P}} P'(E) \longleftarrow \textbf{pessimistic} \\ \overline{P}(E) = \max\limits_{P' \in \mathcal{P}} P'(E) \longleftarrow \textbf{optimistic} \end{array} \end{array}$$

QUESTION: Why should we relax **additivity**?

- By de Finetti's theory, if we start from an **incomplete probabilistic description** of a problem, we get:
  - a class $\mathcal{P}$ of compatible probability measures
  - a pair $\underline{P}, \overline{P}$ of non-additive set functions

$\Longrightarrow$ Basis of modern theories of **imprecise probabilities** by Williams (1975) and Walley (1981)

- In decision theory, Ellsberg (1961) called **ambiguity** the lack of a complete probabilistic description and showed that agents' behavior in such cases is not consistent with the **EU paradigm**

$\Longrightarrow$ This led to the **CEU paradigm** by Schmeidler (1989) and the **maximin EU paradigm** by Gilboa and Schmeidler (1989)

Pessimistic/optimistic uncertainty measures

Pessimistic uncertainty measures (Denneberg 1994, Dubois-Prade 1988)

Given an algebra $\mathcal{F} \subseteq 2^{\Omega}$, a capacity $\nu : \mathcal{F} \to [0,1]$ is said to be a:

(coherent) lower probability iff there exists a (closed and convex) set $\mathcal{P}$ of probability measures on $\mathcal{F}$ such that

$$\nu(E) = \min_{P \in \mathcal{P}} P(E).$$

belief function iff it is k-monotone capacity (for any $k \geq 2$)

$$\nu\left(\bigcup_{i=1}^{k} E_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} E_i\right)$$

$\Longrightarrow$ **Dual optimistic measure:** $\overline{\nu}(E) = 1 - \nu(E^c)$

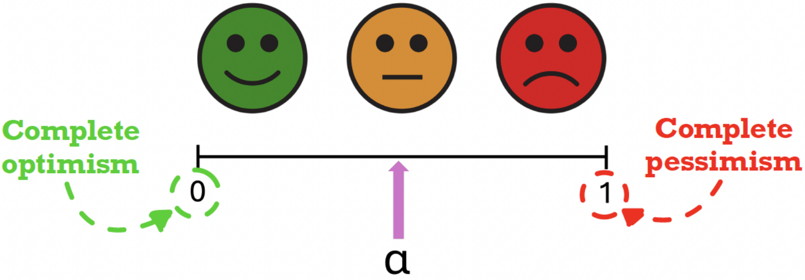
Legacy of de Finetti's theory in non-additive theories

Coherence as a bridge from additivity to non-additivity

- Lower/upper probabilities (Walley 1981)
- k-monotone/k-alternating capacities (Grabish 2012)
- Belief/plausibility functions (Fagin-Halpern 1991)
- Necessity/possibility measures (Coletti et al. 2013, 2016)
- α-DS mixtures (Petturiti-Vantaggi 2023)

Hurwicz criterion (Hurwicz (1951))

We fix a **pessimism index** $\alpha \in [0, 1]$:



For a gamble X, if we learn the piece of information $B \in \mathcal{U}$:

$$[\![X]\!]^{\alpha}(B) = \alpha \underbrace{\min_{\omega \in B} X(\omega)}_{\text{Worst result on } B} + (1-\alpha) \underbrace{\max_{\omega \in B} X(\omega)}_{\text{Best result on } B}$$

Betting scheme under PRU and the Hurwicz criterion[5]

Betting scheme

Let $\alpha \in [0,1]$. Given an arbitrary set of events $\mathcal{E} = \{E_1, \ldots, E_n\}$, that is $\mathcal{E} \subseteq 2^\Omega$, we consider an α-**DS mixture assessment**

$$\begin{array}{cccccl} \varphi_\alpha & : & \mathcal{E} & \to & [0,1] & \\ & & \Cup & & \Cup & \\ & & E_i & \mapsto & \varphi_\alpha(E_i) = & \text{“}\alpha\text{-mixture price for betting on } E_i\text{”} \end{array}$$

A **combination of bets** under PRU and the Hurwicz criterion on $E_1, \ldots, E_n$ with fixed **stakes** $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ produces the **random gain**

$$G(B) = \sum_{i=1}^{n} \lambda_i([\![\mathbf{1}_{E_i}]\!]^\alpha(B) - \varphi_\alpha(E_i)), \quad \forall B \in \mathcal{U}.$$

[5] Petturiti-Vantaggi 2023, Coletti et al. 2024

Coherence under PRU and the Hurwicz criterion

Definition

Let $\alpha \in [0,1]$. $\varphi_\alpha : \mathcal{E} \to [0,1]$ is **coherent** iff for all choices of stakes $\lambda_1, \ldots, \lambda_n$ we do not have a sure loss or a sure win (i.e., a **Dutch book**) under PRU and the Hurwicz criterion, that is

$$\min_{B \in \mathcal{U}} G(B) \leq 0 \leq \max_{B \in \mathcal{U}} G(B). \quad \longleftarrow \text{Normative axiom}$$

Theorem

Let $\alpha \in [0,1]$. Given an α-DS mixture assessment $\varphi_\alpha : \mathcal{E} \to [0,1]$, the following statements are equivalent:

(i) φ_α is coherent;

(ii) there exists an α-DS mixture such that $\varphi'_\alpha(E_i) = \varphi_\alpha(E_i)$, $i = 1, \ldots, n$.

Existing literature on recovering coherence in different framework

1 Probability

- Vantaggi (2008), Capotorti-Regoli-Vattari 2010, Brozzi-Capotori-Vantaggi (2012),
- Gilio-Sanfilippo 2011, Lad-Sanfilippo-Agro 2018
- Zhou-Deng 2023, Xue-Deng 2023
- Miranda-Montes 2017, Cozman 2012

1. AI Calculi

- Montes-Miranda-Vicig (2018, 2019), Miranda-Montes-Presa (2023)
- Petturiti-Vantaggi (2022), de Cooman (2005)

Dempster-Shafer theory: complete pessimism ($\alpha = 1$)

$$\varphi_1 \equiv \nu \text{ belief function}$$

PROS:

1. They are the non-additive measures "closest" to probability measures in terms of properties
2. Every ν is the **lower envelope** of a **closed** (in $[0,1]^{\mathcal{F}}$ endowed with the product topology) and **convex** set of probability measures

$$\mathrm{core}(\nu) = \{P : P \text{ is a probability measure on } \mathcal{F},\ P \geq \nu\}$$

CONS:

1. Their representation has generally exponential size in $d = |\Omega|$

Choquet expectation

A ν on $\mathcal{F}$ induces a **Choquet expectation** functional on $\mathbb{R}^{\Omega}$:

$$\oint X \mathrm{d}\nu = \int \nu(\omega | X(\omega) \geq x) dx$$

and in the discrete case

$$\sum_{i=1}^{d} [X(\omega_{\sigma(i)}) - X(\omega_{\sigma(i+1)})] \nu(\{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(i)}\})$$

where σ is a permutation such that $X(\omega_{\sigma(1)}) \geq \cdots \geq X(\omega_{\sigma(d)})$.

$\Longrightarrow$ If ν reduces to a P, then $\oint X \mathrm{d}P = \int X \mathrm{d}P$

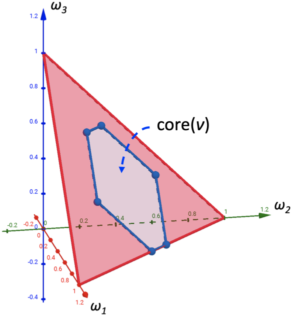
Lower and upper expectations (Schmeidler (1986))

$$\begin{aligned} \oint X \mathrm{d}\nu &= \min_{P \in \mathrm{core}(\nu)} \int X \mathrm{d}P \\ -\oint (-X) \mathrm{d}\nu &= \max_{P \in \mathrm{core}(\nu)} \int X \mathrm{d}P \end{aligned}$$

An example

Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and take:

$\mathcal{F}$	$\emptyset$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω
ν	0	0.1	0.2	0	0.4	0.4	0.5	1



Ω	ω_1	ω_2	ω_3
X	60	70	90

$$\mathrm{C}\!\!\!\!\int X\,\mathrm{d}\nu = 65$$

$$-\mathrm{C}\!\!\!\!\int (-X)\,\mathrm{d}\nu = 81$$

Conditioning in Dempster-Shafer theory

$\Longrightarrow$ The choice of a conditioning rule has a direct impact on computational issues and bid-ask spreads[6]

Product conditioning rule (Suppes-Zanotti 1977)

$$\nu(E|H) = \frac{\nu(E \cap H)}{\nu(H)} \quad \text{provided } \nu(H) > 0$$

$$\Downarrow$$

$\nu(\cdot|H)$ is a belief function on $\mathcal{F}$ generating $\mathrm{core}(\nu(\cdot|H))$

$$\Downarrow$$

$$\oint X(\omega)\mathrm{d}\nu(\omega|H) = \min_{P \in \mathrm{core}(\nu(\cdot|H))} \int X(\omega)\mathrm{d}P(\omega)$$

[6] Coletti et al 2013, 2016

Other popular conditioning rules

Dempster-Shafer rule (Dempster (1967))

$$\nu_D(E|H) = 1 - \frac{\overline{\nu}(E^c \cap H)}{\overline{\nu}(H)} = \frac{\nu((E \cap H) \cup H^c) - \nu(H^c)}{1 - \nu(H^c)}$$

Bayesian rule (Fagin and Halpern (1991))

$$\nu_B(E|H) = \min\left\{\frac{P(E \cap H)}{P(H)} \,:\, P \in \mathrm{core}(\nu)\right\} = \frac{\nu(E \cap H)}{\nu(E \cap H) + \overline{\nu}(E^c \cap H)}$$

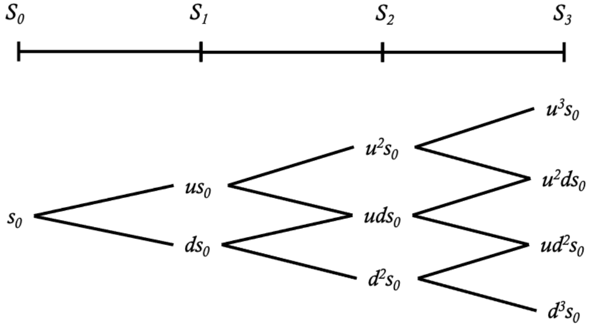
Dilation effect of Bayesian conditioning (Coletti et al. (2016))

$$\oint X(\omega)\mathrm{d}\nu_B(\omega|H) \leq \min\left\{\oint X(\omega)\mathrm{d}\nu(\omega|H), \oint X(\omega)\mathrm{d}\nu_D(\omega|H)\right\}$$

$\Longrightarrow$ No dominance relation between $\nu(\cdot|H)$ and $\nu_D(\cdot|H)$

Multiplicative binomial process

- $\{S_0, \ldots, S_T\}$, discrete-time finite-horizon process with $T \in \mathbb{N}$
- $S_0 = s_0 > 0$ and $S_n = \begin{cases} uS_{n-1} & \text{if "up"}, \\ dS_{n-1} & \text{if "down"}, \end{cases}$ with $u > d > 0$
- $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$



Literature on imprecise stochastic processes

- Hartfiel (1998)
- Kozine-Utkin (2002)
- Skulj (2006, 2009)
- de Cooman et al. (2009, 2016)
- Joens et al. (2021)
- de Cooman (2021)
- Persiau et al. (2022)

Remark

We are looking for a theory based on belief functions that can be described by few parameters.

Evaluating transitions in Dempster-Shafer theory

PROBLEM: Given the history $\{S_0 = s_0, \dots, S_n = s_n\}$, how to evaluate our beliefs on S_{n+t} in Dempster-Shafer theory?

Filtered belief space

$(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{T}, \nu)$ where $\nu : \mathcal{F} \to [0,1]$ is a belief function.

Set of t-step multiplicative coefficients for S_n:

$$\mathcal{A}_t = \{a_k = u^k d^{t-k} \ : \ k = 0, \dots, t\}, \quad \text{with } a_0 < a_1 < \dots < a_t$$

Transition belief function: for all $A \in \mathcal{P}(\mathcal{A}_t)$

$$A \mapsto \nu(S_{n+t} \in A s_n | S_0 = s_0, \dots, S_n = s_n)$$

DS-multiplicative binomial process[7]

Definition

Given a filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{T}, \nu)$, the process $\{S_0, \ldots, S_T\}$ is a **DS-multiplicative binomial process** when:

Markov property: if for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, $A \in \mathcal{P}(\mathcal{A}_t)$ it holds that

$$\nu(S_{n+t} \in As_n | S_0 = s_0, \ldots, S_n = s_n) = \nu(S_{n+t} \in As_n | S_n = s_n);$$

Time-homogeneity property: if for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, $A \in \mathcal{P}(\mathcal{A}_t)$ it holds that

$$\nu(S_{n+t} \in As_n | S_0 = s_0, \ldots, S_n = s_n) = \beta_t(A),$$

where $\beta_t : \mathcal{P}(\mathcal{A}_t) \to [0, 1]$ is a fixed belief function.

[7] Cinfrigni et al. 2023, 2024

Issues of existence and uniqueness

$\Longrightarrow$ Chapman-Kolmogorov equations do not hold for a non-additive ν

$\Longrightarrow$ One-step Markov and time-homogeneity properties do not imply global ones

$\Longrightarrow$ We need the entire family $\{\beta_t : t = 1, \ldots, T\}$ of **transition belief functions** that characterizes ν

We want a family $\{\beta_t : t = 1, \ldots, T\}$ of **transition belief functions**:

1. It is characterized by $b_u, b_d > 0$ and $b_u + b_d \leq 1$;
2. It has an interpretation.

Canonical family of transition belief functions

$$\beta_t(A) = \sum_{a_k \in A} \binom{t}{k} b_u^k b_d^{t-k} + \underbrace{\sum_{\substack{[a_k, a_{k+j}] \subseteq A \\ j \geq 1}} \binom{t-j}{k} b_u^k b_d^{t-j-k} (1 - b_u - b_d)}_{\substack{\text{Binomial-like weights of partial trajectories} \\ \text{with decreasing length starting from node } s_n \\ \text{supporting the evidence of having} \\ \text{a } t\text{-step state in } As_n}} \quad (1)$$

Proposition

The function $\beta_t : \mathcal{P}(\mathcal{A}_t) \to [0, 1]$ defined as in equation (1) is a belief function on $\mathcal{P}(\mathcal{A}_t)$.

Existence theorem

Theorem

There exists a belief function $\nu : \mathcal{F} \to [0, 1]$ *such that a multiplicative binomial process on the filtered belief space* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{T}, \nu)$ *meets the following properties:*

(i) $\nu(B) > 0$*, for every* $B \in \mathcal{F} \setminus \{\emptyset\}$*;*

(ii) $\{S_0, \ldots, S_T\}$ *is a DS-multiplicative binomial process whose transition belief functions* $\{\beta_t \,:\, t = 1, \ldots, T\}$ *satisfy* (1)*.*

ASSUMPTION: We assume the belief function ν meeting conditions *(i)*–*(ii)* of Theorem above to be fixed.

Conditionat Choquet expectation operator

Definition

Let $\{S_0, \ldots, S_T\}$ be a DS-multiplicative binomial process on the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{T}, \nu)$. Then, for every random variable $X \in \mathbb{R}^{\Omega}$, define:

- for all $\omega \in \{S_n = s_n\}$ set
$$\mathbb{C}[X|S_n](\omega) := \oint X \mathrm{d}\nu(\cdot|S_n = s_n)$$
- for all $\omega \in \{S_0 = s_0, \ldots, S_n = s_n\}$ set
$$\mathbb{C}[X|S_0, \ldots, S_n](\omega) := \oint X \mathrm{d}\nu(\cdot|S_0 = s_0, \ldots, S_n = s_n).$$

NOTATION: $\mathbb{C}[\cdot|\mathcal{F}_n] := \mathbb{C}[\cdot|S_0, \ldots, S_n]$.

Properties of $\mathbb{C}[\cdot|\mathcal{F}_n]$

$\Longrightarrow$ $\mathbb{C}[\cdot|\mathcal{F}_n]$ is positively homogeneous, monotone, comonotone additive, translation invariant and superadditive

$\Longrightarrow$ **Complete monotonicity:** for $k \geq 2$ and $X_1, \ldots, X_k \in \mathbb{R}^\Omega$,

$$\mathbb{C}\left[\bigvee_{i=1}^{k} X_i \middle| \mathcal{F}_n\right] \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \mathbb{C}\left[\bigwedge_{i \in I} X_i \middle| \mathcal{F}_n\right],$$

$\Longrightarrow$ **Conditional constant:** for $X, Y \in \mathbb{R}^\Omega$, $\mathcal{F}_n$-measurable X,

$$\mathbb{C}[X|\mathcal{F}_n] = X,$$
$$\mathbb{C}[XY|\mathcal{F}_n] = X\mathbb{C}[Y|\mathcal{F}_n], \quad \text{if } X \geq 0.$$

FAILURE OF THE TOWER PROPERTY: In general

$$\mathbb{C}[\mathbb{C}[X|\mathcal{F}_{n+t}]|\mathcal{F}_n] \neq \mathbb{C}[X|\mathcal{F}_n].$$

Closed-form expression for $X = \varphi(S_{n+t})$

Proposition

For every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, and every real-valued function of one real variable $\varphi(x)$ defined on the range of S_{n+t}, we have that

$$\mathbb{C}[\varphi(S_{n+t})|S_n = s_n] = \sum_{h=0}^{t} \varphi(a_h s_n) \binom{t}{h} b_u^h b_d^{t-h} + \sum_{j=1}^{t} \sum_{h=0}^{t-j} \left[\min_{a_i \in [a_h, a_{h+j}]} \varphi(a_i s_n) \right] \binom{t-j}{h} b_u^h b_d^{t-j-h} (1 - b_u - b_d)$$

and $\mathbb{C}[\varphi(S_{n+t})|S_0 = s_0, \ldots, S_n = s_n] = \mathbb{C}[\varphi(S_{n+t})|S_n = s_n]$.

$\Longrightarrow$ We have a simpler expression when $\varphi(x)$ is monotone

Choquet martingales

Definition

An adapted process $\{X_0, \ldots, X_T\}$ on the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{T}, \nu)$ is said to be a:

Choquet martingale: if, for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, it holds that

$$\mathbb{C}[X_{n+t}|\mathcal{F}_n] = X_n.$$

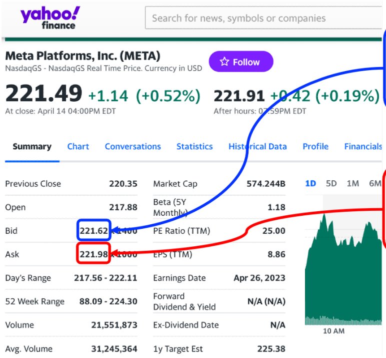
Choquet super[sub]-martingale: if, for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, it holds that

$$\mathbb{C}[X_{n+t}|\mathcal{F}_n] \leq [\geq] X_n.$$

$\Longrightarrow$ The above properties are called **one-step** if restricted to $t = 1$

A financial application

Is the absence of frictions hypothesis realistic?



Bid price (if we want to SELL):
The highest price that a buyer is willing to pay.

Ask price (if we want to BUY):
The lowest price that a seller is willing to accept.

OUR GOAL: Formulate a multi-period pricing problem in Dempster-Shafer theory so as to model bid-ask prices.

Existing literature on bid-ask pricing

1. Approaches based on probability theory:
   - Jouini and Kallal (1995)
   - Jouini (2000)
   - Bion-Nadal (2009)
   - Roux (2011)
2. Approaches based on Choquet theory:
   - Chateauneuf et al. (1996)
   - Cerreia-Vioglio et al. (2015)
   - Lécuyer and Lefort (2021)
   - Chateauneuf and Cornet (2022)
   - Cinfrignini et al. (2023, 2024)
   - Bastianello et al. (2024)
   - Petturiti and Vantaggi (2023, 2024)

Problem of approaches

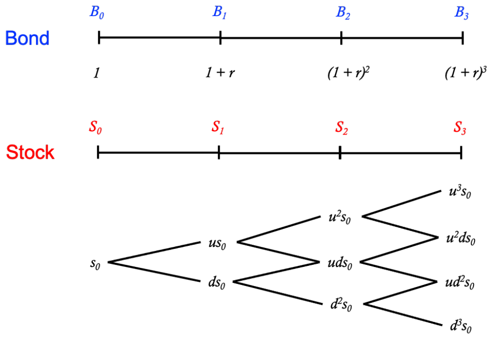
Many models focus on the single period case. **DS-multiplicative binomial processes!**

Market structure with bid-ask spreads

Fix $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{T}, \nu)$ and consider a market formed by:

- $\{B_0, \ldots, B_T\}$, lower ($\equiv$ upper) price of a **frictionless** bond
  $\Longrightarrow$ it is a deterministic process

- $\{S_0, \ldots, S_T\}$, lower price of a **frictional** stock (no dividends)
  $\Longrightarrow$ it is a DS-multiplicative binomial process



Discounted conditional Choquet expectation representation

PROBLEM: Can we find another belief function $\widehat{\nu}$ such that

$$S_n = \frac{1}{1+r}\widehat{\mathbb{C}}[S_{n+1}|\mathcal{F}_n], \quad \text{for } n = 0, \dots, T-1,$$

where $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ is computed with respect to $\widehat{\nu}$?

Real-world belief function

ν

versus

Equivalent one-step Choquet martingale belief function

$\widehat{\nu}$

$\{S_0, \dots, S_T\}$ **Discounting:** Bond as numéraire $\{S_0^*, \dots, S_T^*\}$

Theorem of change of measure

Theorem

The condition $u > 1 + r > d > 0$ *is necessary and sufficient to the existence of a belief function* $\widehat{\nu} : \mathcal{F} \to [0, 1]$ *equivalent to* ν *such that the discounted process* $\{S_0^*, \ldots, S_T^*\}$ *on the filtered belief space* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \widehat{\nu})$ *satisfies the following properties:*

(a) *it is a* **DS-multiplicative binomial process** *with transition belief functions* $\{\widehat{\beta}_t \, : \, t = 1, \ldots, T\}$ *satisfying* (1) *with*

$$u^* = \frac{u}{1+r}, \; d^* = \frac{d}{1+r}, \; \widehat{b_u} = \frac{(1+r) - d}{u - d}, \; \widehat{b_d} \in (0, 1 - \widehat{b_u}],$$

(b) *it is a one-step Choquet martingale,*

(c) *it is a Choquet super-martingale,*

(d) *it is a Choquet martingale if and only if* $\widehat{b_d} = 1 - \widehat{b_u}$.

Ongoing research ad future research

- Dynamic portfolio selection under ambiguity;
- Risk measures under partial knowledge and capital requirements;
- Optimal transport under partially specified marginal probabilities and related Wasserstein pseudo-distances;
- Dynamic mean field games under ambiguity.

Thanks for your attention!